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ON THE CIRCULAR INVERSION IN MAXIMUM PLANE

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Abstract

In this paper, we investigate general properties and basic concepts of circular inversions in the maximum plane. We delve into cross-ratio and harmonic conjugates under maximum circular inversion. Furthermore, we illustrated figures related to inversions obtained in the maximum plane via Mathematica.

Keywords: Inversion; maximum metric; cross-ratio; harmonic conjugates.

1. Introduction

Whatever we are working on these days, we think according to Euclidean geometry. Especially considering the distance between two points, the first thing that comes to mind is the Euclidean metric and its distance function. There are so many useful metrics for measuring distance. One of them is the maximum metric which is defined in maximum metric geometry. In [5], the maximum metric is defined as follows.

Definition 1.1. $X=(x_1,y_1)$ and $Y=(x_2,y_2)$ are two points in the Cartesian plane, the maximum metric distance is given by

$$d_M(X,Y)=\max\{|x_2-x_1|,|y_2-y_1|\}. \tag{1}$$

We use this distance function to define circular inversion in the maximum plane. Inversion in geometry is a transformation, that is not an isometry and not even an affine transformation. Inversions have the property that they transform certain circles in lines and that they preserve the angles. According to [5], inversion, which is a kind of study of transformations in the Euclidean plane, is also called "circular inversion" since it is defined on a circle. Inversion can be thought of as a reflection in the circle. Inversion can map the circle into the circle, circle into the line, or line into the circle. It is possible to apply inversion which has different transformation examples from subjects previously studied, to the solution of many problems in geometry. For more details about concepts and properties of inversions in different planes, see [1], [2] and [4].

In this paper, we define an inversion in the maximum plane. After giving the definition, we examine basic concepts and general properties of circular inversions in this plane. Cross-ratio and harmonic conjugates under maximum circular inversion are also studied. Moreover, we draw figures related to the properties of inversions that we obtained during this study.

2. Basic Concepts

In this section, we briefly mention some basic concepts. By the maximum metric d_M , the shortest path between the points P_1 and P_2 is a line segment that is parallel to a coordinate axis.

Proposition 2.1. Every Euclidean translation preserves the distance in the maximum plane. Thence, each of them is an isometry in $\mathbb{R}_M^2$.

Proposition 2.2. Let d_E denote the Euclidean distance function and l be the line passing through the points P_1 and P_2 in the analytical plane. If l has the slope m, then

$$d_M(P_1,P_2) = \frac{max\{1,|m|\}}{\sqrt{1+m^2}} d_E(P_1,P_2). \tag{2}$$

Proposition 2.2 states that d_M-distance along any line is a positive constant multiple of Euclidean distance along the same line.

Corollary 2.3. Let P_1, P_2 and X be three collinear points in $\mathbb{R}^2$. Then, $d_E(P_1,X) = d_E(P_2,X)$ if and only if $d_M(P_1,X) = d_M(P_2,X)$.

Corollary 2.4. Let P_1, P_2 and X be three distinct collinear points in $\mathbb{R}^2$. Then,

$$d_E(P_1,X)/d_E(P_2,X) = d_M(P_1,X)/d_M(P_2,X). \tag{3}$$

That is the ratios of the Euclidean and d_M-distance along a line are the same.

Definition 2.5. Let $\mathcal{C}$ be a circle centered at a point O with radius r. If P is any point other than O, the inverse of P with respect to $\mathcal{C}$ is the point P' on the ray $\overrightarrow{OP}$ such that the product of the distances of P and P' from O is equal to r^2 , that is

$$d_E(O,P).\, d_E(O,P') = r^2, \tag{4}$$

see [3].

Clearly, if P' is the inverse point of P, then P is the inverse point of the P'. Note that if P is in the interior of $\mathcal{C}$, P' is exterior to $\mathcal{C}$; and vice-versa. So, the interior of $\mathcal{C}$ except for O is mapped to the exterior and the exterior to the interior. $\mathcal{C}$ itself is left by the inversion pointwise fixed. O has no image, and no point of the plane is mapped to O. However, points close to O are mapped to points far from O, and points far from O mapped to points close to O. By adjoining one "ideal point", or "point at infinity", to the Euclidean plane, we can include O in the domain and range of an inversion.

