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Review Article

Local abelian Kato-Parshin reciprocity law: A survey

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-In memory of Seydin Serbest-

Abstract

Let K denote an n-dimensional local field. The aim of this expository paper is to survey the basic arithmetic theory of the n-dimensional local field K together with its Milnor K-theory and Parshin topological K-theory; to review Kato's ramification theory for finite abelian extensions of the n-dimensional local field K, and to state the local abelian higher-dimensional K-theoretic generalization of local abelian class field theory of Hasse, which is developed by Kato and Parshin. The paper is geared toward non-abelian generalization of this theory.

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1. Introduction

The aim of this paper is to survey the local abelian higher-dimensional K-theoretic generalization of the local abelian class field theory of Hasse [16] developed by Parshin (in positive characteristic) [42, 44, 45] and by Kato (in general) [22, 23, 25] in the late 1970s and early 80's; namely, the local abelian Kato-Parshin class field theory, which has later been simplified, made explicit, and cohomology free by Fesenko [7–9].

For a field K, let K^{ab} denote the maximal abelian extension of K in a fixed separable closure K^{sep} of K. Then the maximal abelian Hausdorff quotient G_K^{ab} of the absolute Galois group $G_K = \text{Gal}(K^{\text{sep}}/K)$ is naturally isomorphic to $\text{Gal}(K^{\text{ab}}/K)$. In particular, if K is a non-archimedean local field; i.e., a complete discrete valuation field with finite residue-class field $\kappa_K = O_K/\mathfrak{p}_K$ of $q = p^f$ elements, where O_K denotes the ring of integers of K, $\mathfrak{p}_K$ its unique maximal ideal, and p a prime number; that is, if K is either a finite extension of $\mathbb{Q}_p$ in case $\text{char}(K) = 0$, or a finite extension of $\mathbb{F}_p((X))$ in case $\text{char}(K) = p > 0$, then

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$\mathrm{Gal}(K^{\mathrm{ab}}/K)$ and the profinite completion $\widehat{K^\times}$ of the multiplicative group $K^\times$ of the non-archimedean local field K are both algebraically and topologically isomorphic via local abelian Hasse reciprocity law

$$\mathsf{Rec}_K : \widehat{K^\times} \xrightarrow{\sim} \mathrm{Gal}(K^{\mathrm{ab}}/K)$$

of K. This isomorphism has many salient features. For instance, via this arrow, $\widehat{K^\times}$ encodes all of the arithmetic information on the abelian extensions of the non-archimedean local field K, which is the subject matter of local abelian class field theory of K. A detailed exposition of local fields and local abelian class field theory in modern terms can be found in [15, 21].

Now, let F be a global field; that is, F is either a finite extension of $\mathbb{Q}$ in case $\mathrm{char}(F) = 0$, or a finite extension of $\mathbb{F}_p(X)$ in case $\mathrm{char}(F) = p > 0$. The completion F_ν of F with respect to a finite place ν of F is a non-archimedean local field. Following the "idèlic philosophy" of Chevalley, global class field theory of F can be constructed by glueing the local abelian class field theories of F_ν for all ν [2]. In recent years however, the arithmetic study of global fields extended its scope and instead of considering only global fields; that is, integral schemes X of absolute dimension 1, higher-dimensional integral schemes X are taken into consideration. In this setting, let F denote the field of rational functions on an integral scheme X of absolute dimension n. Then to any flag of irreducible non-singular subschemes $X_0 \subset X_1 \subset \cdots \subset X_n = X$ of X with $\dim(X_i) = i$ for $i = 0, 1, \cdots, n$, Parshin introduced a completion $F_{(X_0,\cdots,X_n)}$ of F, which is an example of an n-dimensional local field. Recall that, an n-dimensional local field has an inductive definition: for $n \geq 1$, an n-dimensional local field is a complete discrete valuation field whose residue field is an $(n-1)$-dimensional local field, where in this terminology 0-dimensional local fields are finite fields and 1-dimensional local fields are the "classical" non-archimedean local fields. The collection of such n-dimensional local fields $F_{(X_0,\cdots,X_n)}$ over all possible flags $(X_0, \cdots, X_n)$ of the scheme X plays a central role in the global class field theory of the scheme X, a grand theory again created by Parshin[†], Bloch[‡], Kato and S. Saito[§], which is constructed, following the "higher-dimensional idèlic philosophy" of Beilinson and Parshin [18], by glueing the local abelian n-dimensional class field theories of $F_{(X_0,\cdots,X_n)}$ for all $(X_0, \cdots, X_n)$.

The aim of this work is to survey the local abelian n-dimensional class field theory; namely, the study of arithmetic information on the abelian extensions of an n-dimensional local field K encoded in the local abelian n-dimensional reciprocity law

$$\mathsf{Rec}_K : \widehat{\mathsf{K}}_n^{\mathrm{top}}(K) \xrightarrow{\sim} \mathrm{Gal}(K^{\mathrm{ab}}/K)$$

of the n-dimensional local field K, where $\widehat{\mathsf{K}}_n^{\mathrm{top}}(K)$ is the profinite completion of the n-th Parshin topological K-group $\mathsf{K}_n^{\mathrm{top}}(K)$ of K, which is an algebraic, analytic and topological object depending only and solely to the ground field K. Moreover, in the particular case $n = 1$, this arrow reduces to the ordinary local abelian Hasse reciprocity law of K. In the local abelian n-dimensional theory:

– Non-archimedean local fields K are replaced by n-dimensional local fields K

$$\text{non-archimedean local fields } K \rightsquigarrow n\text{-dimensional local fields } K,$$

[†] Parshin developed the global class field theory of algebraic surfaces using his 2-dimensional adèles [43, 44].

[‡] Bloch is one of the first researchers who used algebraic K-theory to construct the class field theory of arithmetic surfaces [4].

[§] Kato and S. Saito studied the global class field theory of arithmetic surfaces and then extended their results to arbitrary dimensional arithmetic schemes [26, 27].

– and multiplicative groups $K^\times$ of non-archimedean local fields K are replaced by the n-th Parshin topological K-groups $\mathsf{K}_n^{\mathrm{top}}(K)$ of n-dimensional local fields K

$$\text{the group } \widehat{K^\times} \rightsquigarrow \text{the group } \widehat{\mathsf{K}}_n^{\mathrm{top}}(K),$$

hence the name "K-theoretic generalization" of local abelian class field theory, or the local abelian Kato-Parshin class field theory.

The paper is organised as follows.In Sections 2 and 3, we shall respectively review the basic arithmetical theory and the topological theory of n-dimensional local fields. Next, in Sections 4 and 5, Milnor K-theory and Parshin topological K-theory of n-dimensional local fields are discussed. In Section 6, after reviewing ramification theory for non-archimedean local fields, we sketch Kato's ramification theory, which is defined only for abelian extensions of n-dimensional local fields introduced in [24, 28], and note that Kato's ramification theory¶ for finite abelian extensions of n-dimensional local fields is compatible with the local abelian Kato-Parshin reciprocity law. Finally, in Section 7, we state the local abelian K-theoretic class field theory of Kato and Parshin. In this section, we stick to the methods introduced by Fesenko, as his methods have advantages for the non-abelian generalization of this theory [20].

2. n-dimensional local fields

The main references for this section are [33] and the excellent reviews [36, 37, 41, 50]. Let K be an *n-dimensional local field*. That is, attached to K, there exists a sequence of fields

$$K_0, K_1, \cdots, K_{n-1}, K_n = K,$$

called the *Parshin chain of* K, where

– K_{i+1} is a complete discrete valuation field endowed with a discrete valuation

$$\nu_{K_{i+1}} : K_{i+1} \to \mathbb{Z} \cup \{\infty\}$$

with the ring of integers $O_{\nu_{K_{i+1}}} = O_{K_{i+1}}$ having the unique maximal ideal $\mathfrak{p}_{\nu_{K_{i+1}}} = \mathfrak{p}_{K_{i+1}}$ for every $i = 0, \cdots, n-1$;

– The residue-class field $\kappa_{\nu_{K_{i+1}}} = \kappa_{K_{i+1}}$ of K_{i+1} is K_i for every $i = 0, \cdots, n-1$;

– $K_0 = \mathbb{F}_q$ the finite field with $q = p^s$ elements, where p denotes a prime number (we could have assumed K_0 is a perfect field instead).

The residue-class field K_{n-1} of K_n is called the *first residue-class field* of the *n-dimensional local field* K, and the residue-class field $K_0 = \mathbb{F}_q$ of K_1 is called the *last residue-class field* of the *n-dimensional local field* K. Moreover, K is said to be a *mixed-characteristic* n-dimensional local field if $\mathrm{char}(K) = 0$ and $\mathrm{char}(K_{n-1}) = p > 0$, and called an *equal-characteristic* n-dimensional local field if $\mathrm{char}(K) = \mathrm{char}(K_{n-1})$.

Here are some examples of n-dimensional local fields:

Example 2.1. Observe that,

$$K = L((X_1))\cdots((X_{n-1})),$$

where L is a non-archimedean local field, is a natural example of an n-dimensional local field.

¶Note that, Kato's ramification theory introduced in [24] is for abelian extensions of n-dimensional local fields, while Abbes and T. Saito's ramification theory [1] is for general Galois extensions of n-dimensional local fields [49]. On the other hand, Abbes-Saito ramification theory for abelian extensions of n-dimensional local fields coincides with Kato's ramification filtration [28].

Example 2.2. Let k be a complete discrete valuation field with respect to a discrete valuation $\nu_k : k \to \mathbb{Z} \cup \{\infty\}$. The field

$$K = k\{\{X\}\} = \left\{ \sum_{i=-\infty}^{+\infty} c_i X^i \mid c_i \in k, \inf\{\nu_k(c_i) \mid i \in \mathbb{Z}\} > -\infty, \lim_{i \to -\infty} \nu_k(c_i) = +\infty \right\}$$

endowed with a discrete valuation

$$\nu_K : K \to \mathbb{Z} \cup \{\infty\}$$

defined by

$$\nu_K \left(\sum_{i=-\infty}^{+\infty} c_i X^i \right) = \inf\{\nu_k(c_i) \mid i \in \mathbb{Z}\},$$

for every $\sum_{i=-\infty}^{+\infty} c_i X^i \in K$, is a complete discrete valuation field with residue class field $\kappa_K = \kappa_k((X))$.

Therefore, for a non-archimedean local field L, and for $0 \leq j \leq n-1$,

$$K = L\{\{X_1\}\} \cdots \{\{X_j\}\}((X_{j+2})) \cdots ((X_n))$$

is an n-dimensional local field, called a *standard n-dimensional local field*, following [36, 50]. The extreme cases $j = 0$ and $j = n-1$ mean $K = L((X_2)) \cdots ((X_n))$ and $K = L\{\{X_1\}\} \cdots \{\{X_{n-1}\}\}$, respectively.

Remark 2.3. Let k be a complete discrete valuation field with respect to a discrete valuation $\nu_k : k \to \mathbb{Z} \cup \{\infty\}$. Then, $k((X_1))\{\{X_2\}\}$ is isomorphic to $k((X_2))((X_1))$. So, it suffices to consider standard higher-dimensional local fields. For details, look at the classification theorem for n-dimensional local fields that we recall below.

