*Fundamental Journal of Mathematics and Applications*, 4 (1) (2021) 17-24
**Research Article**

FUJMA

# Fundamental Journal of Mathematics and Applications

Journal Homepage: www.dergipark.gov.tr/fujma
ISSN: 2645-8845
doi: 10.33401/fujma.831447

# Faber Polynomial Expansion for a New Subclass of Bi-univalent Functions Endowed with $(p,q)$ Calculus Operators

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**Article Info**

***Keywords:*** *Bi-univalent functions, Faber polynomial expansion, $(p,q)-$calculus*
***2010 AMS:*** *30C45, 30C80, 05A30*
***Received:*** *25 November 2020*
***Accepted:*** *30 January 2021*
***Available online:*** *31 January 2021*

**Abstract**

In this paper, we use the Faber polynomial expansion techniques to get the general Taylor-Maclaurin coefficient estimates for $|a_n|$, $(n \geq 4)$ of a generalized class of bi-univalent functions by means of $(p,q)-$calculus, which was introduced by Chakrabarti and Jagannathan. For functions in such a class, we get the initial coefficient estimates for $|a_2|$ and $|a_3|$. In particular, the results in this paper generalize or improve (in certain cases) the corresponding results obtained by recent researchers.

## 1. Introduction

Let $\mathscr{A}$ indicate the class of functions $f$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disc $\mathbb{D} = \{z : |z| < 1\}$ and satisfy the conditions $f(0) = 0, f'(0) = 1$ for every $z \in \mathbb{D}$. Denote by $\mathscr{S}$ the subclass of $\mathscr{A}$ containing of all univalent functions. Let $\Omega$ be the class of Schwarz functions $\phi$, which are analytic in $\mathbb{D}$ satisfying the conditions $\phi(0) = 0$ and $|\phi(z)| < 1$ for all $z \in \mathbb{D}$. If $f_1$ and $f_2$ are analytic functions in $\mathbb{D}$, then we state $f_1$ is subordinate to $f_2$, denoted by $f_1 \prec f_2$, if there exists a Schwarz function $\phi \in \Omega$ such that $f_1(z) = f_2(\phi(z))$ (see [1]).

According to the Koebe 1/4 Theorem [1], the range of $\mathbb{D}$ under every function $f$ in the univalent function class $\mathscr{S}$ contains a disc $\{w : |w| < 1/4\}$ of radius $1/4$. Thus, every univalent function $f$ has an inverse $f^{-1}$ satisfying the conditions

$$f^{-1}(f(z)) = z, \quad (z \in \mathbb{D})$$

and

$$f(f^{-1}(w)) = w, \quad (|w| < r_0(f);\ r_0(f) \geq 1/4),$$

where

$$f^{-1}(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \cdots .$$

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If both $f$ and $f^{-1}$ are univalent in $\mathbb{D}$, then a function $f \in \mathscr{A}$ is said to be bi-univalent in $\mathbb{D}$. The class of bi-univalent functions will be denoted by $\Sigma$ in $\mathbb{D}$.

Not much is known about the bounds for $|a_n|$ of Faber polynomials in quantum calculus because the bi-univalency requirement makes the behaviour of the coefficients of the functions $f$ and $f^{-1}$ unpredictable. The quantum calculus has a great number of applications in the fields of special functions and other areas (see [2], [3]). There is a possibility to extend some of the results in quantum calculus to post quantum calculus in geometric function theory.

Let us first recall certain notations of the $(p,q)-$calculus. The $(p,q)-$twin-basic number $[n]_{p,q}$ is defined by

$$[n]_{p,q} = \frac{p^n - q^n}{p-q}, \ \ (0 < q < p \leq 1, n = 0,1,2,...).$$

The $(p,q)-$derivative operator of a function $f$ is given by

$$(D_{p,q}f)(z) = \frac{f(pz) - f(qz)}{(p-q)z}, \ \ (z \neq 0) \tag{1.2}$$

and $(D_{p,q}f)(0) = f'(0)$ provided that the function $f$ is differentiable at $z = 0$ (see [4]). For a function $f$ given by (1.1), it can be easily concluded that

