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A new study on focal surface of a given surface

İlim Kişi[a], Günay Öztürk[b]

[a]*Department of Mathematics, Kocaeli University, Kocaeli, Turkey.*
[b]*Department of Mathematics, İzmir Demokrasi University, İzmir, Turkey.*

Abstract. Focal surfaces are special cases of line congruences. With the aid of the definiton of a focal surface of a given surface M, we obtain a new type of focal surface in Galilean 3-space $\mathbb{G}_3$. We show that the focal surface we found is not the same type of surface as the given surface. We present the visualizations of the focal surface and the given surface with an example. Lastly, by searching the curvature functions, we give the minimality conditions of the focal surface.

1. Introduction

The concept of line congruences is first defined in the area of visualization by Hagen et al in 1991 [8]. Actually, line congruences are surfaces which are obtained from by transforming one surface to another by lines. Focal surface is one of these congruences. For a given surface M with the parametrization $X(u, v)$, the line congruence is defined as

$$C(u, v, z) = X(u, v) + zE(u, v). \tag{1}$$

Here $E(u, v)$ is the set of unit vectors and z is a distance. For each pair (u, v), the equation (1), expresses a line of the congruence and called as generatrix. On every generatrix of C, there are two points called as focal points and the focal surface is the locus of the focal points. If $E(u, v) = N(u, v)$, the unit normal vector field of the surface, then C is a normal congruence. In this case, the parametric equation of the focal surface $C = X^*(u, v)$ of $X(u, v)$ is given as

$$X^*(u, v) = C(u, v, z) = X(u, v) + {\kappa_i}^{-1}N(u, v); \quad i = 1, 2 \tag{2}$$

where κ_is; $(i = 1, 2)$ are the principal curvature functions of $X(u, v)$ [7]. Focal surfaces are the subject of many studies such as [7, 15–17, 23].

Galilean geometry is a non-Euclidean geometry and associated with Galilei principle of relativity. This principle can be explained briefly as ”in all inertial frames, all law of physics are the same.” (Except for the Euclidean geometry in some cases), Galilean geometry is the easiest of all Klein geometries, and it is revelant to the theory of relativity of Galileo and Einstein. One can have a look at the studies [20, 24] for Galilean geometry. Recently, many works related to Galilean geometry have been done by several authors in [2, 6, 21].

Tubular surfaces are special cases of canal surfaces which are the envelopes of a family of spheres. In canal surfaces, center of the spheres are on a given space curve (spine curve), and the radius of the spheres are different. As to tubular surfaces, the radius functions are constant. These surfaces have been widely studied in recent times [4, 10, 11, 13, 14, 18]. In Galilean 3-space, tubular surfaces are studied in [5].

Corresponding author: İK mail address: ilim.ayvaz@kocaeli.edu.tr ORCID:0000-0002-4785-8165, GÖ ORCID:0000-0002-1608-0354
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2. Preliminaries

In Galilean 3-space $\mathbb{G}_3$, we can give the following basic concepts.

The vector $a = (a_1, a_2, a_3)$ is isotropic if $a_1 = 0$ and non-isotropic otherwise. Thus, for the standard coordinates (x, y, z), the x-axis is non-isotropic while the others are isotropic. The yz-plane, i.e. $x = 0$, is Euclidean and the xy-plane and xz-plane are isotropic. The scalar product of the vectors $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3)$ and the length of the vector $a = (a_1, a_2, a_3)$ in $\mathbb{G}_3$ are respectively defined as

$$\langle a, b\rangle = \begin{cases} a_1 b_1, & \text{if } \; a_1 \neq 0 \;\vee b_1 \neq 0 \\ a_2 b_2 + a_3 b_3, & \text{if } \; a_1 = 0 \;\wedge b_1 = 0, \end{cases} \tag{3}$$

$$\|a\| = \begin{cases} |a_1|, & \text{if } \; a_1 \neq 0 \\ a_2^2 + a_3^2, & \text{if } \; a_1 = 0. \end{cases} \tag{4}$$

The cross product of the vectors $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3)$ in $\mathbb{G}_3$ is also defined as

$$a \wedge b = \begin{vmatrix} 0 & e_2 & e_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \tag{5}$$