Assumption 2.4. *From now on, all through the paper, K denotes an n-dimensional local field with the corresponding Parshin chain*

$$\mathbb{F}_q = K_0, K_1, \cdots, K_{n-1}, K_n = K.$$

Notation 2.5. To simplify the discussion, for $a \in O_{K_n}$ and for an integer i satisfying $0 \leq i \leq n-1$, let $\overline{a}^{(n,\cdots,n-i)}$ denote the element in K_{n-i-1} defined by "succesive reductions of a modulo maximal ideals $\mathfrak{p}_{K_n}, \cdots, \mathfrak{p}_{K_{n-i}}$ respectively" as

$$a \pmod{\mathfrak{p}_{K_n}} \pmod{\mathfrak{p}_{K_{n-1}}} \cdots \pmod{\mathfrak{p}_{K_{n-i}}}$$

provided that $\overline{a}^{(n)} \in O_{K_{n-1}}$, $\overline{a}^{(n,n-1)} \in O_{K_{n-2}}$, $\cdots$, $\overline{a}^{(n,\cdots,n-i+1)} \in O_{K_{n-i}}$. Note that, $\overline{a}^{(n,\cdots,n-i)}$ is a non-zero element of K_{n-i-1} if $\overline{a}^{(n)} \in O_{K_{n-1}}^{\times} = U_{K_{n-1}}$, $\overline{a}^{(n,n-1)} \in O_{K_{n-2}}^{\times} = U_{K_{n-2}}$, $\cdots$, $\overline{a}^{(n,\cdots,n-i+1)} \in O_{K_{n-i}}^{\times} = U_{K_{n-i}}$.

An n-tuple $\Pi_K = (t_{1,K}, \cdots, t_{n,K})$ in K^n is called a *system of local parameters* of K, if

(1) $t_{n,K}$ is a prime element of K_n with respect to ν_{K_n};

(2) $t_{n-1,K} \in U_{K_n}$ and its residue class $\overline{t}_{n-1,K}^{(n)} := t_{n-1,K} \pmod{\mathfrak{p}_{K_n}}$ modulo $\mathfrak{p}_{K_n}$ is a prime element of K_{n-1} with respect to $\nu_{K_{n-1}}$;

$\vdots$

(n) $t_{1,K} \in U_{K_n}$ such that $\overline{t}_{1,K}^{(n)} \in U_{K_{n-1}}, \cdots, \overline{t}_{1,K}^{(n,\cdots,3)} \in U_{K_2}$ and $\overline{t}_{1,K}^{(n,\cdots,2)}$ is a prime element of K_1 with respect to ν_{K_1}.

So, following [33] and [42, 44, 45], n-dimensional local fields can be classified as follows. For the n-dimensional local field K:

– If $\mathrm{char}(K) = p$, then it is possible to choose $t_1, \cdots, t_n \in K$, such that

$$K \xrightarrow{\sim} \mathbb{F}_q((t_1)) \cdots ((t_n)).$$

Moreover, $(t_1, \cdots, t_n) \in K^n$ becomes a system of local parameters of K;

– If $\mathrm{char}(K_1) = 0$, then it is possible to choose $t_2, \cdots, t_n \in K$, such that

$$K \xrightarrow{\sim} K_1((t_2))\cdots((t_n)).$$

Moreover, choosing $\pi_{1,K} \in U_{K_n}$ such that $\overline{\pi}_{1,K}^{(n)} \in U_{K_{n-1}}, \cdots, \overline{\pi}_{1,K}^{(n,\cdots,3)} \in U_{K_2}$ and $\overline{\pi}_{1,K}^{(n,\cdots,2)}$ is a prime element π_{K_1} of K_1 with respect to ν_{K_1}, the n-tuple $(\pi_{K_1}, t_2, \cdots, t_n) \in K^n$ becomes a system of local parameters of K;

– If none of the above holds, there exists a unique $r \in \{1, \cdots, n-1\}$ such that $\mathrm{char}(K_{r+1}) \neq \mathrm{char}(K_r)$. Then, there exists a unique non-archimedean local field L of char. 0, and there exist $n-1$ elements $t_1, \cdots, t_r, t_{r+2}, \cdots, t_n \in K$, such that K is a finite extension of the standard field

$$L\{\{t_1\}\}\cdots\{\{t_r\}\}((t_{r+2}))\cdots((t_n)).$$

Moreover, if $\mathrm{char}(K_0) = p$, then L may be chosen to be the unique unramified extension of $\mathbb{Q}_p$ with residue-class field K_0.

Now, fix a system of local parameters $\Pi_K = (t_{1,K}, \cdots, t_{n,K}) \in K^n$ of K. This system of local parameters Π_K of K naturally determines a mapping

$$\rho_{\Pi_K} : K \to K_1 \times \cdots \times K_n$$

defined by

$$\rho_{\Pi_K} : a \mapsto (a_1, \cdots, a_n),$$

where $a_n = a \in K_n$ and $a_i = \overline{a}_{i+1}^{(i+1)} \left(\overline{t}_{i+1,K}^{(n,\cdots,i+1)}\right)^{-\nu_{K_{i+1}}(a_{i+1})} \in K_i$ for $1 \leq i \leq n-1$. Then, there exists a *rank n discrete valuation*

$$\overline{v}_K = (\nu_{K_1}, \cdots, \nu_{K_n}) \circ \rho_{\Pi_K} : K \xrightarrow{\rho_{\Pi_K}} K_1 \times \cdots \times K_n \xrightarrow{(\nu_{K_1},\cdots,\nu_{K_n})} \mathbb{Z}^n \cup \{\infty\}$$

on K defined by

$$\overline{v}_K(a) := (\nu_{K_1}, \cdots, \nu_{K_n}) \circ \rho_{\Pi_K}(a) = (\nu_{K_1}(a_1), \cdots, \nu_{K_n}(a_n))$$

for $a \in K^\times$. Here, $\mathbb{Z}^n$ is assumed to be *lexicographically ordered in the sense of Madunts and Zhukov* as follows: For $\boldsymbol{i} = (i_1, \cdots, i_n)$, $\boldsymbol{j} = (j_1, \cdots, j_n) \in \mathbb{Z}^n$,

$$\boldsymbol{i} \prec \boldsymbol{j} \iff i_\ell < j_\ell, i_{\ell+1} = j_{\ell+1}, \cdots, i_n = j_n \text{ for some } 0 \leq \ell \leq n.$$

Recall that, this rank n discrete valuation $\overline{v}_K$ on K depends on the system of local parameters Π_K of K. However, if $\Pi'_K \in K^n$ is another system of local parameters of K, then the corresponding rank n discrete valuation $\overline{v}'_K$ on K is *equivalent* to $\overline{v}_K$ in the following sense:

$$\overline{v}'_K(a) = \overline{v}_K(a)T, \ {}^\forall a \in K,$$

where $T = \left(v'_{K_j}(\mathrm{Proj}_j \circ \rho_{\Pi_K}(t_{i,K}))\right)_{1\leq i,j\leq n} \in \mathrm{M}(n, \mathbb{Z})$, which is a lower triangular square integral matrix of size n with the unit element 1 on the main diagonal. Here, $\mathrm{Proj}_j : K_1 \times \cdots \times K_n \to K_j$ denotes the projection map on the j^{th} coordinate. As usual, $\mathrm{M}(n, \mathbb{Z})$ denotes the set of all integral square matrices of size n. The rank n discrete valuation $\overline{v}_K$ on K is called *normalized*, if $\overline{v}_K(K^\times) = \mathbb{Z}^n$.

For a rank n discrete valuation $\overline{v}_K : K \to \mathbb{Z}^n \cup \{\infty\}$ defined on K, introduce the subring $O_{\overline{v}_K}$ of K by

$$O_{\overline{v}_K} = \{a \in K \mid \overline{v}_K(a) \succeq \mathbf{0}\},$$

where $\mathbf{0} = (0, \cdots, 0) \in \mathbb{Z}^n$, which is called the ring of integers of K with respect to the rank n discrete valuation $\overline{v}_K$. Note that $O_{\overline{v}_K}$ has the unique maximal ideal $\mathfrak{p}_{\overline{v}_K}$ defined by

$$\mathfrak{p}_{\overline{v}_K} = \{a \in O_{\overline{v}_K} \mid \overline{v}_K(a) \succ \mathbf{0}\}.$$

The quotient field $O_{\overline{v}_K}/\mathfrak{p}_{\overline{v}_K} =: \kappa_{\overline{v}_K}$, called the residue class field of K with respect to the rank n discrete valuation $\overline{v}_K$, is isomorphic to $K_0 = \mathbb{F}_q$.

The arithmetic structure of $O_{\overline{v}_K}$ has the following description. Introduce for each $\ell = 1, 2, \cdots, n$, the rank $n - \ell + 1$ discrete valuation

$$\overline{v}_{K,\geq \ell} : K \to \mathbb{Z}^{n-\ell+1} \cup \{\infty\}$$

on K induced from the rank n valuation $\overline{v}_K$ of K by the rule

$$\overline{v}_{K,\geq \ell}(a) = \mathrm{Pr}_{\geq \ell}(\overline{v}_K(a))$$

for each $a \in K$, where

$$\mathrm{Pr}_{\geq \ell} : \mathbb{Z}^n \to \mathbb{Z}^{n-\ell+1}$$

is the projection map defined by

$$\mathrm{Pr}_{\geq \ell} : (m_1, \cdots, m_n) \mapsto (m_\ell, \cdots, m_n)$$

for every $(m_1, \cdots, m_n) \in \mathbb{Z}^n$. In particular, the rank 1 valuation $\overline{v}_{K,\geq n}$ on K is nothing but the first valuation ν_{K_n} of K_n. Now, define a family of ideals of K, for each $\ell = 1, 2, \cdots, n$, by

$$P_{\overline{v}_K}^{(i_\ell, \cdots, i_n)} := \{a \in K \mid \overline{v}_{K,\geq \ell}(a) \succeq (i_\ell, \cdots, i_n)\}$$

for every $(i_\ell, \cdots, i_n) \in \mathbb{Z}^{n-\ell+1}$. Observe that

$$P_{\overline{v}_K}^{\underbrace{(0,0,\cdots,0)}_{n\text{-tuple}}} = O_{\overline{v}_K}, \qquad P_{\overline{v}_K}^{\underbrace{(1,0,\cdots,0)}_{n\text{-tuple}}} = \mathfrak{p}_{\overline{v}_K}.$$

Note that, the collection of all non-zero ideals of $O_{\overline{v}_K}$ consists of all ideals $P_{\overline{v}_K}^{(i_\ell, \cdots, i_n)}$ satisfying $(i_\ell, \cdots, i_n) \succeq \overbrace{(0, \cdots, 0)}^{(n-\ell+1)\text{-tuple}}$, for each $1 \leq \ell \leq n$. Thus, we see that $O_{\overline{v}_K}$ is not a Noetherian ring for $n > 1$.

Now, the unit group $U_{\overline{v}_K}$ and the group of principal units $V_{\overline{v}_K}$ of K relative to the rank n discrete valuation $\overline{v}_K$ are defined by

$$U_{\overline{v}_K} = O_{\overline{v}_K}^\times, \qquad V_{\overline{v}_K} = 1 + \mathfrak{p}_{\overline{v}_K}.$$

It is also possible to define the higher-unit groups $U_{\overline{v}_K}^{(i_\ell, \cdots, i_n)}$ of K relative to the rank n discrete valuation $\overline{v}_K$ by

$$U_{\overline{v}_K}^{(i_\ell, \cdots, i_n)} = 1 + P_{\overline{v}_K}^{(i_\ell, \cdots, i_n)} = \{a \in K \mid \overline{v}_{K,\geq \ell}(a-1) \succeq (i_\ell, \cdots, i_n)\},$$

where $(i_\ell, \cdots, i_n) \in \mathbb{Z}^{n-\ell+1}$ satisfying $(i_\ell, \cdots, i_n) \succeq \overbrace{(0, \cdots, 0)}^{(n-\ell+1)\text{-tuple}}$, for each ℓ satisfying $1 \leq \ell \leq n$.