$$D_{p,q}f(z) = 1 + \sum_{n=2}^{\infty} [n]_{p,q} a_n z^{n-1}. \tag{1.3}$$

Note that, for $p = 1$, $(p,q)-$derivative operator reduces to the Jackson $q-$derivative ([5], [6]) given by

$$(D_q f)(z) = \frac{f(z) - f(qz)}{(1-q)z}, \ \ (z \neq 0). \tag{1.4}$$

Also, for $p = 1$, $q-$bracket $[n]_q$ is given by

$$[n]_q = \frac{1-q^n}{1-q}, \ \ (n = 0,1,2,...).$$

In 1903, G. Faber [7] in his thesis, introduced the polynomials which have since proved useful in analysis, and hence are known as Faber polynomials. By using the Faber polynomial expansion of functions $f \in \mathscr{A}$, researchers in [8] got the following useful results.

**Lemma 1.1.** *If $f$ is of the form (1.1), then the coefficients of its inverse functions $g = f^{-1}$ are given by*

$$g(w) = f^{-1}(w) = w + \sum_{n=2}^{\infty} \frac{1}{n} K_{n-1}^{-n}(a_2, a_3, ...)w^n := w + \sum_{n=2}^{\infty} b_n w^n,$$

*where*

$$\begin{aligned} K_{n-1}^{-n} = & \tfrac{(-n)!}{(-2n+1)!(n-1)!} a_2^{n-1} + \tfrac{(-n)!}{(2(-n+1))!(n-3)!} a_2^{n-3} a_3 + \tfrac{(-n)!}{(-2n+3)!(n-4)!} a_2^{n-4} a_4 \\ & + \tfrac{(-n)!}{(2(-n+2))!(n-5)!} a_2^{n-5} [a_5 + (-n+2)a_3^2] \\ & + \tfrac{(-n)!}{(-2n+5)!(n-6)!} a_2^{n-6} [a_6 + (-2n+5)a_3 a_4] + \textstyle\sum_{l \geq 7} a_2^{n-l} V_l \end{aligned}$$

*such that $V_l, (7 \leq l \leq n)$ is a homogeneous polynomial in the variables $a_2, a_3, ..., a_n$. The first three terms of $K_{n-1}^{-n}$ are given below:*

$$K_1^{-2} = -2a_2, \quad K_2^{-3} = 3(2a_2^2 - a_3), \quad K_3^{-4} = -4(5a_2^3 - 5a_2 a_3 + a_4).$$

Making use of $(p,q)-$derivative operator defined in (1.2), we define the class $\mathscr{N}_\Sigma(p,q;\lambda,\delta,A,B)$ as below:

**Definition 1.2.** *Let $A$ and $B$ be real numbers such that $-1 \leq B < A \leq 1$. For $0 < q < p \leq 1, \lambda \geq 1, \delta \geq 0$, a bi-univalent function $f \in \Sigma$ is said to be in $\mathscr{N}_\Sigma(p,q;\lambda,\delta,A,B)$ if*

$$(1-\lambda)\frac{f(z)}{z} + \lambda(D_{p,q}f)(z) + \delta z D_{p,q}(D_{p,q}f)(z) \prec \frac{1+Az}{1+Bz}, \ \ (z \in \mathbb{D}) \tag{1.5}$$

*and*

$$(1-\lambda)\frac{g(w)}{w} + \lambda(D_{p,q}g)(w) + \delta w D_{p,q}(D_{p,q}g)(w) \prec \frac{1+Aw}{1+Bw}, \ \ (w \in \mathbb{D}) \tag{1.6}$$

*where $g(w) = f^{-1}(w)$ for $w \in \mathbb{D}$.*

By taking different values of the parameters $p,q,\lambda,\delta,A,B$, we may obtain several new and known subclasses of the family $\mathscr{N}_\Sigma(p,q;\lambda,\delta,A,B)$; for instance we have