Now in $\mathbb{R}_M^2$, the definition of inversion with respect to a M-circle (maximum circle) can be given as follows.

Definition 2.6. Let $\mathcal{C}$ be a M-circle centered at point O with radius r in $\mathbb{R}_M^2$, and P_∞ be the ideal point adjoined to the maximum plane. In $\mathbb{R}_M^2$, the maximum circular inversion with respect to $\mathcal{C}$ is the transformation

$$I_M(O,r): \mathbb{R}_M^2 \cup \{P_\infty\} \to \mathbb{R}_M^2 \cup \{P_\infty\}$$

given by

$$d_M(O,P).d_M(O,P') = r^2, \tag{5}$$

where $I_M(O,r)(O) = P_\infty,\ I_M(O,r)(P_\infty) = O\ ,\ I_M(O,r)(P) = P'$ for $P \neq 0$ and P' is on the ray $\overrightarrow{OP}$.

Lemma 2.7. Let C be the M-circle which is centered at the origin and the radius is r. If the point P is in the interior of C, the point P' is in the exterior to C, and vice-versa.

Proof. Let us consider that the point P is in the interior of C. Thus,

$$d_M(O,P) < r\ .$$

Since $P' = I_M(O,r)$ and from Equation 5, then it is obtained

$$r^2 = d_M(O,P).d_M(O,P') < r.d_M(O,P')$$

and

$$d_M(O,P') > r.$$

So, the point P' is in the exterior of C.

Proposition 2.8. Let $I_M(O,r)$ be the maximum circular inversion, with respect to a M-circle, C centered at the orijin and the radius is r in $\mathbb{R}_M^2$. Therefore, the maximum circular inverse of the point $P = (x,y)$ is the point $P' = (x',y')$, whose coordinates are

$$x' = \frac{r^2 x}{(\max\{|x|,|y|\})^2}\ ,\ y' = \frac{r^2 y}{(\max\{|x|,|y|\})^2}\ . \tag{6}$$

Proof. The M-circle C, which is centered at the origin and has the radius r, is the set of points satisfies the equation $\max\{|x|,|y|\} = r$. Let $P = (x,y)$ and $P' = (x',y')$ are inverse points with respect to C. Since the points O, P and P' are collinear and the rays $\overrightarrow{OP}$ and $\overrightarrow{OP'}$ have the same direction, then $\overrightarrow{OP'} = k.\overrightarrow{OP}$ for $k \in \mathbb{R}^+$, and $(x',y') = (kx,ky)$. Using by $d_M(O,P).d_M(O,P') = r^2$, $k = \frac{r^2}{(\max\{|x|,|y|\})^2}$ is obtained and by substituting the obtained value of k, the required results are obtained. Note that, if the point P' is inverse of P, then P is the inverse of P'. As a result of this, the equivalent form would be written as

$$x = \frac{r^2 x'}{(\max\{|x'|,|y'|\})^2}\ ,\ y = \frac{r^2 y'}{(\max\{|x'|,|y'|\})^2}\ . \tag{7}$$

Corollary 2.6. Let $I_M(O',r)$ be the maximum circular inversion, with respect to a M-circle centered at $O' = (a,b)$ and the radius is r in $\mathbb{R}_M^2$, then the maximum circular inverse of the point $P = (x,y)$ is the point $P' = (x',y')$, whose coordinates are

$$x' = a + \frac{r^2(x-a)}{(\max\{|x-a|,|y-b|\})^2}\ ,\ y' = b + \frac{r^2(y-b)}{(\max\{|x-a|,|y-b|\})^2}\ . \tag{8}$$

Proof. The proof is obvious by the fact that all translations are isometries of the maximum plane.

Remark 2.7. It is clear that the interior of C_M , except the center O, is mapped to the exterior and exterior to the interior under maximum circular inversion.