In particular, in case $\ell = n$, as already mentioned the rank 1 discrete valuation $\overline{v}_{K,\geq n} : K \to \mathbb{Z} \cup \{\infty\}$ is $\nu_{K_n} : K_n \to \mathbb{Z} \cup \{\infty\}$, and in this setting:

$$U_{\overline{v}_K}^{(i_n)} = \{a \in K \mid \overline{v}_{K,\geq n}(a-1) \succeq i_n\} = \{a \in K \mid \nu_{K_n}(a-1) \geq i_n\},$$

where $i_n \in \mathbb{Z}$ satisfies $i_n \geq 0$. Thus, we shall use the standard notation for $U_{\overline{v}_K}^{(i_n)}$, and set

$$U_{\overline{v}_K}^{(i_n)} = U_{K_n}^{i_n},$$

for each $i_n \in \mathbb{Z}$ such that $i_n \geq 0$. Moreover, in the specific case $i_n = 1$, we further denote the group of principal units $U_{\overline{v}_K}^{(i_n=1)}$ of $K = K_n$ relative to the rank 1 discrete valuation $\overline{v}_{K,\geq n} = \nu_{K_n}$ of $K = K_n$ by

$$U_{\overline{v}_K}^{(i_n=1)} = V_{K_n}.$$

Remark 2.6. The objects $O_{\overline{v}_K}$, $\mathfrak{p}_{\overline{v}_K}$, $U_{\overline{v}_K}$, $V_{\overline{v}_K}$, and $P_{\overline{v}_K}^{(i_\ell, \cdots, i_n)}$, $U_{\overline{v}_K}^{(i_\ell, \cdots, i_n)}$ introduced so far do not depend on the choice of a system of local parameters Π_K of the n-dimensional local field K.

If $\Pi_K = (t_{1,K}, \cdots, t_{n,K})$ is a system of local parameters of K, then as in the classical 1-dimensional case, we can describe the multiplicative group $K^\times$ of the n-dimensional local field K by

$$K^\times \simeq \mathbb{Z}t_{n,K} \oplus \mathbb{Z}t_{n-1,K} \oplus \cdots \oplus \mathbb{Z}t_{1,K} \oplus U_{\overline{v}_K}$$

and

$$U_{\overline{v}_K} \simeq R_{\overline{v}_K} \oplus V_{\overline{v}_K},$$

where $R_{\overline{v}_K}$ is the subgroup in $K^\times$ consisting of Teichmüller representatives of all non-zero elements of the last-residue field $K_0 = \mathbb{F}_q$ of K. Moreover, any $a \in K$ has a unique expression as a formal power series

$$a = \sum_{\boldsymbol{b}=(b_1,\cdots,b_n)} [\theta_{\boldsymbol{b}}] t_{1,K}^{b_1} \cdots t_{n,K}^{b_n},$$

where all coefficients $[\theta_{\boldsymbol{b}}]$ are from the Teichmüller representatives of all non-zero elements of the last residue field $K_0 = \mathbb{F}_q$ of K and the summation over $\boldsymbol{b}$ runs over the admissible set $\{\boldsymbol{b} \in \mathbb{Z}^n \mid \theta_{\boldsymbol{b}} \neq 0\}$, which is well-ordered in $\mathbb{Z}^n$.

For any algebraic extension L of K, there exists a unique extension $\overline{w}_L$ of the rank n discrete valuation $\overline{v}_K$ of K to L. Now, let in particular, L/K be a finite extension. Then, L has an n-dimensional local field structure with the corresponding Parshin chain

$$\begin{array}{ccccc} L_0 & \cdots & L_{n-1} & & L_n = L \\ \Big| & & \Big| & & \Big| \\ K_0 & \cdots & K_{n-1} & & K_n = K. \end{array}$$

Let $\Pi_K = (t_{1,K}, \cdots, t_{n,K}) \in K^n$ and $\Pi_L = (t_{1,L}, \cdots, t_{n,L}) \in L^n$ be systems of local parameters of K and of L respectively. As usual, let $\overline{v}_K$ and $\overline{v}_L$ be the corresponding rank n discrete valuations on K and on L respectively. Then, for every $a \in K \subseteq L$, the n-tuples

$$\overline{v}_K(a) := (\nu_{K_1}, \cdots, \nu_{K_n}) \circ \rho_{\Pi_K}(a)$$

and

$$\overline{v}_L(a) := (\nu_{L_1}, \cdots, \nu_{L_n}) \circ \rho_{\Pi_L}(a)$$

are both in $\mathbb{Z}^n$, and they are related by

$$\overline{v}_L(a) = \overline{v}_K(a) E(L/K; \Pi_K, \Pi_L),$$

where $E(L/K; \Pi_K, \Pi_L) \in \mathrm{M}(n, \mathbb{Z})$ is the lower-triangular integral matrix given by

$$E(L/K; \Pi_K, \Pi_L) = \left(v_{Lj}(t_{i,K})\right)_{i,j}.$$

The diagonal entries of $E(L/K; \Pi_K, \Pi_L)$ do not depend on the choice of the systems of local parameters Π_K and Π_L. Therefore, the diagonal elements of $E(L/K; \Pi_K, \Pi_L)$ will be denoted simply by $e_1(L/K), \cdots, e_n(L/K)$. As a notation, let $[L_0 : K_0] = f(L/K)$. It is then easy to prove that

$$[L_i : K_i] = f(L/K) e_1(L/K) \cdots e_i(L/K),$$

for $i = 1 \cdots, n$, where

$$e_\ell(L/K) = e(L_\ell/K_\ell),$$

for $\ell = 1, \cdots, i \leq n$. Moreover, the finite extension L/K is called:

- *totally ramified*, if $f(L/K) = 1$ (or equivalently $L_0 = K_0$);
- *semi ramified* if $e_n(L/K) = 1$ and L_{n-1}/K_{n-1} is separable;
- *purely unramified*, if the equality $[L : K] = f(L/K)$ (or equivalently $\prod_{i=1}^n e_i(L/K) = 1$) holds.

If K satisfies $\mathrm{char}(K_{n-1}) = p > 0$, then $[L_{n-1} : K_{n-1}]$ has an expression of the form

$$[L_{n-1} : K_{n-1}] = f_0.p^s,$$

where f_0 is the separable degree of L_{n-1}/K_{n-1} denoted by $f_0(L/K)$, and p^s is the inseparable degree of L_{n-1}/K_{n-1} denoted by $s(L/K)$.

Now, assume that L/K is an infinite algebraic extension in a fixed algebraic closure $\overline{K}$. The infinite algebraic extension L/K is called:

– *totally ramified*, if every finite subextension F/K of L/K inside $\overline{K}$ is totally ramified. Thus, if M/K is any subextension of a totally ramified extension L/K, then M/K is totally ramified as well. Moreover, L/K is called *maximal totally ramified*, if there is no totally ramified extension E/K satisfying $L \subsetneq E \subset \overline{K}$. A maximal totally ramified extension of K in $\overline{K}$ exists but it is not unique. Note that, the compositum of a collection of totallly ramified extensions of K inside $\overline{K}$ is not necessarily totally ramified over K.
– *purely unramified*, if every finite subextension F/K of L/K inside $\overline{K}$ is purely unramified. Thus, if M/K is any subextension of a purely unramified extension L/K, then M/K is purely unramified as well. The compositum of a collection of purely unramified extensions of K in $\overline{K}$ is again purely unramified over K. Therefore, the compositum K^{pur} of all purely unramified extensions of K in $\overline{K}$ is the maximal purely unramified extension of K in $\overline{K}$. Moreover,

$$K^{\mathrm{pur}} = \bigcup_{(m,p)=1} K(\zeta_m),$$

where ζ_m is a primitive m^{th} root of unity with m relatively prime to p. Thus, it follows that K^{pur} is Galois over K. A topological generator φ_K of $\mathrm{Gal}(K^{\mathrm{pur}}/K)$ which is mapped on the topological generator Frob_q of $\mathrm{Gal}(\mathbb{F}_q^{\mathrm{sep}}/\mathbb{F}_q)$ is called the *Frobenius automorphism* of K. So, for each $0 < d \in \mathbb{Z}$, there exists a unique purely unramified extension $K^{\mathrm{pur},d}$ of degree d over K, which is the splitting field of the polynomial $X^{p^d} - X \in K[X]$ over K. Moreover, note that, if L/K is purely unramified, and $\Pi_K \in K^n$ is a system of local parameters of K, then $\Pi_K \in L^n$ remains a system of local parameters of L as well.

The proof of the following proposition is clear.

Proposition 2.7. *If L/K is any algebraic extension, then its unique maximal purely unramified subextension L_o/K is nothing but $L_o = L \cap K^{\mathrm{pur}}$.*

Moreover,

Proposition 2.8. *Let L/K be a finite extension. Then the unique maximal purely unramified subextension L_o/K of L/K is the splitting field of the polynomial $X^{p^{f(L/K)}} - X \in K[X]$ over K. Moreover, L/L_o is a totally ramified extension, and*

$$[L : L_o] = e_1(L/K) \cdots e_n(L/K), \quad [L_o : K] = f(L/K).$$

A special case of this proposition reads as follows: Let L/K be an algebraic extension. Then,

$$L/K : \text{totally ramified} \Leftrightarrow L_o = K. \tag{2.1}$$

3. Topologies on an n-dimensional local field

There are several topologies related to the n-dimensional local field K with the corresponding Parshin chain

$$K_0, K_1, \cdots, K_{n-1}, K_n = K.$$

- The complete discrete valuation $\nu_{K_n} : K_n \to \mathbb{Z} \cup \{\infty\}$ on K_n defines a natural topology on $K_n = K$, called *the discrete valuation topology on* K, denoted by $\mathscr{V}_{K_n}$. With respect to $\mathscr{V}_{K_n}$:
  - K has a natural complete and Hausdorff topological field structure;
  - As a topological field, K is not locally compact in case $n \geq 2$ as $\kappa_{K_n} = K_{n-1}$ is not a finite field;
  - Moreover, again in case $n \geq 2$, the elements of K, which can be considered as formal series $\sum_i a_i t_{n,K}^i$ in the first local parameter $t_{n,K}$ of K via the structure theorem of higher-dimensional local fields, do not converge, as $\mid a_i \mid_{\nu_{K_n}} = 1$ whenever $a_i \neq 0$.

Let $\Pi_K = (t_{1,K}, \cdots, t_{n,K}) \in K^n$ be a system of local parameters of K and $\overline{v}_K : K \to \mathbb{Z}^n \cup \{\infty\}$ be the corresponding rank n discrete valuation on K introduced in Section 2.