(i) $\mathscr{G}_\Sigma(q;\lambda,\delta,A,B) \equiv \mathscr{N}_\Sigma(1,q;\lambda,\delta,A,B)$.
(ii) $\mathscr{D}_\Sigma(p,q;\lambda,\frac{1+Az}{1+Bz}) \equiv \mathscr{N}_\Sigma(p,q;\lambda,0,A,B)$.
(iii) $\mathscr{R}_\Sigma(\lambda,\delta,\alpha) \equiv \mathscr{N}_\Sigma(1,1;\lambda,\delta,1-2\alpha,-1),(0\leq\alpha<1)$, [9].
(iv) $\mathscr{T}_\Sigma(\lambda,\alpha) \equiv \mathscr{N}_\Sigma(1,1;\lambda,0,1-2\alpha,-1),(0\leq\alpha<1)$, [10].
(v) $\mathscr{H}_\Sigma(\alpha) \equiv \mathscr{N}_\Sigma(1,1;1,0,1-2\alpha,-1),(0\leq\alpha<1)$, [11].
(vi) $\mathscr{M}_\Sigma(\delta,\alpha) \equiv \mathscr{N}_\Sigma(1,1;1,\delta,1-2\alpha,-1),(0\leq\alpha<1)$, [12].

**Remark 1.3.** *Note that the class $\mathscr{G}_\Sigma(q;\lambda,\delta,A,B)$ in (i) is a new generalized class of bi-uivalent functions defined by $D_q=\lim_{p\to 1} D_{p,q}$ given in (1.4).*

**Remark 1.4.** *The class $\mathscr{D}_\Sigma(p,q;\lambda,\frac{1+Az}{1+Bz})$ in (ii) may be obtained by letting $\varphi=\frac{1+Az}{1+Bz}$ in the class $\mathscr{D}_\Sigma(p,q;\lambda,\varphi)$ which was studied in 2017 by Altınkaya and Yalçın [13]. The results in our paper improve the estimates of the corresponding bounds in [13]. Similarly, our results are also better than those determined in [11].*

In view of the relations witnessed in (i) to (vi) and Remarks 1.3 and 1.4, we conclude that the generalized class $\mathscr{N}_\Sigma(p,q;\lambda,\delta,A,B)$ unifies several subclasses of $\Sigma$.

## 2. Main results

We first give coefficient estimates of a function $f$ in the class $\mathscr{N}_\Sigma(p,q;\lambda,\delta,A,B)$ for all the coefficients except for the first initial coefficients $a_2$ and $a_3$.

**Theorem 2.1.** *For $0<q<p\leq 1$, $\delta\geq 0$, $\lambda\geq 1$, $-1\leq B<A\leq 1$, let the function $f$ given by (1.1) be in the class $\mathscr{N}_\Sigma(p,q;\lambda,\delta,A,B)$. If $a_m=0,(2\leq m\leq n-1)$, then*

$$|a_n|\leq \frac{A-B}{|1+([n]_{p,q}-1)\lambda+[n]_{p,q}[n-1]_{p,q}\delta|},\quad (n\geq 4). \tag{2.1}$$

*Proof.* If a function $f$ given by (1.1) is in $\mathscr{N}_\Sigma(p,q;\lambda,\delta,A,B)$, then by using (1.2) and (1.3), the left side of (1.5) gives

$$(1-\lambda)\frac{f(z)}{z}+\lambda(D_{p,q}f)(z)+\delta z D_{p,q}(D_{p,q}f)(z)=1+\sum_{n=2}^{\infty}\left[1+([n]_{p,q}-1)\lambda+[n]_{p,q}[n-1]_{p,q}\delta\right]a_n z^{n-1}. \tag{2.2}$$

In view of (1.2), (1.3) and Lemma 1.1, the left side of (1.6) yields

$$\begin{aligned}&(1-\lambda)\tfrac{g(w)}{w}+\lambda(D_{p,q}g)(w)+\delta w D_{p,q}(D_{p,q}g)(w)\\ &=1+\textstyle\sum_{n=2}^{\infty}\left[1+([n]_{p,q}-1)\lambda+[n]_{p,q}[n-1]_{p,q}\delta\right]b_n w^{n-1}\\ &=1+\textstyle\sum_{n=2}^{\infty}\left[1+([n]_{p,q}-1)\lambda+[n]_{p,q}[n-1]_{p,q}\delta\right]\times\frac{1}{n}K_{n-1}^{-n}(a_2,a_3,\ldots,a_n)w^{n-1},\end{aligned} \tag{2.3}$$

where $K_{n-1}^{-n}(a_2,a_3,\ldots,a_n)$ are given in Lemma 1.1.