[19]. An admissible unit speed curve $\alpha : I \subset \mathbb{R} \to \mathbb{G}_3$ is given with the parametrization

$$\alpha(u) = (u, y(u), z(u)). \tag{6}$$

The associated Frenet frame vectors $\mathbf{t}, \mathbf{n}, \mathbf{b}$ on the curve is given as

$$\begin{aligned} \mathbf{t}(u) &= (1, y'(u), z'(u)), \\ \mathbf{n}(u) &= \frac{1}{\kappa(u)}(0, y''(u), z''(u)), \\ \mathbf{b}(u) &= \frac{1}{\kappa(u)}(0, -z''(u), y''(u)), \end{aligned} \tag{7}$$

where $\kappa(u) = \sqrt{(y''(u))^2 + (z''(u))^2}$ and $\tau(u) = \frac{\det(\alpha'(u), \alpha''(u), \alpha'''(u))}{\kappa^2(u)}$ are the curvature and the torsion of the curve, respectively. Thus, the famous Frenet formulas can be written as

$$\begin{aligned} \mathbf{t}' &= \kappa \mathbf{n}, \\ \mathbf{n}' &= \tau \mathbf{b}, \\ \mathbf{b}' &= -\tau \mathbf{n}. \end{aligned} \tag{8}$$

Let M be a surface parametrized with

$$X(u_1, u_2) = (x(u_1, u_2), y(u_1, u_2), z(u_1, u_2)) \tag{9}$$

in $\mathbb{G}_3$. To represent the partial derivatives, we use

$$x_{,i} = \frac{\partial x}{\partial u_i}, \quad x_{,ij} = \frac{\partial^2 x}{\partial u_i \partial u_j}, 1 \leq i, j \leq 2. \tag{10}$$

If $x_{,i} \neq 0$ for some $i = 1, 2$, then the surface is admissible (i.e. having not any Euclidean tangent planes). The first fundamental form I of the surface M is defined as

$$I = (g_1 d_{u_1} + g_2 d_{u_2})^2 + \varepsilon(h_{11} d_{u_1}^2 + 2h_{12} d_{u_1} d_{u_2} + h_{22} d_{u_2}^2), \tag{11}$$

where $g_i = x_{,i}$, $h_{ij} = y_{,i}\, y_{,j} + z_{,i}\, z_{,j}$; $i, j = 1, 2$ and

$$\varepsilon = \begin{cases} 0, & \text{if} \quad d_{u_1} : d_{u_2} \quad \text{is non-isotropic}, \\ 1, & \text{if} \quad d_{u_1} : d_{u_2} \quad \text{is isotropic}. \end{cases} \tag{12}$$

Let a function W is given by

$$W = \sqrt{(x_{,1}\, z_{,2} - x_{,2}\, z_{,1})^2 + (x_{,2}\, y_{,1} - x_{,1}\, y_{,2})^2}. \tag{13}$$

Then, the unit normal vector field is given as

$$N = \frac{1}{W}(0, -x_{,1}\, z_{,2} + x_{,2}\, z_{,1}, x_{,1}\, y_{,2} - x_{,2}\, y_{,1}). \tag{14}$$

Similarly, the second fundamental form II of the surface M is defined as

$$II = L_{11} d_{u_1}^2 + 2L_{12} d_{u_1} d_{u_2} + L_{22} d_{u_2}^2, \tag{15}$$

where

$$L_{ij} = \frac{1}{g_1}\left\langle g_1(0, y_{,ij}, z_{,ij}) - g_{i,j}(0, y_{,1}, z_{,1}), N\right\rangle, \quad g_1 \neq 0 \tag{16}$$

or

$$L_{ij} = \frac{1}{g_2}\left\langle g_2(0, y_{,ij}, z_{,ij}) - g_{i,j}(0, y_{,2}, z_{,2}), N\right\rangle, \quad g_2 \neq 0.$$

The Gaussian and the mean curvatures of M are defined as

$$K = \frac{L_{11}L_{22} - L_{12}^2}{W^2}, \quad H = \frac{g_2^2 L_{11} - 2g_1 g_2 L_{12} + g_1^2 L_{22}}{2W^2}. \tag{17}$$

A surface is called as flat (resp. minimal) if its Gaussian (resp. mean) curvatures vanish [2, 20]. The principal curvatures κ_1 and κ_2 of the surface M are given as

$$\kappa_1 = \frac{g_2^2 L_{11} - 2g_1 g_2 L_{12} + g_1^2 L_{22}}{W^2}, \quad \kappa_2 = \frac{L_{11}L_{22} - L_{12}^2}{g_2^2 L_{11} - 2g_1 g_2 L_{12} + g_1^2 L_{22}}, \tag{18}$$

respectively [22].