- There is a natural topology $\mathscr{T}_K$ on K, called *the higher topology on* K, which is defined recursively by the higher topologies on the residue fields $K_{n-1}, \cdots, K_1$ and K_0, where the higher topology $\mathscr{T}_{K_1}$ on K_1 coincides with the discrete valuation topology $\mathscr{V}_{K_1}$ on K_1, look at [50] for details. With respect to the topology $\mathscr{T}_K$:
  - K does not have a topological field structure. In fact, K is a complete and Hausdorff *sequential ring*; that is, the additive group K^+ is a topological group, multiplication $K \times K \xrightarrow{\times} K$ on K is sequentially continuous. In general, the inversion $K^\times \xrightarrow{\iota} K^\times$ on $K^\times$ is *not* sequentially continuous with respect to the induced topology of $\mathscr{T}_K$ on $K^\times$. Look at [5,6] for an overview of sequential algebraic structures;
  - The map $K \to K$ defined by multiplication with a fixed non-zero $a_o \in K$ as $a \mapsto a_o.a$ for every $a \in K$ is a homeomorphism;
  - The residue homomorphism $O_{\overline{v}_K} \to K_{n-1}$ is continuous and open, where $O_{\overline{v}_K}$ is equipped with the subspace topology induced from the higher topology $\mathscr{T}_K$ of K and K_{n-1} is endowed with its higher topology $\mathscr{T}_{K_{n-1}}$;
  - The unique formal power series expression of $a \in K$ given by
  $$a = \sum_{\boldsymbol{b}=(b_1,\cdots,b_n)} [\theta_{\boldsymbol{b}}] t_{1,K}^{b_1} \cdots t_{n,K}^{b_n},$$
  where all coefficients $[\theta_{\boldsymbol{b}}]$ are from the Teichmüller representatives of all non-zero elements of the last residue field $K_0 = \mathbb{F}_q$ of K and the summation is over the admissible well-ordered set $\{\boldsymbol{b} \in \mathbb{Z}^n \mid \theta_{\boldsymbol{b}} \neq 0\}$, is absolutely convergent.
- There is also a natural topology $\mathscr{T}_{K^\times}$ on the multiplicative group $K^\times$, called *the higher topology on* $K^\times$, which is defined as the initial (that is, weakest) topology on $K^\times$ that makes the map
$$K^\times \to K \times K$$
given by
$$a \mapsto (a, a^{-1}),$$
for every $a \in K^\times$ sequentially continuous. Equivalently, the topology $\mathscr{T}_{K^\times}$ on $K^\times$ is defined as follows:
If $\mathrm{char}(K_{n-1}) = p > 0$, then the topology $\mathscr{T}_{K^\times}$ on $K^\times$ is defined to be the unique topology on $K^\times$ that turns the isomorphism
$$K^\times \xrightarrow{\sim} \langle t_{n,K} \rangle \times \cdots \times \langle t_{1,K} \rangle \times R_{\overline{v}_K} \times V_{\overline{v}_K}$$
into a topological group isomorphism. Here, as introduced in the previous section, $R_{\overline{v}_K}$ is the subgroup of $K^\times$ consisting of Teichmüller representatives of all non-zero elements of the last-residue field $K_0 = \mathbb{F}_q$ of K, $V_{\overline{v}_K}$ is the group of principal units of K relative to $\overline{v}_K$, and the topology on $\langle t_{n,K} \rangle \times \cdots \times \langle t_{1,K} \rangle \times R_{\overline{v}_K} \times V_{\overline{v}_K}$ is the

product topology defined by the discrete topology on $\langle t_{n,K}\rangle \times \cdots \times \langle t_{1,K}\rangle \times R_{\overline{v}_K}$ and the topology on $V_{\overline{v}_K}$ induced from the topology $\mathscr{T}_K$ on K.

If $\mathrm{char}(K) = \cdots = \mathrm{char}(K_{m+1}) = 0$, $\mathrm{char}(K_m) = p > 0$ for some $m \leq n-2$, then the natural topology $\mathscr{T}_{K^\times}$ on $K^\times$ is defined to be the unique topology on $K^\times$ that turns the isomorphism

$$K^\times \xrightarrow{\sim} \langle t_{n,K}\rangle \times \cdots \times \langle t_{1,K}\rangle \times R_{\overline{v}_K} \times V_{\overline{v}_K}$$

into a topological group isomorphism, where the topology on $\langle t_{n,K}\rangle \times \cdots \times \langle t_{1,K}\rangle \times R_{\overline{v}_K} \times V_{\overline{v}_K}$ is the product topology defined by the discrete topology on $\langle t_{n,K}\rangle \times \cdots \times \langle t_{1,K}\rangle$ and the topology on $U_{\overline{v}_K} = R_{\overline{v}_K} \times V_{\overline{v}_K}$ induced from the natural subspace topology on $U_{\overline{v}_{K_{m+1}}}$ given by $\mathscr{T}_{K_{m+1}^\times}$ via the canonical short exact sequence

$$1 \to 1 + P_{\overline{v}_K}^{(1,0,\cdots,0)} \to U_{\overline{v}_K} \to U_{\overline{v}_{K_{m+1}}} \to 1.$$

The basic properties of the topology $\mathscr{T}_{K^\times}$ on $K^\times$ are the following :

- Every Cauchy sequence in $K^\times$ with respect to the topology $\mathscr{T}_{K^\times}$ converges in $K^\times$;
- Multiplication $K^\times \times K^\times \xrightarrow{\times} K^\times$ on $K^\times$ is sequentially continuous and the inversion $K^\times \xrightarrow{\iota} K^\times$ on $K^\times$ is sequentially continuous. That is, $K^\times$ becomes a *sequential group*;
- If $n \leq 2$, then the multiplicative group $K^\times$ is furthermore a topological group with respect to $\mathscr{T}_{K^\times}$ with a countable base of open subgroups. If $n \geq 3$, then the multiplicative group $K^\times$ is not a topological group with respect to $\mathscr{T}_{K^\times}$;
- The unique formal product expression of $a \in K^\times$ given by

$$a = t_{1,K}^{r_1} \cdots t_{n,K}^{r_n} \theta \prod_{\boldsymbol{b}=(b_1,\cdots,b_n)} (1 + [\theta_{\boldsymbol{b}}] t_{1,K}^{b_1} \cdots t_{n,K}^{b_n}),$$

  where $r_1, \cdots, r_n \in \mathbb{Z}$, all coefficients $[\theta_{\boldsymbol{b}}]$ and θ are from the Teichmüller representatives of all non-zero elements of the last residue field $K_0 = \mathbb{F}_q$ of K and the product is over the admissible well-ordered set $\{\boldsymbol{b} \in \mathbb{Z}^n \mid \theta_{\boldsymbol{b}} \neq 0\}$, is absolutely convergent.

For details about the topology $\mathscr{T}_{K^\times}$, look at [50].

As Fesenko points out, the higher topology $\mathscr{T}_K$ on K and the higher topology $\mathscr{T}_{K^\times}$ on $K^\times$ are indeed “the appropriate topologies” for class field theoretic investigations for K. It is also quite possible, as suggested by Braunling, that a totally new theory, like “condensed mathematics” of Clausen and Scholze [46] or “pyknotic mathematics” of Barwick and Haine [3], is needed to settle the topological problems of K.

4. Milnor K-theory

Let F be *any* field. For any integer $m > 0$, the m^{th} *Milnor K-group* $\mathrm{K}_m^{\mathrm{Milnor}}(F)$ of F is defined by the quotient

$$\mathrm{K}_m^{\mathrm{Milnor}}(F) := F^{\times \otimes m}/J_m(F),$$

where $F^{\times\otimes m} = \overbrace{F^\times \otimes \cdots \otimes F^\times}^{m\text{-copies}}$ is the m-fold tensor product of $F^\times$ and $J_m(F)$ is the subgroup of $F^{\times\otimes m}$ defined by

$$\left\langle x_1 \otimes \cdots \otimes x_m \mid x_i + x_j = 1, {}^\exists i, j,\ 1 \leq i \neq j \leq m \right\rangle.$$

For $x_1, \cdots, x_m \in F^\times$, the element $x_1 \otimes \cdots \otimes x_m \pmod{J_m(F)}$ in $\mathrm{K}_m^{\mathrm{Milnor}}(F)$ is simply denoted by $\{x_1, \cdots, x_m\}$ and called the *generalized Steinberg symbol of* $x_1, \cdots, x_m$. In case $m = 0$, we set $\mathrm{K}_{m=0}^{\mathrm{Milnor}}(F) = \mathbb{Z}$.

Milnor K-theory $\mathrm{K}_m^{\mathrm{Milnor}}$ defines a functor from the category of fields to the category of abelian groups. Let L/F be *any* extension. Then the natural embedding $j_{L/F} : F \hookrightarrow L$ induces a group homomorphism

$$\mathrm{K}_m^{\mathrm{Milnor}}(j_{L/F}) = j_{L/F}^{\mathrm{Milnor}} : \mathrm{K}_m^{\mathrm{Milnor}}(F) \to \mathrm{K}_m^{\mathrm{Milnor}}(L).$$

In case $m = 0$, the homomorphism $j_{L/F}^{\mathrm{Milnor}}$ is the identity arrow $\mathrm{id}_{\mathbb{Z}} : \mathbb{Z} \to \mathbb{Z}$.

By a theorem of Bass, Tate and Kato, there exists, for each finite extension L/F, a group homomorphism

$$\mathrm{N}_{L/F}^{\mathrm{Milnor}} : \mathrm{K}_m^{\mathrm{Milnor}}(L) \to \mathrm{K}_m^{\mathrm{Milnor}}(F),$$

called the *(K-theoretic) norm map from L to F*. The basic properties of this arrow are the following:

- The norm map $\mathrm{N}_{L/F}^{\mathrm{Milnor}} : \mathrm{K}_m^{\mathrm{Milnor}}(L) \to \mathrm{K}_m^{\mathrm{Milnor}}(F)$ from L to F is transitive in the sense that, for every chain $F \subset M \subset L$ of extensions of F, the equality
$$\mathrm{N}_{L/F}^{\mathrm{Milnor}} = \mathrm{N}_{M/F}^{\mathrm{Milnor}} \circ \mathrm{N}_{L/M}^{\mathrm{Milnor}}$$
holds;
- In the low-dimensional cases, the homomorphism
$$\mathrm{N}_{L/F}^{\mathrm{Milnor}} : \mathrm{K}_m^{\mathrm{Milnor}}(L) \to \mathrm{K}_m^{\mathrm{Milnor}}(F)$$
reduces to the multiplication by $[L : F]$ mapping if $m = 0$, and to the usual norm map of fields $\mathrm{N}_{L/F} : L^\times \to F^\times$ if $m = 1$;
- The composition
$$\mathrm{K}_m^{\mathrm{Milnor}}(F) \xrightarrow{j_{L/F}^{\mathrm{Milnor}}} \mathrm{K}_m^{\mathrm{Milnor}}(L) \xrightarrow{\mathrm{N}_{L/F}^{\mathrm{Milnor}}} \mathrm{K}_m^{\mathrm{Milnor}}(F)$$
is the mapping defined as the multiplication by $[L : F]$;
- If $\sigma \in \mathrm{Aut}_F(L)$, then
$$\mathrm{N}_{L/F}^{\mathrm{Milnor}} \circ \mathrm{K}_m^{\mathrm{Milnor}}(\sigma) = \mathrm{N}_{L/F}^{\mathrm{Milnor}},$$
where $\mathrm{K}_m^{\mathrm{Milnor}}(\sigma) : \mathrm{K}_m^{\mathrm{Milnor}}(L) \to \mathrm{K}_m^{\mathrm{Milnor}}(L)$ is the homomorphism induced by the F-automorphism $\sigma : L \to L$.

For details about Milnor K-theory, look at Chapter IX of [15].