On the other hand, (1.5) and (1.6) imply the existence of two Schwarz functions $\phi(z)=\sum_{n=1}^{\infty}c_n z^n$, $(z\in\mathbb{D})$ and $\psi(w)=\sum_{n=1}^{\infty}d_n z^n,(w\in\mathbb{D})$ so that

$$(1-\lambda)\frac{f(z)}{z}+\lambda(D_{p,q}f)(z)+\delta z D_{p,q}(D_{p,q}f)(z)=\frac{1+A\phi(z)}{1+B\phi(z)} \tag{2.4}$$

and

$$(1-\lambda)\frac{g(w)}{w}+\lambda(D_{p,q}g)(w)+\delta w D_{p,q}(D_{p,q}g)(w)=\frac{1+A\psi(w)}{1+B\psi(w)}. \tag{2.5}$$

Moreover, by using the method given in [8] and [14], Jahangiri and Hamidi in [15] observed that

$$\frac{1+A\phi(z)}{1+B\phi(z)}=1-\sum_{n=1}^{\infty}(A-B)K_n^{-1}(c_1,c_2,\ldots,c_n,B)z^n, \tag{2.6}$$

and

$$\frac{1+A\psi(w)}{1+B\psi(w)}=1-\sum_{n=1}^{\infty}(A-B)K_n^{-1}(d_1,d_2,\ldots,d_n,B)w^n, \tag{2.7}$$

where $K_n^{-1}(k_1,k_2,...,k_n,B)$ are obtained by the general coefficients $K_n^j(k_1,k_2,...,k_n,B)$ for all $j\in\mathbb{Z}$ given by

$$\begin{aligned}K_n^j(k_1,k_2,...,k_n,B) &= \tfrac{j!}{(j-n)!(n)!}k_1^nB^{n-1}+\tfrac{j!}{(j-n+1)!(n-2)!}k_1^{n-2}k_2B^{n-2}\\&+\tfrac{j!}{(j-n+2)!(n-3)!}k_1^{n-3}k_3B^{n-3}\\&+\tfrac{j!}{(j-n+3)!(n-4)!}k_1^{n-4}[k_4B^{n-4}+\tfrac{j-n+3}{2}k_3^2B]\\&+\tfrac{j!}{(j-n+4)!(n-5)!}k_1^{n-5}[k_5B^{n-5}+(j-n+4)k_3k_4B]+\textstyle\sum_{j\ge6}k_1^{n-j}V_j,\end{aligned}$$

and where $V_j$ is a homogeneous polynomial of degree $j$ in the variables $k_2,k_3,...,k_n$; (see [8], [14], [15]).

In view of (2.2), (2.4) and (2.6), for every $n\geq 2$, we get

$$[1+([n]_{p,q}-1)\lambda+[n]_{p,q}[n-1]_{p,q}\delta]a_n=-(A-B)K_{n-1}^{-1}(c_1,c_2,...,c_{n-1},B). \tag{2.8}$$

Similarly, because of (2.3), (2.5) and (2.7), for every $n\geq 2$, we have

$$[1+([n]_{p,q}-1)\lambda+[n]_{p,q}[n-1]_{p,q}\delta]b_n=-(A-B)K_{n-1}^{-1}(d_1,d_2,...,d_{n-1},B). \tag{2.9}$$