3. Circular Inversions in $\mathbb{R}_M^2$

In this section, the results and the definitions obtained by maximum circular inversion are given. First, inversions of lines and circles according to their positions in $\mathbb{R}_M^2$ are investigated. In addition, properties of inversions in the Euclidean and the Maximum planes are compared. First, the following properties of inversion in the Euclidean plane, which are well known, will be given as:

i. The inverse of a line through the center of inversion is the line itself.
ii. The inverse of a line not passing through the center of inversion is a circle passing through the center of inversion and conversely.
iii. The inverse of a circle not passing through the center of inversion is a circle not passing through the center of inversion.
iv. Circles with center of inversion are mapped into circles with center of inversion.

All of the properties of inversion in the Euclidean space which are given above are not valid in the maximum plane. We now give the theorem to show which properties given above are satisfied or not in the maximum plane.

Theorem 3.1.

i. Lines passing through the center O are mapped onto themselves under the maximum circular inversion $I_M(O,r)$.
ii. Lines not containing the center of the maximum circular inversion circle are not mapped onto maximum circles centered O under the maximum circular inversion $I_M(O,r)$.
iii. Maximum circles centered O are mapped onto maximum circles with the center O under the maximum circular inversion $I_M(O,r)$.
iv. Maximum circles not through O are not mapped onto any maximum circles under the maximum circular inversion $I_M(O,r)$.
v. Maximum circles containing the center of inversion circle are not mapped onto straight lines not containing the center O under the maximum circular inversion $I_M(O,r)$.

Proof. By examining all possible cases the properties in the Theorem 3.1. are obtained.
For i. and ii. let $ax+by+c=0$ be a line in the maximum plane. By using Equation 7, it is acquired that

$$\frac{ax'r^2}{(\max\{|x'|,|y'|\})^2}+\frac{by'r^2}{(\max\{|x'|,|y'|\})^2}+c=0. \tag{9}$$

Thus, it can be written as

$$ax'r^2+by'r^2+c(\max\{|x'|,|y'|\})^2=0. \tag{10}$$

Now, Equation 10 would be considered under cases which are given below:

Case 1. If $|x'| \geq |y'|$, then $c(x')^2 + ar^2x' + br^2y' = 0$.

1.1. If $c = 0$, the inverse of the line $ax + by = 0$ is $ax' + by' = 0$ that means both lines are the same.

1.2. If $c \neq 0, a = 0$, the inverse of the line $by + c = 0$ is the parabola $c(x')^2 + br^2y' = 0$

1.3. If $c \neq 0, b = 0$, the inverse of the line $ax + c = 0$ is the line $x' = 0$ or $x' = \frac{-ar^2}{c}$.

1.4. If $c \neq 0, a \neq 0$ and $b \neq 0$, the inverse of the line $ax + by + c = 0$ is the parabola $c(x')^2 + ar^2x' + br^2y' = 0$.

Case 2. If $|y'| \geq |x'|$, then $c(y')^2 + br^2y' + ar^2x' = 0$.

2.1. If $c = 0, a \neq 0$ and $b \neq 0$, the inverse of the line $ax + by = 0$ is $ax' + by' = 0$ which means both lines are the same.

2.2. If $c \neq 0, a = 0$, the inverse of the line $by + c = 0$ is the line $y' = 0$ or $y' = \frac{-br^2}{c}$.

2.3. If $c \neq 0, b = 0$, the inverse of the line $ax + c = 0$ is the parabola $c(y')^2 + ar^2x' = 0$.

2.4. If $c \neq 0, a \neq 0$ and $b \neq 0$, the inverse of the line $ax + by + c = 0$ is the parabola $c(y')^2 + br^2y' + ar^2x' = 0$.