In case, K is the n-dimensional local field with the corresponding Parshin chain

$$\mathbb{F}_q = K_0, K_1, \cdots, K_{n-1}, K_n = K,$$

there exists a surjective homomorphism called the *(K-theoretic) valuation map*

$$\nu_{\mathrm{K}_n^{\mathrm{Milnor}}(K)} : \mathrm{K}_n^{\mathrm{Milnor}}(K) \to \mathbb{Z} \tag{4.1}$$

on $\mathrm{K}_n^{\mathrm{Milnor}}(K)$ defined by the composition

$$\nu_{\mathrm{K}_n^{\mathrm{Milnor}}(K)} : \mathrm{K}_n^{\mathrm{Milnor}}(K_n) \xrightarrow{\partial_{n-1}^n} \mathrm{K}_{n-1}^{\mathrm{Milnor}}(K_{n-1}) \xrightarrow{\partial_{n-2}^{n-1}} \cdots \xrightarrow{\partial_0^1} \mathrm{K}_0^{\mathrm{Milnor}}(K_0) = \mathbb{Z}, \tag{4.2}$$

where the arrows

$$\partial_{i-1}^i : \mathrm{K}_i^{\mathrm{Milnor}}(K_i) \to \mathrm{K}_{i-1}^{\mathrm{Milnor}}(K_{i-1}),$$

for $i = 1, 2, 3, \cdots, n$, are the *boundary homomorphisms in Milnor* K*-theory* defined by

$$\partial_{i-1}^i(\{u_1, \cdots, u_{i-1}, x\}) = \nu_{K_i}(x)\{\overline{u}_1, \cdots, \overline{u}_{i-1}\}$$

for each $u_1, \cdots, u_{i-1} \in O_{K_i}^\times = U_{K_i}$ and $x \in K_i^\times$, where $\overline{u}_1, \cdots, \overline{u}_{i-1} \in K_{i-1}$ are defined by reduction modulo $\mathfrak{p}_{K_i}$ of the elements $u_1, \cdots, u_{i-1}$ in K_i. Let L be a finite extension of K. Then the K-theoretic valuation map

$$\nu_{\mathrm{K}_n^{\mathrm{Milnor}}(L)} : \mathrm{K}_n^{\mathrm{Milnor}}(L) \to \mathbb{Z}$$

on $\mathrm{K}_n^{\mathrm{Milnor}}(L)$ satisfies

$$\nu_{\mathrm{K}_n^{\mathrm{Milnor}}(L)} = \frac{1}{f(L/K)}\nu_{\mathrm{K}_n^{\mathrm{Milnor}}(K)} \circ \mathrm{N}_{L/K}^{\mathrm{Milnor}},$$

where $f(L/K) = [L_0 : K_0]$, because the diagram

$$\begin{array}{ccc}
\mathrm{K}_n^{\mathrm{Milnor}}(L) & \xrightarrow{\partial_{n-1}^n} & \mathrm{K}_{n-1}^{\mathrm{Milnor}}(L_{n-1}) \\
{\scriptstyle \mathrm{N}_{L/K}^{\mathrm{Milnor}}}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{N}_{L_{n-1}/K_{n-1}}^{\mathrm{Milnor}}} \\
\mathrm{K}_n^{\mathrm{Milnor}}(K) & \xrightarrow{\partial_{n-1}^n} & \mathrm{K}_{n-1}^{\mathrm{Milnor}}(K_{n-1})
\end{array}$$

is commutative. An element $\Pi_{\mathrm{K}_n^{\mathrm{Milnor}}(K)}$ of $\mathrm{K}_n^{\mathrm{Milnor}}(K)$ is called a *prime element* of $\mathrm{K}_n^{\mathrm{Milnor}}(K)$ if

$$\nu_{\mathrm{K}_n^{\mathrm{Milnor}}(K)}(\Pi_{\mathrm{K}_n^{\mathrm{Milnor}}(K)}) = 1.$$

Note that, a prime element $\Pi_{\mathrm{K}_n^{\mathrm{Milnor}}(K)}$ of $\mathrm{K}_n^{\mathrm{Milnor}}(K)$ can be expressed as

$$\Pi_{\mathrm{K}_n^{\mathrm{Milnor}}(K)} = \{t_{1,K}, \cdots, t_{n,K}\} + \varepsilon,$$

where $\Pi_K = (t_{1,K}, \cdots, t_{n,K}) \in K^n$ is a system of local parameters of the n-dimensional local field K and the element ε lies in $\mathrm{Ker}(\nu_{\mathrm{K}_n^{\mathrm{Milnor}}(K)})$.

Continue to assume that K is an n-dimensional local field. Then, choosing a system of local parameters $\Pi_K = (t_{1,K}, \cdots, t_{n,K}) \in K^n$ of K, Π_K determines a rank n discrete valuation

$$\overline{v}_K : K \to \mathbb{Z}^n \cup \{\infty\}$$

of K, which determines a collection $\left\{U_{\overline{v}_K}^{(i_\ell,\cdots,i_n)}\mathrm{K}_m^{\mathrm{Milnor}}(K)\right\}_{(i_\ell,\cdots,i_n)}$ consisting of subgroups $U_{\overline{v}_K}^{(i_\ell,\cdots,i_n)}\mathrm{K}_m^{\mathrm{Milnor}}(K)$ of $\mathrm{K}_m^{\mathrm{Milnor}}(K)$ given by

$$U_{\overline{v}_K}^{(i_\ell,\cdots,i_n)}\mathrm{K}_m^{\mathrm{Milnor}}(K) = \left\langle \{x_1, \cdots, x_m\} \in \mathrm{K}_m^{\mathrm{Milnor}}(K) \mid x_1 \in U_{\overline{v}_K}^{(i_\ell,\cdots,i_n)} \right\rangle,$$

where $(i_\ell, \cdots, i_n) \in \mathbb{Z}^{n-\ell+1}$ satisfies $(i_\ell, \cdots, i_n) \succeq \overbrace{(0, \cdots, 0)}^{(n-\ell+1)\text{-tuple}}$, for each ℓ satisfying $1 \leq \ell \leq n$.

In particular, in case $\ell = n$, the group $U_{\overline{v}_K}^{(i_n)}\mathrm{K}_m^{\mathrm{Milnor}}(K)$ is denoted by $U_{K_n}^{i_n}\mathrm{K}_m^{\mathrm{Milnor}}(K)$ for each $i_n \in \mathbb{Z}$ satisfying $i_n \geq 0$. Moreover,

- if $i_n = 0$, the group $U_{\overline{v}_K}^{(i_n=0)}\mathrm{K}_m^{\mathrm{Milnor}}(K)$ is denoted by $U_{K_n}\mathrm{K}_m^{\mathrm{Milnor}}(K)$;
- if $i_n = 1$, the group $U_{\overline{v}_K}^{(i_n=1)}\mathrm{K}_m^{\mathrm{Milnor}}(K)$ is denoted by $V_{K_n}\mathrm{K}_m^{\mathrm{Milnor}}(K)$.

In case L is an algebraic extension of K and $\overline{w}_L$ is the unique extension of the rank n discrete valuation $\overline{v}_K$ of K to L, the subgroup

$$U_{\overline{w}_L}^{(i_\ell,\cdots,i_n)}\mathrm{K}_m^{\mathrm{Milnor}}(L) = \left\langle \{x_1, \cdots, x_m\} \in \mathrm{K}_m^{\mathrm{Milnor}}(L) \mid x_1 \in U_{\overline{w}_L}^{(i_\ell,\cdots,i_n)} \right\rangle,$$

of $\mathrm{K}_m^{\mathrm{Milnor}}(L)$ is denoted by $U_{\overline{v}_K}^{(i_\ell,\cdots,i_n)}\mathrm{K}_m^{\mathrm{Milnor}}(L)$, where $(i_\ell, \cdots, i_n) \in \mathbb{Z}^{n-\ell+1}$ satisfies $(i_\ell, \cdots, i_n) \succeq \overbrace{(0, \cdots, 0)}^{(n-\ell+1)\text{-tuple}}$, for each ℓ satisfying $1 \leq \ell \leq n$.

In particular, in case $\ell = n$, the group $U_{\overline{v}_K}^{(i_n)}\mathrm{K}_m^{\mathrm{Milnor}}(L)$ is denoted by $U_{K_n}^{i_n}\mathrm{K}_m^{\mathrm{Milnor}}(L)$ for each $i_n \in \mathbb{Z}$ satisfying $i_n \geq 0$. Moreover,

- if $i_n = 0$, the group $U_{\overline{v}_K}^{(i_n=0)}\mathrm{K}_m^{\mathrm{Milnor}}(L)$ is denoted by $U_{K_n}\mathrm{K}_m^{\mathrm{Milnor}}(L)$;
- if $i_n = 1$, the group $U_{\overline{v}_K}^{(i_n=1)}\mathrm{K}_m^{\mathrm{Milnor}}(L)$ is denoted by $V_{K_n}\mathrm{K}_m^{\mathrm{Milnor}}(L)$.

5. $\mathrm{K}_*^{\mathrm{top}}$-groups

Let F be a field such that $F^\times$ is endowed with a topology $\mathscr{T}$. The topology $\mathscr{T}$ on $F^\times$ introduces a natural topology $\mathscr{T}_{\mathrm{K}_m^{\mathrm{Milnor}}(F)}$ on $\mathrm{K}_m^{\mathrm{Milnor}}(F)$. The sequential saturation[‖] $(\mathscr{T}_{\mathrm{K}_m^{\mathrm{Milnor}}(F)})_{\mathrm{seq}}$ of $\mathscr{T}_{\mathrm{K}_m^{\mathrm{Milnor}}(F)}$ is the strongest topology on $\mathrm{K}_m^{\mathrm{Milnor}}(F)$ that makes the mappings

$$(\alpha, \beta) \mapsto \alpha - \beta, \ {}^\forall \alpha, \beta \in \mathrm{K}_m^{\mathrm{Milnor}}(F)$$

and

$$(a_1, \cdots, a_m) \mapsto \{a_1, \cdots, a_m\}, \ {}^\forall a_1, \cdots, a_m \in F^\times$$

continuous. Look at Remark 1 in [14]. With respect to the topology $(\mathscr{T}_{\mathrm{K}_m^{\mathrm{Milnor}}(F)})_{\mathrm{seq}}$ defined on $\mathrm{K}_m^{\mathrm{Milnor}}(F)$, Parshin introduced

$$\Lambda_{\mathrm{K}_m^{\mathrm{Milnor}}(F)} := \bigcap_{\mathscr{O}} \mathscr{O},$$

where $\mathscr{O}$ runs over all open neighbourhoods of the identity element $0_{K_m^{\mathrm{Milnor}}(F)}$ of $K_m^{\mathrm{Milnor}}(F)$, which is a closed subgroup of $\mathrm{K}_m^{\mathrm{Milnor}}(F)$. The quotient group

$$\mathrm{K}_m^{\mathrm{top}}(F) := \mathrm{K}_m^{\mathrm{Milnor}}(F)/\Lambda_{\mathrm{K}_m^{\mathrm{Milnor}}(F)}$$

endowed with the quotient topology of $(\mathscr{T}_{\mathrm{K}_m^{\mathrm{Milnor}}(F)})_{\mathrm{seq}}$; that is, the maximal Hausdorff quotient of $\mathrm{K}_m^{\mathrm{Milnor}}(F)$ with respect to $(\mathscr{T}_{\mathrm{K}_m^{\mathrm{Milnor}}(F)})_{\mathrm{seq}}$, is called the m^{th} *Parshin topological K-group* of the field F. For $x_1, \cdots, x_m \in F^\times$, the element $\{x_1, \cdots, x_m\} \pmod{\Lambda_{\mathrm{K}_m^{\mathrm{Milnor}}(F)}}$ is denoted by $\{x_1, \cdots, x_m\}^{\mathrm{top}}$ and called the *topological Steinberg symbol of* $x_1, \cdots, x_m$.

Let L be *any "compatible"* extension of F in the sense that:

- $L^\times$ is endowed with a topology $\mathscr{T}'$;
- The topology $\mathscr{T}$ on $F^\times$ is induced from $\mathscr{T}'$.