Since $a_m=0$ for $2\leq m\leq n-1$, we have $b_n=-a_n$ and thus,

$$[1+([n]_{p,q}-1)\lambda+[n]_{p,q}[n-1]_{p,q}\delta]a_n=-(A-B)c_{n-1},$$

$$-[1+([n]_{p,q}-1)\lambda+[n]_{p,q}[n-1]_{p,q}\delta]a_n=-(A-B)d_{n-1}.$$

Recall that for the Schwarz functions $\phi$ and $\psi$, we have $|c_{n-1}|\leq 1$ and $|d_{n-1}|\leq 1$ (see [1]). Taking absolute values of the last two equalities and using $|c_{n-1}|\leq 1$ and $|d_{n-1}|\leq 1$, we obtain

$$|a_n|=\frac{(A-B)|c_{n-1}|}{|1+([n]_{p,q}-1)\lambda+[n]_{p,q}[n-1]_{p,q}\delta|}=\frac{(A-B)|d_{n-1}|}{|1+([n]_{p,q}-1)\lambda+[n]_{p,q}[n-1]_{p,q}\delta|},$$

thus we arrive at

$$|a_n|\leq\frac{A-B}{|1+([n]_{p,q}-1)\lambda+[n]_{p,q}[n-1]_{p,q}\delta|}.$$

This completes the proof. □

Setting $p=1$ in (2.1) and using (i), we get the $q-$coefficient bounds of the Faber polynomials of the class $\mathscr{G}_\Sigma(q;\lambda,\delta,A,B)$.

**Corollary 2.2.** *Let $q\in(0,1)$, $\delta\geq 0$, $\lambda\geq 1$ and $-1\leq B<A\leq 1$. If $f\in\mathscr{G}_\Sigma(q;\lambda,\delta,A,B)$ and $a_m=0,(2\leq m\leq n-1)$, then*

$$|a_n|\leq\frac{A-B}{|1+([n]_q-1)\lambda+[n]_q[n-1]_q\delta|},\quad (n\geq 4).$$

Setting $\delta=0$ in (2.1) and in view of (ii) together with Remark 1.4, we get the following:

**Corollary 2.3.** *If $f\in\mathscr{D}_\Sigma(p,q;\lambda,\frac{1+Az}{1+Bz})$ and $a_m=0,(2\leq m\leq n-1)$, then*

$$|a_n|\leq\frac{A-B}{|1+([n]_{p,q}-1)\lambda|},\quad (n\geq 4).$$

**Remark 2.4.** *In [13], the authors found that if $f\in\mathscr{D}_\Sigma(p,q;\lambda,\varphi)$ and $a_m=0,(2\leq m\leq n-1)$, then*

$$|a_n|\leq\frac{2}{|1+([n]_{p,q}-1)\lambda|},\quad (n\geq 4). \tag{2.10}$$

*However, we find that the coefficient estimates in Corollary 2.3 further improve the estimates in (2.10) because*

$$|a_n|\leq\frac{A-B}{|1+([n]_{p,q}-1)\lambda|}\leq\frac{2}{|1+([n]_{p,q}-1)\lambda|},\quad (n\geq 4)$$

*for all $\lambda\geq 1,-1\leq B<A\leq 1$, (see [13]).*

In view of (iii), Theorem 2.1 gives the next corollary:

**Corollary 2.5.** *[9] Let $\lambda \geq 1, \delta \geq 0, 0 \leq \alpha < 1$. If $f \in \mathscr{R}_\Sigma(\lambda,\delta,\alpha)$ and $a_m = 0, (2 \leq m \leq n-1)$, then*

$$|a_n| \leq \frac{2(1-\alpha)}{1+(n-1)\lambda+n(n-1)\delta}, \quad (n \geq 4).$$

Since $\mathscr{T}_\Sigma(\lambda,\alpha) \equiv \mathscr{N}_\Sigma(1,1;\lambda,0,1-2\alpha,-1)$ by (iv), Theorem 2.1 gives the next result:

**Corollary 2.6.** *[16] Let $\lambda \geq 1$, $0 \leq \alpha < 1$ and $a_m = 0, (2 \leq m \leq n-1)$. If $f \in \mathscr{T}_\Sigma(\lambda,\alpha)$, then*

$$|a_n| \leq \frac{2(1-\alpha)}{1+(n-1)\lambda}, \quad (n \geq 4).$$

**Remark 2.7.** *In view of (vi), if $f \in \mathscr{M}_\Sigma(\delta,\alpha)$, then we get corresponding result obtained in [12].*

For the next theorem, we need the following lemma.