3. Focal Surface of Tubular Surface in $\mathbb{G}_3$

A tubular surface M in $\mathbb{G}_3$ at a distance r from the points of spine curve $\alpha(u) = (u, y(u), z(u))$ is given with

$$M : X(u, v) = \alpha(u) + r(\cos v\mathbf{n} + \sin v\mathbf{b}). \tag{19}$$

Writing the Frenet vectors of $\alpha(u)$ in (19), the parametrization can be given as

$$M : X(u, v) = (u, y(u), z(u)) + \frac{r}{\kappa}\left[\cos v(0, y''(u), z''(u)) + \sin v(0, -z''(u), y''(u))\right]. \tag{20}$$

From (20),

$$g_1 = u_{,1} = 1, \quad g_2 = u_{,2} = 0. \tag{21}$$

The tangent vectors X_u, X_v and the normal vector N of M are given by

$$\begin{aligned} X_u &= \mathbf{t} - r\tau \sin v\mathbf{n} + r\tau \cos v\mathbf{b}, \\ X_v &= -r \sin v\mathbf{n} + r \cos v\mathbf{b}, \end{aligned} \tag{22}$$

and

$$N = -\cos v\mathbf{n} - \sin v\mathbf{b}. \tag{23}$$

Here $W = r$. The coefficients of the second fundamental form are obtained as

$$L_{11} = -\kappa\cos v + r\tau^2, \quad L_{12} = r\tau, \quad L_{22} = r. \tag{24}$$

From, (21) and (24), the curvature functions of M are obtained as

$$K = \frac{-\kappa\cos v}{r}, \quad H = \frac{1}{2r} \tag{25}$$

[5].

Corollary 3.1. *[5] Tubular surfaces are constant mean curvature surfaces in Galilean space.*

By the equation (18), we obtain the principal curvatures κ_1, κ_2 of M as

$$\kappa_1 = -\kappa\cos v \text{ and } \kappa_2 = \frac{1}{r}. \tag{26}$$

For the function $\kappa_2 = \frac{1}{r}$, the focal surface degenerates to a curve. Thus, we obtain the focal surface M^* of M for the function $\kappa_1 = -\kappa\cos v$ as

$$M^* : X^*(u,v) = \alpha(u) + \left(r + \frac{1}{\kappa(u)\cos v}\right)(\cos v\mathbf{n} + \sin v\mathbf{b}), \tag{27}$$

where $\kappa \neq 0$.

Corollary 3.2. *The focal surface M^* of M is not a canal surface.*

Proposition 3.3. *If the spine curve $\alpha(u)$ is a straight line or equivalently M is flat, we cannot construct the focal surface of M.*

Example 3.4. *Let us consider the cylindrical helix $\alpha(u) = (u, \cos u, \sin u)$ in $\mathbb{G}_3$. The Frenet frame vectors of the spine curve $\alpha(u)$ is given by*

$$\begin{aligned} \mathbf{t}(u) &= (1, -\sin u, \cos u), \\ \mathbf{n}(u) &= (0, -\cos u, -\sin u), \\ \mathbf{b}(u) &= (0, \sin u, -\cos u). \end{aligned}$$

The tubular surface M has the following parametrization

$$X(u,v) = (u, \cos u - r\cos(u+v), \sin u - r\sin(u+v)).$$

[5]. Then from the equation (27), we write the parametrization of the focal surface M^* of M as in the following:

$$X^*(u,v) = (u, -r\cos(u+v) + \tan v\sin u, -r\sin(u+v) - \tan v\cos u).$$

By using the maple programme, we plot the graph of the tubular surface and its focal surface for the value $r = 2$ in $\mathbb{G}_3$.