Then, the inclusion $j_{L/F}^{\mathrm{Milnor}}(\Lambda_{\mathrm{K}_m^{\mathrm{Milnor}}(F)}) \subseteq \Lambda_{\mathrm{K}_m^{\mathrm{Milnor}}(L)}$ clearly follows, and the group homomorphism $j_{L/F}^{\mathrm{Milnor}} : \mathrm{K}_m^{\mathrm{Milnor}}(F) \to \mathrm{K}_m^{\mathrm{Milnor}}(L)$ extends uniquely to a continuous homomorphism

$$j_{L/F}^{\mathrm{top}} : \mathrm{K}_m^{\mathrm{top}}(F) \to \mathrm{K}_m^{\mathrm{top}}(L).$$

For the n-dimensional local field K, let $\mathscr{T}_{K^\times}$ be the higher topology on $K^\times$ introduced in Section 3. As in the preceding paragraph, the strongest topology on $\mathrm{K}_m^{\mathrm{Milnor}}(K)$ that makes the mappings

$$(\alpha, \beta) \mapsto \alpha - \beta, \ {}^\forall \alpha, \beta \in \mathrm{K}_m^{\mathrm{Milnor}}(K)$$

[‖] Let X be a set endowed with a topology $\mathscr{T}$. Recall that $U \subseteq X$ is called sequentially open (with respect to $\mathscr{T}$), if for any sequence (x_n) in X converging to $u \in U$, there exists n_o such that $x_n \in U$ for every $n \geq n_o$. The collection of sequentially open subsets of X (with respect to $\mathscr{T}$) defines a topology $\mathscr{T}_{\mathrm{seq}}$ on X finer than $\mathscr{T}$, called the *sequential saturation of* $\mathscr{T}$, and the topological space $(X, \mathscr{T}_{\mathrm{seq}})$ the *sequential saturation of the topological space* $(X, \mathscr{T})$. To simplify the notation, the sequential saturation of the topological space X is simply denoted by X_{seq}. If $X = X_{\mathrm{seq}}$ (that is, if $\mathscr{T} = \mathscr{T}_{\mathrm{seq}}$), then X is called a sequentially saturated topological space. Note that, the topological space X_{seq} has the following basic properties:

- X_{seq} is sequentially saturated, namely $(X_{\mathrm{seq}})_{\mathrm{seq}} = X_{\mathrm{seq}}$;
- The universal mapping property satisfied by X_{seq}: If Y is a sequentially saturated space, then any continuous map $f : Y \to X$ factors naturally as

$$\begin{tikzcd} & X_{\mathrm{seq}} \arrow[dr, "\mathrm{id}"] & \\ Y \arrow[ur, "f_{\mathrm{seq}}"] \arrow[rr, "f"'] & & X, \end{tikzcd}$$

  where the induced map $f_{\mathrm{seq}} : Y \to X_{\mathrm{seq}}$ is continuous;
- If Y is a topological space and $f : Y \to X$ is any sequentially continuous map, then the induced map $f_{\mathrm{seq}} : Y_{\mathrm{seq}} \to X_{\mathrm{seq}}$ is continuous.

and

$$(a_1, \cdots, a_m) \mapsto \{a_1, \cdots, a_m\}, \ {}^\forall a_1, \cdots, a_m \in K^\times$$

continuous is the sequential saturation $(\mathscr{T}_{\mathrm{K}_m^{\mathrm{Milnor}}(K)})_{\mathrm{seq}}$ of the topology $\mathscr{T}_{\mathrm{K}_m^{\mathrm{Milnor}}(K)}$ on $\mathrm{K}_m^{\mathrm{Milnor}}(K)$, where $\mathscr{T}_{\mathrm{K}_m^{\mathrm{Milnor}}(K)}$ denotes the topology on $\mathrm{K}_m^{\mathrm{Milnor}}(K)$ induced from the higher topology $\mathscr{T}_{K^\times}$ of $K^\times$.

Note that, by [14], the closed subgroup $\Lambda_{\mathrm{K}_m^{\mathrm{Milnor}}(K)}$ of $\mathrm{K}_m^{\mathrm{Milnor}}(K)$ is also equal to

$$\Lambda_{\mathrm{K}_m^{\mathrm{Milnor}}(K)} = \bigcap_{\ell \neq p} \ell \mathrm{K}_m^{\mathrm{Milnor}}(K),$$

where ℓ runs over all primes different than $p = \mathrm{char}(K_0)$. Therefore, the boundary homomorphism in Milnor K-theory

$$\partial_{i-1}^i : \mathrm{K}_i^{\mathrm{Milnor}}(K_i) \to \mathrm{K}_{i-1}^{\mathrm{Milnor}}(K_{i-1})$$

naturally induces the following morphism

$$\partial_{i-1}^i : \Lambda_{\mathrm{K}_i^{\mathrm{Milnor}}(K_i)} \to \Lambda_{\mathrm{K}_{i-1}^{\mathrm{Milnor}}(K_{i-1})},$$

and thereby defines the *boundary homomorphism in topological Milnor K-theory*

$$(\partial_{i-1}^i)^{\mathrm{top}} : \mathrm{K}_i^{\mathrm{top}}(K_i) \to \mathrm{K}_{i-1}^{\mathrm{top}}(K_{i-1}),$$

where

$$\begin{aligned}(\partial_{i-1}^i)^{\mathrm{top}}(\{u_1, \cdots, u_{i-1}, x\}^{\mathrm{top}}) &= (\partial_{i-1}^i)^{\mathrm{top}} \left(\{u_1, \cdots, u_{i-1}, x\} \pmod{\Lambda_{\mathrm{K}_i^{\mathrm{Milnor}}(K_i)}}\right) \\ &= \nu_{K_i}(x)\{\overline{u}_1, \cdots, \overline{u}_{i-1}\} \pmod{\Lambda_{\mathrm{K}_{i-1}^{\mathrm{Milnor}}(K_{i-1})}} \\ &= \nu_{K_i}(x)\{\overline{u}_1, \cdots, \overline{u}_{i-1}\}^{\mathrm{top}},\end{aligned}$$

for each $u_1, \cdots, u_{i-1} \in O_{K_i}^\times = U_{K_i}$ and $x \in K_i^\times$, where $\overline{u}_1, \cdots, \overline{u}_{i-1} \in K_{i-1}$ are defined by reduction modulo $\mathfrak{p}_{K_i}$ of the elements $u_1, \cdots, u_{i-1}$ in K_i, for each $i = 1, 2, \cdots, n$. Therefore, there exists a surjective homomorphism called the *(topological K-theoretic) valuation map*

$$\nu_{\mathrm{K}_n^{\mathrm{top}}(K)} : \mathrm{K}_n^{\mathrm{top}}(K) \to \mathbb{Z} \tag{5.1}$$

on $\mathrm{K}_n^{\mathrm{top}}(K)$ defined by the composition

$$\nu_{\mathrm{K}_n^{\mathrm{top}}(K)} : \mathrm{K}_n^{\mathrm{top}}(K_n) \xrightarrow{(\partial_{n-1}^n)^{\mathrm{top}}} \mathrm{K}_{n-1}^{\mathrm{top}}(K_{n-1}) \xrightarrow{(\partial_{n-2}^{n-1})^{\mathrm{top}}} \cdots \xrightarrow{(\partial_0^1)^{\mathrm{top}}} \mathrm{K}_0^{\mathrm{top}}(K_0) = \mathbb{Z}. \tag{5.2}$$

Clearly, the valuation $\nu_{\mathrm{K}_n^{\mathrm{Milnor}}(K)} : \mathrm{K}_n^{\mathrm{Milnor}}(K) \to \mathbb{Z}$ factors through

$$\nu_{\mathrm{K}_n^{\mathrm{Milnor}}(K)} : \mathrm{K}_n^{\mathrm{Milnor}}(K) \xrightarrow{\mathrm{red}_{\Lambda_{\mathrm{K}_n^{\mathrm{Milnor}}(K)}}} \mathrm{K}_n^{\mathrm{top}}(K) \xrightarrow{\nu_{\mathrm{K}_n^{\mathrm{top}}(K)}} \mathbb{Z}$$

as the diagram

$$\begin{tikzcd}
\mathrm{K}_n^{\mathrm{top}}(K) \arrow[r, "(\partial_{n-1}^n)^{\mathrm{top}}"] \arrow[rrr, bend left, "\nu_{\mathrm{K}_n^{\mathrm{top}}(K)}"] & \mathrm{K}_{n-1}^{\mathrm{top}}(K_{n-1}) \arrow[r, "(\partial_{n-2}^{n-1})^{\mathrm{top}}"] & \cdots \arrow[r, "(\partial_0^1)^{\mathrm{top}}"] & \mathrm{K}_0^{\mathrm{top}}(K_0) = \mathbb{Z} \arrow[d, equal] \\
\mathrm{K}_n^{\mathrm{Milnor}}(K) \arrow[u, "\mathrm{red}_{\Lambda_{\mathrm{K}_n^{\mathrm{Milnor}}(K)}}"] \arrow[r, "\partial_{n-1}^n"] \arrow[rrr, bend right, "\nu_{\mathrm{K}_n^{\mathrm{Milnor}}(K)}"'] & \mathrm{K}_{n-1}^{\mathrm{Milnor}}(K_{n-1}) \arrow[u, "\mathrm{red}_{\Lambda_{\mathrm{K}_{n-1}^{\mathrm{Milnor}}(K_{n-1})}}"'] \arrow[r, "\partial_{n-2}^{n-1}"] & \cdots \arrow[r, "\partial_0^1"] & \mathrm{K}_0^{\mathrm{Milnor}}(K_0) = \mathbb{Z}
\end{tikzcd}$$

is commutative. An element $\Pi_{\mathrm{K}_n^{\mathrm{top}}(K)}$ of $\mathrm{K}_n^{\mathrm{top}}(K)$ is called a *prime element* of $\mathrm{K}_n^{\mathrm{top}}(K)$ if

$$\nu_{\mathrm{K}_n^{\mathrm{top}}(K)}(\Pi_{\mathrm{K}_n^{\mathrm{top}}(K)}) = 1.$$

Note that, a prime element $\Pi_{\mathrm{K}_n^{\mathrm{top}}(K)}$ of $\mathrm{K}_n^{\mathrm{top}}(K)$ can be expressed as

$$\Pi_{\mathrm{K}_n^{\mathrm{top}}(K)} = \{t_{1,K}, \cdots, t_{n,K}\}^{\mathrm{top}} + \varepsilon,$$

where $\Pi_K = (t_{1,K}, \cdots, t_{n,K})$ is a system of local parameters of K and the element ε lies in $\mathrm{Ker}(\nu_{\mathrm{K}_n^{\mathrm{top}}(K)})$.