**Lemma 2.8.** *[15] Let $\phi(z) = \sum_{n=1}^{\infty} c_n z^n$ be a Schwarz function satisfying $|\phi(z)| < 1$ for $|z| < 1$. If $\gamma \geq 0$, then*

$$|c_2 + \gamma c_1^2| \leq 1 + (\gamma - 1)|c_1|^2.$$

**Theorem 2.9.** *For $0 < q < p \leq 1$, $\delta \geq 0$, $\lambda \geq 1$, $-1 \leq B \leq A \leq 1$, let the function $f$ given by (1.1) be in the class $\mathscr{N}_\Sigma(p,q;\lambda,\delta,A,B)$. If*

$$\begin{aligned} t &= [3]_{p,q} = p^2+pq+q^2 \\ \mu &= [2]_{p,q} = p+q, \end{aligned} \tag{2.11}$$

*then*

$$|a_2| \leq \min \begin{cases} \dfrac{A-B}{\sqrt{\left|(A-B)\left[1+(t-1)\lambda+t\mu\delta\right]+(1+B)\left[1+(\mu-1)\lambda+\mu\delta\right]^2\right|}}, & B \leq 0 \\[2ex] \dfrac{A-B}{|1+(\mu-1)\lambda+\mu\delta|}, & \end{cases} \tag{2.12}$$

$$|a_3| \leq \frac{(A-B)^2}{[1+(\mu-1)\lambda+\mu\delta]^2} + \frac{A-B}{|1+(t-1)\lambda+t\mu\delta|} \tag{2.13}$$

*and*

$$|a_3 - 2a_2^2| \leq \frac{(A-B)\left[1-(1+B)\dfrac{(1+(\mu-1)\lambda+\mu\delta)^2|a_2|^2}{(A-B)^2}\right]}{|1+(t-1)\lambda+t\mu\delta|} \quad (B \leq 0). \tag{2.14}$$

*These results are sharp.*

*Proof.* Upon setting 2 in place of $n$ in (2.8), we obtain

$$[1+([2]_{p,q}-1)\lambda+[2]_{p,q}\delta]a_2 = (A-B)K_1^{-1}(c_1) = -(A-B)c_1. \tag{2.15}$$

Again, replacing $n = 3$ in (2.8), we have

$$[1+([3]_{p,q}-1)\lambda+[3]_{p,q}[2]_{p,q}\delta]a_3 = (A-B)K_2^{-1}(c_2) = -(A-B)(Bc_1^2-c_2). \tag{2.16}$$

Similarly, by substituting $n = 2$ and $n = 3$, respectively in (2.9), we observe

$$-[1+([2]_{p,q}-1)\lambda+[2]_{p,q}\delta]a_2 = -(A-B)d_1, \tag{2.17}$$

and

$$[1+([3]_{p,q}-1)\lambda+[3]_{p,q}[2]_{p,q}\delta](2a_2^2-a_3) = -(A-B)(Bd_1^2-d_2). \tag{2.18}$$

Using $|c_1| \leq 1$ and $|d_1| \leq 1$, it follows from (2.15) and (2.17) that $c_1 = -d_1$ and

$$|a_2| = \frac{(A-B)|c_1|}{|1+([2]_{p,q}-1)\lambda+[2]_{p,q}\delta|} = \frac{(A-B)|d_1|}{|1+([2]_{p,q}-1)\lambda+[2]_{p,q}\delta|},$$

then we get

$$|a_2| \leq \frac{A-B}{|1+(\mu-1)\lambda+\mu\delta|},$$

where $\mu$ is given by (2.11) and for $-1 \leq B \leq A \leq 1$.