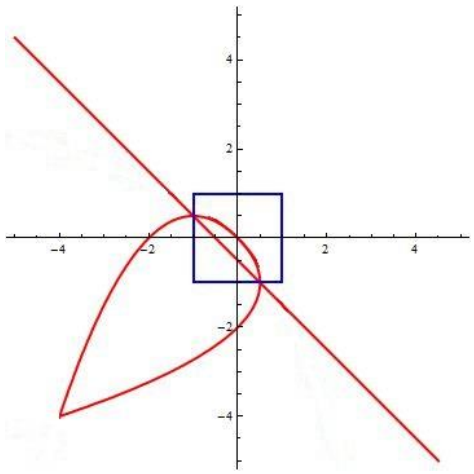

Figure 1. A line not passing through O isn't mapped onto a maximum circle with center O

For iii. let $\max\{|x|, |y|\} = r_1$ be the radius of the circle C' whose center is the same with C the maximum circular inversion circle. The inversion of this circle respect to C is

$$\max\left\{\left|\frac{|x'|r^2}{(\max\{|x'|,|y'|\})^2}\right|, \left|\frac{|y'|r^2}{(\max\{|x'|,|y'|\})^2}\right|\right\} = r_1 \tag{11}$$

and

$$\max\{|x'|, |y'|\} = \frac{r^2}{r_1} = r_2. \tag{12}$$

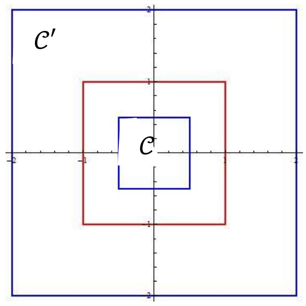

Figure 2. A maximum circle centered O is mapped onto a maximum circle with center O.

The other properties would be obtained similarly by using the definition of the maximum circular inversion and Proposition 2.8.

4. The Cross Ratio and Harmonic Conjugates in $\mathbb{R}_M^2$

The inversion in maximum plane is not an isometry. Thence, the distance is not preserved under maximum circular inversion. However, related to the concept of the distance, it can be shown that the cross-ratio is preserved under maximum circular inversion. Thus, in this section, the cross-ratio and harmonic conjugates in $\mathbb{R}_M^2$ are investigated.

Proposition 4.1. Let P, Q and O be three different collinear points in $\mathbb{R}_M^2$. If P' and Q' are inverses of P and Q respectively with respect to the maximum inversion circle $I_M(O, r)$, then

$$d_M(P', Q') = \frac{r^2 d_M(P,Q)}{d_M(O,P).d_M(O,Q)} \tag{13}$$

is obtained.

Proof. Let P, Q and O be three different collinear points, $P = (x_1, y_1), Q = (x_2, y_2), P' = (x'_1, y'_1)$ and $Q' = (x'_2, y'_2)$. Note that inverse points P' and Q' lies on the same line l with P, Q and O. If the slope of line l is m, then two cases would be considered; $|m| \geq 1$ and $|m| \leq 1$.

If $|m| \geq 1$, then

$$d_M(P', Q') = \max\{|x_2' - x_1'|, |y_2' - y_1'|\}.$$

$$=\max\left\{\left|\frac{r^2.x_2}{(\max\{|x_2|,|y_2|\})^2} - \frac{r^2.x_1}{(\max\{|x_1|,|y_1|\})^2}\right|, \left|\frac{r^2.y_2}{(\max\{|x_2|,|y_2|\})^2} - \frac{r^2.y_1}{(\max\{|x_1|,|y_1|\})^2}\right|\right\}$$

$$=\max\left\{\left|\frac{r^2.x_2}{(y_2)^2} - \frac{r^2.x_1}{(y_1)^2}\right|, \left|\frac{r^2}{y_2} - \frac{r^2}{y_1}\right|\right\}$$

$$= \left|\frac{r^2}{y_2} - \frac{r^2}{y_1}\right| = \frac{r^2|y_1 - y_2|}{|y_1||y_2|}$$

$$= \frac{r^2 d_M(P,Q)}{d_M(O,P).d_M(O,Q)} \tag{14}$$

is acquired.

The case $|m| \leq 1$ can be easily shown with a similar method.