Let L be a finite extension of the n-dimensional local field K. As $\mathrm{N}_{L/K}^{\mathrm{Milnor}}(\Lambda_{\mathrm{K}_m^{\mathrm{Milnor}}(L)}) \subseteq \Lambda_{\mathrm{K}_m^{\mathrm{Milnor}}(K)}$, the norm map

$$\mathrm{N}_{L/K}^{\mathrm{Milnor}} : \mathrm{K}_m^{\mathrm{Milnor}}(L) \to \mathrm{K}_m^{\mathrm{Milnor}}(K)$$

induces a unique homomorphism

$$\mathrm{N}_{L/K}^{\mathrm{top}} : \mathrm{K}_m^{\mathrm{top}}(L) \to \mathrm{K}_m^{\mathrm{top}}(K)$$

satisfying, following [38],

$$\mathrm{N}_{L/K}^{\mathrm{top}}(\text{open subgroup of } \mathrm{K}_m^{\mathrm{top}}(L)) \subseteq \text{open subgroup of } \mathrm{K}_m^{\mathrm{top}}(K)$$

with the usual transitivity property; namely, $\mathrm{N}_{L/K}^{\mathrm{top}} = \mathrm{N}_{M/K}^{\mathrm{top}} \circ \mathrm{N}_{L/M}^{\mathrm{top}}$ for every chain $K \subseteq M \subseteq L$ of finite extensions of K. Moreover,

- The composition

$$\mathrm{K}_m^{\mathrm{top}}(K) \xrightarrow{j_{L/K}^{\mathrm{top}}} \mathrm{K}_m^{\mathrm{top}}(L) \xrightarrow{\mathrm{N}_{L/K}^{\mathrm{top}}} \mathrm{K}_m^{\mathrm{top}}(K) \tag{5.3}$$

  is the multiplication by $[L : K]$ mapping;
- If $\sigma \in \mathrm{Aut}_K(L)$, then

$$\mathrm{N}_{L/K}^{\mathrm{top}} \circ \mathrm{K}_m^{\mathrm{top}}(\sigma) = \mathrm{N}_{L/K}^{\mathrm{top}}, \tag{5.4}$$

  where $\mathrm{K}_m^{\mathrm{top}}(\sigma) : \mathrm{K}_m^{\mathrm{top}}(L) \to \mathrm{K}_m^{\mathrm{top}}(L)$ is the homomorphism induced by the K-automorphism $\sigma : L \to L$.

For any finite extension L of K, the topological K-theoretic valuation map

$$\nu_{\mathrm{K}_n^{\mathrm{top}}(L)} : \mathrm{K}_n^{\mathrm{top}}(L) \to \mathbb{Z}$$

on $\mathrm{K}_n^{\mathrm{top}}(L)$ satisfies

$$\nu_{\mathrm{K}_n^{\mathrm{top}}(L)} = \frac{1}{f(L/K)} \nu_{\mathrm{K}_n^{\mathrm{top}}(K)} \circ \mathrm{N}_{L/K}^{\mathrm{top}},$$

where $f(L/K) = [L_0 : K_0]$, because the diagram

$$\begin{array}{ccc} \mathrm{K}_n^{\mathrm{top}}(L) & \xrightarrow{(\partial_{n-1}^n)^{\mathrm{top}}} & \mathrm{K}_{n-1}^{\mathrm{top}}(L_{n-1}) \\ {\scriptstyle \mathrm{N}_{L/K}^{\mathrm{top}}}\downarrow & & \downarrow{\scriptstyle \mathrm{N}_{L_{n-1}/K_{n-1}}^{\mathrm{top}}} \\ \mathrm{K}_n^{\mathrm{top}}(K) & \xrightarrow{(\partial_{n-1}^n)^{\mathrm{top}}} & \mathrm{K}_{n-1}^{\mathrm{top}}(K_{n-1}) \end{array} \tag{5.5}$$

is commutative.

Note that, the commutative diagram (5.5) naturally induces a homomorphism

$$(\partial_{n-1}^n)_*^{\mathrm{top}} : \mathrm{K}_n^{\mathrm{top}}(K)/\mathrm{N}_{L/K}^{\mathrm{top}}(\mathrm{K}_n^{\mathrm{top}}(L)) \to \mathrm{K}_{n-1}^{\mathrm{top}}(K_{n-1})/\mathrm{N}_{L_{n-1}/K_{n-1}}^{\mathrm{top}}(\mathrm{K}_{n-1}^{\mathrm{top}}(L_{n-1}))$$

for every finite extension L of K.

The system of local parameters $\Pi_K = (t_{1,K}, \cdots, t_{n,K}) \in K^n$ of K determines a rank n discrete valuation

$$\overline{v}_K : K \to \mathbb{Z}^n \cup \{\infty\}$$

of K, which naturally determines a collection $\left\{U_{\overline{v}_K}^{(i_\ell,\cdots,i_n)}\mathrm{K}_m^{\mathrm{top}}(K)\right\}_{(i_\ell,\cdots,i_n)}$ of subgroups $U_{\overline{v}_K}^{(i_\ell,\cdots,i_n)}\mathrm{K}_m^{\mathrm{top}}(K)$ of $\mathrm{K}_m^{\mathrm{top}}(K)$, where $U_{\overline{v}_K}^{(i_\ell,\cdots,i_n)}\mathrm{K}_m^{\mathrm{top}}(K)$ is defined by the image

$$\mathrm{red}_{\Lambda_{\mathrm{K}_m^{\mathrm{Milnor}}(K)}} : U_{\overline{v}_K}^{(i_\ell,\cdots,i_n)}\mathrm{K}_m^{\mathrm{Milnor}}(K) \mapsto \mathrm{red}_{\Lambda_{\mathrm{K}_m^{\mathrm{Milnor}}(K)}}\left(U_{\overline{v}_K}^{(i_\ell,\cdots,i_n)}\mathrm{K}_m^{\mathrm{Milnor}}(K)\right)$$

in $\mathrm{K}_m^{\mathrm{top}}(K)$ of $U_{\overline{v}_K}^{(i_\ell,\cdots,i_n)}\mathrm{K}_m^{\mathrm{Milnor}}(K)$ under the natural homomorphism

$$\mathrm{red}_{\Lambda_{\mathrm{K}_m^{\mathrm{Milnor}}(K)}} : \mathrm{K}_m^{\mathrm{Milnor}}(K) \to \mathrm{K}_m^{\mathrm{top}}(K),$$

where $(i_\ell,\cdots,i_n) \in \mathbb{Z}^{n-\ell+1}$ satisfies $(i_\ell,\cdots,i_n) \succeq \overbrace{(0,\cdots,0)}^{(n-\ell+1)\text{-tuple}}$, for each ℓ satisfying $1 \leq \ell \leq n$. The collection $\left\{U_{\overline{v}_K}^{(i_\ell,\cdots,i_n)}\mathrm{K}_m^{\mathrm{top}}(K)\right\}_{(i_\ell,\cdots,i_n)}$ is a neighborhood basis of the identity element of $\mathrm{K}_m^{\mathrm{top}}(K)$.

In particular, in case $\ell = n$, the group $U_{\overline{v}_K}^{(i_n)}\mathrm{K}_m^{\mathrm{top}}(K)$ is denoted by $U_{K_n}^{i_n}\mathrm{K}_m^{\mathrm{top}}(K)$ for each $i_n \in \mathbb{Z}$ satisfying $i_n \geq 0$. Moreover,

- if $i_n = 0$, the group $U_{\overline{v}_K}^{(i_n=0)}\mathrm{K}_m^{\mathrm{top}}(K)$ is denoted by $U_{K_n}\mathrm{K}_m^{\mathrm{top}}(K)$;
- if $i_n = 1$, the group $U_{\overline{v}_K}^{(i_n=1)}\mathrm{K}_m^{\mathrm{top}}(K)$ is denoted by $V_{K_n}\mathrm{K}_m^{\mathrm{top}}(K)$.

In case L is an algebraic extension of K and $\overline{w}_L$ is the unique extension of the rank n discrete valuation $\overline{v}_K$ of K to L, the subgroup $U_{\overline{w}_L}^{(i_\ell,\cdots,i_n)}\mathrm{K}_m^{\mathrm{top}}(L)$ of $\mathrm{K}_m^{\mathrm{top}}(L)$, which is denoted by $U_{\overline{v}_K}^{(i_\ell,\cdots,i_n)}\mathrm{K}_m^{\mathrm{top}}(L)$, is defined by the image $\mathrm{red}_{\Lambda_{\mathrm{K}_m^{\mathrm{Milnor}}(L)}}\left(U_{\overline{v}_K}^{(i_\ell,\cdots,i_n)}\mathrm{K}_m^{\mathrm{Milnor}}(L)\right)$ in $\mathrm{K}_m^{\mathrm{top}}(L)$ of $U_{\overline{v}_K}^{(i_\ell,\cdots,i_n)}\mathrm{K}_m^{\mathrm{Milnor}}(L)$ under the natural homomorphism

$$\mathrm{red}_{\Lambda_{\mathrm{K}_m^{\mathrm{Milnor}}(L)}} : \mathrm{K}_m^{\mathrm{Milnor}}(L) \to \mathrm{K}_m^{\mathrm{top}}(L),$$

where $(i_\ell,\cdots,i_n) \in \mathbb{Z}^{n-\ell+1}$ satisfies $(i_\ell,\cdots,i_n) \succeq \overbrace{(0,\cdots,0)}^{(n-\ell+1)\text{-tuple}}$, for each ℓ satisfying $1 \leq \ell \leq n$.

In particular, in case $\ell = n$, the group $U_{\overline{v}_K}^{(i_n)}\mathrm{K}_m^{\mathrm{top}}(L)$ is denoted by $U_{K_n}^{i_n}\mathrm{K}_m^{\mathrm{top}}(L)$ for each $i_n \in \mathbb{Z}$ satisfying $i_n \geq 0$. Moreover,

- if $i_n = 0$, the group $U_{\overline{v}_K}^{(i_n=0)}\mathrm{K}_m^{\mathrm{top}}(L)$ is denoted by $U_{K_n}\mathrm{K}_m^{\mathrm{top}}(L)$.
- if $i_n = 1$, the group $U_{\overline{v}_K}^{(i_n=1)}\mathrm{K}_m^{\mathrm{top}}(L)$ is denoted by $V_{K_n}\mathrm{K}_m^{\mathrm{top}}(L)$.

The structure of $\mathrm{K}_n^{\mathrm{top}}(K)$ is well-known (look at [12, 14]). In fact,

$$\mathrm{K}_n^{\mathrm{top}}(K) \xrightarrow{\sim} \mathbb{Z}_p \oplus V_{K_n}\mathrm{K}_n^{\mathrm{top}}(K) \tag{5.6}$$

where, as introduced above, $V_{K_n}\mathrm{K}_n^{\mathrm{top}}(K)$ is the image of $V_{K_n}\mathrm{K}_n^{\mathrm{Milnor}}(K)$ under $\mathrm{red}_{\Lambda_{\mathrm{K}_n^{\mathrm{Milnor}}(K)}}$. Now, introduce the subset $\mathbb{I}_{p,n}$ of $\mathbb{Z}^n$ by

$$\mathbb{I}_{p,n} = \{\boldsymbol{a} = (a_1,\cdots,a_n) \in \mathbb{Z}^n : \boldsymbol{a} \notin (p\mathbb{Z})^n,\ \mathbf{0} \prec \boldsymbol{a}\}.$$

For each $\boldsymbol{a} \in \mathbb{I}_{p,n}$, consider the integer $1 \leq i(\boldsymbol{a}) \leq n$ defined uniquely by the conditions :

- $a_{i(\boldsymbol{a})+1} \equiv \cdots \equiv a_n \equiv 0 \pmod p$;
- $a_{i(\boldsymbol{a})} \not\equiv 0 \pmod p$.