Adding (2.16) to (2.18), and simple calculations gives

$$2[1+([3]_{p,q}-1)\lambda+[3]_{p,q}[2]_{p,q}\delta]a_2^2 = (A-B)(c_2+(-B)c_1^2+d_2+(-B)d_1^2).$$

Taking absolute values of both sides, we get

$$\left|2[1+([3]_{p,q}-1)\lambda+[3]_{p,q}[2]_{p,q}\delta]\right||a_2|^2 \leq (A-B)\left[\left|c_2+(-B)c_1^2\right|+\left|d_2+(-B)d_1^2\right|\right].$$

If $B \leq 0$, then by Lemma 2.8 we have

$$2\left|1+([3]_{p,q}-1)\lambda+[3]_{p,q}[2]_{p,q}\delta\right||a_2|^2 \leq (A-B)\left[2-(B+1)(|c_1|^2+|d_1|^2)\right].$$

Upon substituting $c_1$ and $d_1$ from (2.15) and (2.17), we obtain

$$2\left|1+([3]_{p,q}-1)\lambda+[3]_{p,q}[2]_{p,q}\delta\right||a_2|^2 \leq (A-B)\left[2-2(B+1)\frac{\left(1+([2]_{p,q}-1)\lambda+[2]_{p,q}\delta\right)^2|a_2|^2}{(A-B)^2}\right],$$

or equivalently

$$|a_2| \leq \frac{A-B}{\sqrt{\left|(A-B)\big(1+(t-1)\lambda+t\mu\delta\big)+(1+B)\big(1+(\mu-1)\lambda+\mu\delta\big)^2\right|}},$$

where $t$ and $\mu$ are given by (2.11). This completes the proof of (2.12).

In order to obtain the coefficient estimates for $|a_3|$, we subtract (2.18) from (2.16), and we get

$$[1+([3]_{p,q}-1)\lambda+[3]_{p,q}[2]_{p,q}\delta](-2a_2^2+2a_3) = -(A-B)\left[(Bc_1^2-c_2)-(Bd_1^2-d_2)\right],$$

or

$$a_3 = a_2^2+\frac{(A-B)(c_2-d_2)}{2[1+([3]_{p,q}-1)\lambda+[3]_{p,q}[2]_{p,q}\delta]}. \tag{2.19}$$

Upon substituting the value of $a_2^2$ from (2.15) into (2.19), it follows that

$$a_3 = \frac{(A-B)^2c_1^2}{(1+([2]_{p,q}-1)\lambda+[2]_{p,q}\delta)^2}+\frac{(A-B)(c_2-d_2)}{2[1+([3]_{p,q}-1)\lambda+[3]_{p,q}[2]_{p,q}\delta]}.$$

Taking the absolute value and by using $|c_1| \leq 1$, $|c_2| \leq 1$ and $|d_2| \leq 1$, we get

$$|a_3| \leq \frac{(A-B)^2}{(1+(\mu-1)\lambda+\mu\delta)^2}+\frac{A-B}{|1+(t-1)\lambda+t\mu\delta|},$$

where $t$ and $\mu$ are given by (2.11). This proves the inequality in (2.13).

Finally, (2.18) yields

$$2a_2^2-a_3 = \frac{(A-B)(d_2+(-B)d_1^2)}{1+([3]_{p,q}-1)\lambda+[3]_{p,q}[2]_{p,q}\delta}.$$

By taking the absolute value of the above equation, we find

$$|a_3-2a_2^2| \leq \frac{(A-B)|d_2+(-B)d_1^2|}{|1+([3]_{p,q}-1)\lambda+[3]_{p,q}[2]_{p,q}\delta|}.$$

If $B \le 0$, then by Lemma 2.8 we have

$$|a_3 - 2a_2^2| \le \frac{(A-B)(1+(-B-1)|d_1|^2}{|1+([3]_{p,q}-1)\lambda+[3]_{p,q}[2]_{p,q}\delta|}.$$

Upon substituting the value of $d_1$ from (2.17), we get

$$|a_3 - 2a_2^2| \le \frac{(A-B)\left(1-(1+B)\frac{(1+([2]_{p,q}-1)\lambda+[2]_{p,q}\delta)^2|a_2|^2}{(A-B)^2}\right)}{|1+([3]_{p,q}-1)\lambda+[3]_{p,q}[2]_{p,q}\delta|}.$$

This proves the inequality given by (2.14). □

In view of (i) and (ii), Theorem 2.9 leads to the following corollaries.