If P, Q and O are not collinear, then the equality in Proposition 3.2.1 is not valid for all P, Q in $\mathbb{R}_M^2$. For example, let $O = (0,0), P = (-1,1)$ and $Q = (1,2)$ and the radius is $r = 2$. The inversion $I_M(O, 2)$ maps P and Q onto $P' = (-4,4)$ and $Q' = (1,2)$, respectively. Then, it can be easily computed that $d_M(P,Q) = 2,\ d_M(P',Q') = 5, d_M(O,P) = 1$ and $d_M(O,Q) = 2$. So, the equality in Proposition 4.1 is obviously not valid for every points in $\mathbb{R}_M^2$. However, the following two propositions show some conditions that the equality in Proposition 4.1 is satisfied.

Proposition 4.2. Let C be the maximum inversion circle which is centered at origin and the radius is r. Let P, Q and O be any three distinct non-collinear points in $\mathbb{R}_M^2$. If P' and Q' are inverses of P and Q respectively and P and Q lie on the lines with slope $\{0, \infty\}$ or $\{1, -1\}$ passing through the origin, then

$$d_M(P',Q') = \frac{r^2 d_M(P,Q)}{d_M(O,P).d_M(O,Q)} \tag{15}$$

is obtained.

Proof. Note that $P = (p, 0)$ and $Q = (0, q)$ are mapped to $P' = \left(\frac{r^2}{p}, 0\right)$ and $Q' = \left(0, \frac{r^2}{q}\right)$ or $P = (p, -p)$ and $Q = (q, q)$ are mapped to $P' = \left(\frac{r^2}{p}, \frac{-r^2}{p}\right)$ and $Q' = \left(\frac{r^2}{q}, \frac{r^2}{q}\right)$. So, it can be easily shown that

$$d_M(P',Q') = \frac{r^2 d_M(P,Q)}{d_M(O,P).d_M(O,Q)}. \tag{16}$$

Proposition 4.3. Let C be the maximum inversion circle which is centered at origin and the radius is r and let P, Q and O be any three distinct non-collinear points in $\mathbb{R}_M^2$. If the slope of the line passing through P and Q is 1 and $x_P y_Q + y_P x_Q = 0$ where $P = (x_P, y_P)$ and $Q = (x_Q, y_Q)$, then

$$d_M(P',Q') = \frac{r^2 d_M(P,Q)}{d_M(O,P).d_M(O,Q)}. \tag{17}$$

Proof. Let the line passing through P and Q be $l: y = x + c$. Note that $P = (p, p + c)$ and $Q = (q, q + c)$ are mapped to $P' = \left(\frac{r^2}{p}, \frac{(p+c)r^2}{p}\right)$ and $Q' = \left(\frac{r^2}{q}, \frac{(q+c)r^2}{q}\right)$ respectively. Therefore, it can be easily shown that

$$d_M(P',Q') = \frac{r^2 d_M(P,Q)}{d_M(O,P).d_M(O,Q)}. \tag{18}$$

Let $d_M[PQ]$ denotes the maximum directed distance from P to Q along a line in the maximum plane. If the ray with initial point P containing Q has the positive direction of orientation, then $d_M[PQ] = d_M(P,Q)$. If the ray has the opposite direction, then $d_M[PQ] = -d_M(P,Q)$.

Now let P, Q, R and S are four distinct points on an oriented line in the maximum plane. Therefore, their maximum cross ratio $(PQ, RS)_M$ is defined by

$$(PQ, RS)_M = \frac{d_M[PR] d_M[QS]}{d_M[PS] d_M[QR]}. \tag{19}$$

Note that the maximum cross ratio is positive if both R and S are between P and Q or if neither R nor S is between P and Q, whereas the cross ratio is negative if pairs $\{P,Q\}$ and $\{R,S\}$ seperate each other. Also a maximum circular inversion with respect to $\mathcal{C}$ centered at origin which is different from P,Q,R and S preserve the maximum cross ratio.

Theorem 4.4. The maximum circular inversion preserves the maximum cross ratio.