For the focal surface M^*, the tangent space is spanned by the vectors

$$\begin{aligned} (X^*)_u &= \mathbf{t}(u) + \lambda_1(u,v)\mathbf{n}(u) + \lambda_2(u,v)\mathbf{b}(u), \\ (X^*)_v &= -r\sin v\mathbf{n}(u) + \lambda_3(u,v)\mathbf{b}(u), \end{aligned} \tag{28}$$

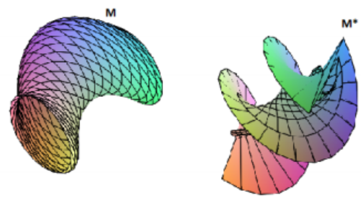

Figure 1: Tubular surface M and the focal surface M^*

where

$$\begin{aligned} \lambda_1(u,v) &= \frac{-\kappa'(u)}{(\kappa(u))^2} - r\tau(u)\sin v - \frac{\tau(u)}{\kappa(u)}\tan v, \\ \lambda_2(u,v) &= \frac{-\kappa'(u)}{(\kappa(u))^2}\tan v + r\tau(u)\cos v + \frac{\tau(u)}{\kappa(u)}, \\ \lambda_3(u,v) &= \frac{1}{\kappa(u)\cos^2 v} + r\cos v. \end{aligned} \tag{29}$$

Thus, from (28), $W^* = ((\lambda_3(u,v))^2 + (r\sin v)^2)^{\frac{1}{2}}$ and the unit normal vector field N^* of M^* is

$$N^* = \frac{-\lambda_3(u,v)\mathbf{n}(u) - r\sin v\mathbf{b}(u)}{W^*}. \tag{30}$$

Further, we get

$$g_1^* = u_{,1} = 1, \quad g_2^* = u_{,2} = 0. \tag{31}$$

The second partial derivatives of X^* are

$$\begin{aligned} (X^*)_{uu} &= \lambda_4(u,v)\mathbf{n}(u) + \lambda_5(u,v)\mathbf{b}(u), \\ (X^*)_{uv} &= \lambda_6(u,v)\mathbf{n}(u) + \lambda_7(u,v)\mathbf{b}(u), \\ (X^*)_{vv} &= -r\cos v\mathbf{n}(u) + \lambda_8(u,v)\mathbf{b}(u), \end{aligned} \tag{32}$$

where

$$\begin{aligned} \lambda_4(u,v) &= \kappa(u) + (\lambda_1(u,v))_u - \tau(u)\lambda_2(u,v), \\ \lambda_5(u,v) &= (\lambda_2(u,v))_u + \tau(u)\lambda_1(u,v), \\ \lambda_6(u,v) &= (\lambda_1(u,v))_v, \\ \lambda_7(u,v) &= (\lambda_2(u,v))_v \\ \lambda_8(u,v) &= (\lambda_3(u,v))_v. \end{aligned} \tag{33}$$

Thus from the equations (30)-(33), the coefficients of the second fundamental form become

$$\begin{aligned} L_{11}^* &= \frac{-\lambda_3(u,v)\lambda_4(u,v) - \lambda_5(u,v)r\sin v}{W^*}, \\ L_{12}^* &= \frac{-\lambda_3(u,v)\lambda_6(u,v) - \lambda_7(u,v)r\sin v}{W^*}, \\ L_{22}^* &= \frac{\lambda_3(u,v)r\cos v - \lambda_8(u,v)r\sin v}{W^*}. \end{aligned} \tag{34}$$

By using the equations (31) and (34), we give the following theorems:

Theorem 3.5. *Let M be a tubular surface given with the parametrization (19) and M^* be the focal surface of M with the parametrization (27) in $\mathbb{G}_3$. Then, the Gaussian and the mean curvatures of M^* are*

$$\begin{aligned} K^* &= \frac{1}{(W^*)^4}\left\{\begin{array}{c} -\lambda_3^2\lambda_4 r\cos v + \lambda_3\lambda_4\lambda_8 r\sin v - \lambda_3\lambda_5 r^2\sin v\cos v \\ +\lambda_5\lambda_8 r^2\sin^2 v - \lambda_3^2\lambda_6^2 - \lambda_7^2 r^2\sin^2 v - 2\lambda_3\lambda_6\lambda_7 r\sin v \end{array}\right. \\ H^* &= \frac{\lambda_3 r\cos v - \lambda_8 r\sin v}{2(W^*)^3}. \end{aligned} \tag{35}$$

Corollary 3.6. *If the focal surface M^* is minimal, then*

$$r = -\frac{1}{\kappa(u)\cos^3 v}.$$

Proof. Let M^* be the focal surface of M with the parametrization (27) in $\mathbb{G}_3$. If M^* is minimal, then $\lambda_3 r\cos v - \lambda_8 r\sin v = 0$. Since the functions $\cos v$ and $\sin v$ are linearly independent, $\lambda_3 = \lambda_8 = 0$ i.e. $\lambda_3 = (\lambda_3)_v = 0$ which corresponds to the last equation. □

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