**Corollary 2.10.** *Let $q \in (0,1), \lambda \ge 1, \delta \ge 0$ and $-1 \le B < A \le 1$. If $f \in \mathscr{G}_\Sigma(q;\lambda,\delta,A,B)$ and $a_m = 0, (2 \le m \le n-1)$, then*

$$|a_2| \le \min \begin{cases} \dfrac{A-B}{\sqrt{\left|(A-B)\left[1+(q+q^2)\lambda+(1+q+q^2)(1+q)\delta\right]+(1+B)\left[1+q\lambda+(1+q)\delta\right]^2\right|}}, & B \le 0 \\ \frac{A-B}{1+q\lambda+(1+q)\delta}, \end{cases}$$

$$|a_3| \le \frac{(A-B)^2}{(1+q\lambda+(1+q)\delta)^2} + \frac{A-B}{1+(q+q^2)\lambda+(1+q+q^2)(1+q)\delta}$$

*and*

$$\left|a_3 - 2a_2^2\right| \le \frac{(A-B)\left[1-(1+B)\frac{\left[1+q\lambda+(1+q)\delta^2\right]|a_2|^2}{(A-B)^2}\right]}{1+(q+q^2)\lambda+(1+q+q^2)(1+q)\delta} \qquad (B \le 0).$$

**Corollary 2.11.** *Let $0 < q < p \le 1, \lambda \ge 1, -1 \le B < A \le 1$. If $f \in \mathscr{D}_\Sigma(p,q;\lambda,\frac{1+Az}{1+Bz})$ and $a_m = 0, (2 \le m \le n-1)$, then*

$$|a_2| \le \min \left\{ \frac{A-B}{|1+(p+q-1)\lambda|}, \frac{A-B}{\sqrt{\left|(A-B)\left[1+(p^2+pq+q^2-1)\lambda\right]+(1+B)\left[1+(p^2+pq+q^2-1)\lambda\right]^2\right|}} \right\},$$

$$|a_3| \le \frac{(A-B)^2}{(1+(p+q-1)\lambda)^2} + \frac{A-B}{|1+(p^2+pq+q^2-1)\lambda|}.$$

**Remark 2.12.** *Let $\lambda \ge 1, \delta \ge 0, 0 \le \alpha < 1$. If $f \in \mathscr{R}_\Sigma(\lambda,\delta,\alpha)$ and $a_m = 0, (2 \le m \le n-1)$, then Theorem 2.9 yields the corresponding results obtained in [9] for coefficients $a_2$, $a_3$ and $a_3 - 2a_2^2$.*

**Remark 2.13.** *Let $\lambda \ge 1, 0 \le \alpha < 1$. If $f \in \mathscr{T}_\Sigma(\lambda,\alpha)$ and $a_m = 0, (2 \le m \le n-1)$, then Theorem 2.9 satisfies the corresponding results obtained in [16] for coefficients $a_2$ and $a_3 - 2a_2^2$.*

**Remark 2.14.** *Setting $p = 1, q \to 1^-, A = 1-2\alpha, (0 \le \alpha < 1), B = -1$ and $\delta = 0$, Theorem 2.9 yields the corresponding results in [10] for coefficients $a_2$ and $a_3$.*

**Remark 2.15.** *Setting $p = 1, q \to 1^-, A = 1-2\alpha, (0 \le \alpha < 1), B = -1, \delta = 0$ and $\lambda = 1$, Theorem 2.9 yields the corresponding results in [11] for coefficient $a_2$.*

## 3. Conclusion

In this paper, we defined a new subclass of bi-univalent functions associated with $(p,q)$-derivative operator and investigated Faber polynomial coefficient estimates for this new class. We also concluded that the results are generalization of the corresponding results obtained by recent researchers.

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