Proof. Suppose that P,Q,R and S are four collinear points in the maximum plane. Let P',Q',R' and S' be inverse points of P,Q,R and S respectively according to the maximum circular inversion $I_M(O,r)$. Note that maximum circular inversion preserves the seperation or non-seperation of the pairs $\{P,Q\}$ and $\{R,S\}$ and also it reverses the maximum-directed distance from the point P to the point Q along a line l to maximum-directed distance from the point Q' to the point P'. The required result follows from Proposition 4.1 as

$$(P'Q',R'S')_M = \frac{d_M(P',R')d_M(Q'S')}{d_M(P'S')d_M(Q'R')}$$

$$= \frac{\frac{r^2 d_M(P,R)}{d_M(O,P)d_M(O,R)} \cdot \frac{r^2 d_M(Q,S)}{d_M(O,Q)d_M(O,S)}}{\frac{r^2 d_M(P,S)}{d_M(O,P)d_M(O,S)} \cdot \frac{r^2 d_M(Q,R)}{d_M(O,Q)d_M(O,R)}}$$

$$= \frac{d_M(P,R)d_M(Q,S)}{d_M(P,S)d_M(Q,R)}$$

$$= (PQ,RS)_M. \tag{20}$$

Let l be a line in $\mathbb{R}^2_M$. Suppose that P,Q,R and S are four points on l. It is called that P,Q,R and S form a harmonic set if $(PQ,RS)_M = -1$, and it is denoted by $H(PQ,RS)_M$. That is, any pair R and S on l for which

$$\frac{d_M[PR]d_M[QS]}{d_M[PS]d_M[QR]} = -1 \tag{21}$$

is said to divide P and Q harmonically. The points R and S are called maximum harmonic conjugates with respect to P and Q.

Theorem 4.5. Let $\mathcal{C}$ be a maximum circle with center O, and line segment $[PQ]$ a diameter of $\mathcal{C}$ in $\mathbb{R}^2_M$. Let R and S be distinct points of the ray $\overrightarrow{OP}$, which divide the segment $[PQ]$ internally and externally. Thus, R and S are maximum harmonic conjugates with respect to P and Q if and only if R and S are inverse points with respect to the maximum circular inversion $I_M(O,r)$.

Proof. Let R and S are maximum harmonic conjugates with respect to P and Q. Then,

$$(PQ,RS)_M = -1 \Rightarrow \frac{d_M[PR]d_M[QS]}{d_M[PS]d_M[QR]} = -1. \tag{22}$$

Since R divides the line segment $[PQ]$ internally and R is on the ray $\overrightarrow{OQ}$,

$$d_M(R,Q) = r - d_M(O,R) \text{ and } d_M(P,R) = r + d_M(O,R). \tag{23}$$

Since S divides the line segment $[PQ]$ externally and S is on the ray $\overrightarrow{OQ}$, it is obtained that

$$d_M(P,S) = r + d_M(O,S) \text{ and } d_M(Q,S) = d_M(O,S) - r. \tag{24}$$

Thus,

$$\frac{(r+d_M(O,R))(d_M(O,S)-r)}{(r+d_M(O,S))(r-d_M(O,R))} = -1 \tag{25}$$

$$\Rightarrow (r + d_M(O,R))(d_M(O,S) - r) = (r + d_M(O,S))(d_M(O,R) - r). \tag{26}$$

By simplifying the last equality, $d_M(O,R).\, d_M(O,S) = r^2$ is obtained. Then, R and S are the maximum inverse points with respect to the maximum circular inversion $I_M(O,r)$. For the other condition (S and R are on the ray $\overrightarrow{OP}$) with similar calculations, the same conclusion is obtained. Conversely, if R and S are maximum inverse points with respect to the maximum circular inversion $I_M(O,r)$, it can be proven with a similar method.

Conclusions

Inversions are not isometries. They transform distances and angles. We examine the way inversions transform the distance in the Maximum plane. We investigate general properties and basic concepts of circular inversions by means of maximum metric. We delve into cross-ratio and harmonic conjugates under maximum circular inversion. In addition, via Mathematica, we illustrated figures related to results we acquired. Drawing their figures reinforce the visualization of the results. Within the knowledge of the maximum metric is the special case of the alpha metric, which is mentioned in [4], we study the maximum circular inversions by examining the special cases in detail. Thence, it is expected to contribute to the literature on inversions.

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