Let $\theta_1,\cdots,\theta_s$ be an $\mathbb{F}_p$-basis of the last residue field $K_0 = \mathbb{F}_q$, where $q = p^s$. Now, for each $\boldsymbol{a} \in \mathbb{I}_{p,n}$ and $1 \leq j \leq s$, introduce the topological Steinberg symbol $\varepsilon_{j,\boldsymbol{a}}$ in $\mathrm{K}_n^{\mathrm{top}}(K)$ by

$$\varepsilon_{j,\boldsymbol{a}} := \left\{1 + \theta_j t_K^{\boldsymbol{a}}, t_{1,K},\cdots,t_{i(\boldsymbol{a})-1,K}, t_{i(\boldsymbol{a})+1,K},\cdots,t_{n,K}\right\}^{\mathrm{top}},$$

where $(t_{1,K}, \cdots, t_{n,K}) \in K^n$ is a system of local parameters of the n-dimensional local field K, and $\underline{t}_K^{\boldsymbol{a}} := t_{1,K}^{a_1} \cdots t_{n,K}^{a_n}$. Then, the collection $\{\varepsilon_{j,\boldsymbol{a}}\}_{\substack{1\le j\le s\\ \boldsymbol{a}\in\mathbb{I}_{p,n}}}$ is a system of free topological generators of $V_{K_n}\mathrm{K}_n^{\mathrm{top}}(K)$. Therefore, any $\xi \in \mathrm{K}_n^{\mathrm{top}}(K)$ can be expressed uniquely as

$$\xi = A_o\{t_{1,K}, \cdots, t_{n,K}\}^{\mathrm{top}} + \sum_{\substack{1\le j\le s\\ \boldsymbol{b}\in\mathbb{I}_{p,n}}} A_{j,\boldsymbol{b}}\varepsilon_{j,\boldsymbol{b}},$$

where A_o, $A_{j,\boldsymbol{b}} \in \mathbb{Z}_p$ for every $1 \le j \le s$, $\boldsymbol{b} \in \mathbb{I}_{p,n}$.

For more details about topological Milnor K-groups, look at [12,14].

6. Ramification theory of n-dimensional local fields

If K is a non-archimedean (=1-dimensional) local field, then there exists a very solid theory, the ramification theory of the non-archimedean local field K [15]. Namely, for a finite Galois extension L/K with Galois group $\mathrm{Gal}(L/K) = G$, there exists a lower filtration $(G_i)_{i\in\mathbb{R}_{\ge -1}}$ of G defined by higher ramification subgroups $G_i := \{\gamma \in G \mid \nu_L(\gamma(x) - x) \ge i + 1,\ \forall x \in O_L\}$ of G in lower numbering, for $i \in \mathbb{R}_{\ge -1}$. The lower filtration $(G_i)_{i\in\mathbb{R}_{\ge -1}}$ of G behaves well with respect to "passing to the subgroups" in the sense that, for any subgroup H of G, $H_i = H \cap G_i$, for every $i \in \mathbb{R}_{\ge -1}$. On the other hand, the lower filtration $(G_i)_{i\in\mathbb{R}_{\ge -1}}$ of G does not behave well with respect to "taking quotients". That is, there exists H a normal subgroup of G such that $(G/H)_i \ne G_iH/H$ for some $i \in \mathbb{R}_{\ge -1}$. In fact, defining $G^j = G_{\psi_{L/K}(j)}$, for all $j \in \mathbb{R}_{\ge -1}$, where $\psi_{L/K} : \mathbb{R}_{\ge -1} \to \mathbb{R}_{\ge -1}$ the Hasse-Herbrand function of the extension L/K is the piecewise linear increasing function with inverse $\psi_{L/K}^{-1} = \phi_{L/K} : \mathbb{R}_{\ge -1} \to \mathbb{R}_{\ge -1}$ defined by $\phi_{L/K}(i) = \int_0^i \frac{dt}{[G_0:G_t]}$ for $i \in \mathbb{R}_{\ge -1}$ produces the upper filtration $(G^j)_{j\in\mathbb{R}_{\ge -1}}$ of G defined by higher ramification subgroups G^j of G in upper numbering, for $j \in \mathbb{R}_{\ge -1}$, which behaves well with respect to "taking quotients" now. That is, for any normal subgroup H of G, $(G/H)^j = G^jH/H$ for every $j \in \mathbb{R}_{\ge -1}$. Thus, higher ramification subgroups G^j of G in upper numbering, for $j \in \mathbb{R}_{\ge -1}$, can be used to define higher ramification subgroups G_K^j of the absolute Galois group G_K in upper numbering, for $j \in \mathbb{R}_{\ge -1}$.

If the finite extension L/K is furthermore assumed to be abelian, the most important property of the upper filtration $(G^j)_{j\in\mathbb{R}_{\ge -1}}$ of G is that the local abelian Hasse reciprocity law

$$\mathsf{Rec}_{L/K_*} : K^\times/\mathrm{N}_{L/K}(L^\times) \xrightarrow{\sim} G$$

of the abelian extension L/K maps the subgroup $U_K^j/(U_K^j \cap \mathrm{N}_{L/K}(L^\times))$ of $K^\times/\mathrm{N}_{L/K}(L^\times)$ to the higher ramification subgroup G^j of G in upper numbering for every $j \in \mathbb{R}_{\ge -1}$. Note that, both filtrations $(U_K^j)_{j\in\mathbb{R}_{\ge -1}}$ and $(G^j)_{j\in\mathbb{R}_{\ge -1}}$ form bases of neighbourhoods of $K^\times$ and of G respectively.

Therefore, in principle, we should be able to define an upper ramification theory on a "valued field" K in the situations where some class field theory for the valued field K is available. For instance, using this principle, Lomadze [35] initiated the ramification theory of abelian extensions of 2-dimensional local fields of characteristic $p > 0$ by defining an upper filtration on corresponding abelian Galois groups using local abelian 2-dimensional class field theory, which is the subject of Section 7. On the other hand, if K is an n-dimensional local field with $n \ge 2$, we observe that there are two different, yet not totally unrelated, valuations on K. Namely, there exists a rank n discrete valuation $\overline{v}_K : K \to \mathbb{Z}^n \cup \{\infty\}$ defined on K, and also a discrete valuation $\nu_{K_n} : K_n \to \mathbb{Z} \cup \{\infty\}$ defined on $K_n = K$. So, there are two "seemingly different" valued field structures on K. Therefore, it is natural to expect different types of ramification theories on K, which are:

- Zhukov type ramification theory on K [49,51], which generalizes [23,35];
- Abbes-Saito type ramification theory on K [1,49], which generalizes [19,24].

In what follows, we shall choose Abbes-Saito type ramification theory on the n-dimensional local field K. In fact, in Abbes-Saito theory on K, there are two filtrations $G^{\bullet}_{K,\mathrm{nlog}}$ and $G^{\bullet}_{K,\log}$ on the absolute Galois group G_K of K both indexed by the set of non-negative rational numbers $\mathbb{Q}_{\geq 0}$, called the upper non-logarithmic ramification filtration of G_K and the upper logarithmic ramification filtration of G_K, respectively [49, Subsection 6.1]. Moreover, specializing only to abelian extensions of K, Abbes-Saito non-logarithmic ramification theory of abelian extensions of K coincides with the ramification theory of Kato, which is defined only for abelian extensions of K [28] and which also behaves well with respect to the existing local abelian Kato-Parshin reciprocity law of K** [24], the main subject of this review.

The ramification theory of Kato on the n-dimensional local field K, which is modelled after the work of Hyodo [19], first constructs a conductor $\mathrm{KSw}(\chi)$ for $\chi \in H^1(K) = \mathrm{Hom}(G_K^{\mathrm{ab}}, \mathbb{Q}/\mathbb{Z})$, called the Kato-Swan conductor for a 1-dimensional representation $\chi : G_K^{\mathrm{ab}} \to \mathbb{Q}/\mathbb{Z}$ of G_K^{ab}, where K is a complete discrete valuation field with *any* residue field κ_K. The conductor $\mathrm{KSw}(\chi)$ for the 1-dimensional representation $\chi : G_K^{\mathrm{ab}} \to \mathbb{Q}/\mathbb{Z}$ of G_K^{ab} is characterized by the smallest integer $f \geq 0$ satisfying

$$U_K^{f+1} \subseteq N_{L_\chi/K} L_\chi^{\times},$$

where L_χ/K is the subextension of K^{ab}/K fixed by $\chi : G_K^{\mathrm{ab}} \to \mathbb{Q}/\mathbb{Z}$. So, there exists an upper filtration $G_K^{\mathrm{ab},\bullet}$ on G_K^{ab}, called the Kato filtration on G_K^{ab}, satisfying

$$\mathrm{KSw}(\chi) = \inf\{a > 0 \mid G_K^{\mathrm{ab},a} \subseteq \mathrm{Ker}(\chi)\},$$

for any $\chi : G_K^{\mathrm{ab}} \to \mathbb{Q}/\mathbb{Z}$.

7. Local abelian K-theoretic class field theory of Kato-Parshin

Fix a separable closure K^{sep} of the n-dimensional local field K and let $K^{\mathrm{ab}} \subset K^{\mathrm{sep}}$ be the maximal abelian extension of K inside K^{sep}.

The *profinite completion* $\widehat{\mathrm{K}}_n^{\mathrm{top}}(K)$ of $\mathrm{K}_n^{\mathrm{top}}(K)$ with respect to the norm map is defined by the projective limit

$$\widehat{\mathrm{K}}_n^{\mathrm{top}}(K) := \varprojlim_{E} \mathrm{K}_n^{\mathrm{top}}(K)/\mathrm{N}_{E/K}^{\mathrm{top}}(\mathrm{K}_n^{\mathrm{top}}(E)),$$

where E runs over all finite extensions of the n-dimensional local field K inside K^{ab}, with respect to the connecting morphisms

$$\mathrm{K}_n^{\mathrm{top}}(K)/\mathrm{N}_{E/K}^{\mathrm{top}}(\mathrm{K}_n^{\mathrm{top}}(E)) \xleftarrow{c_E^{E'}} \mathrm{K}_n^{\mathrm{top}}(K)/\mathrm{N}_{E'/K}^{\mathrm{top}}(\mathrm{K}_n^{\mathrm{top}}(E'))$$

defined for any two finite extensions E and E' of K inside K^{ab} satisfying $E \subseteq E'$ by

$$\alpha \pmod{\mathrm{N}_{E/K}^{\mathrm{top}}(\mathrm{K}_n^{\mathrm{top}}(E))} \xleftarrow{c_E^{E'}} \alpha \pmod{\mathrm{N}_{E'/K}^{\mathrm{top}}(\mathrm{K}_n^{\mathrm{top}}(E'))},$$

for every $\alpha \in \mathrm{K}_n^{\mathrm{top}}(K)$.

Given any finite extension L of K, then the homomorphism $\mathrm{N}_{L/K}^{\mathrm{top}} : \mathrm{K}_n^{\mathrm{top}}(L) \to \mathrm{K}_n^{\mathrm{top}}(K)$ extends to profinite completions, and defines a continuous homomorphism

$$\widehat{\mathrm{N}}_{L/K}^{\mathrm{top}} : \widehat{\mathrm{K}}_n^{\mathrm{top}}(L) \to \widehat{\mathrm{K}}_n^{\mathrm{top}}(K)$$

**So, it is natural to expect that Abbes-Saito type non-logarithmic ramification theory on the n-dimensional local field K behaves well with respect to the *"hypothetical"* local non-abelian Kato-Parshin reciprocity law of K, which still needs construction [20].

satisfying the transitivity condition, as the diagram

$$\begin{tikzcd}
\mathrm{K}_n^{\mathrm{top}}(L)/\mathrm{N}_{T/L}^{\mathrm{top}}(\mathrm{K}_n^{\mathrm{top}}(T)) \arrow[d, "\mathrm{N}_{L/K_*}^{\mathrm{top}}"'] & \mathrm{K}_n^{\mathrm{top}}(L)/\mathrm{N}_{T'/L}^{\mathrm{top}}(\mathrm{K}_n^{\mathrm{top}}(T')) \arrow[l, "c_T^{T'}"'] \arrow[d, "\mathrm{N}_{L/K_*}^{\mathrm{top}}"] \\
\mathrm{K}_n^{\mathrm{top}}(K)/\mathrm{N}_{T\cap K^{\mathrm{ab}}/K}^{\mathrm{top}}(\mathrm{K}_n^{\mathrm{top}}(T\cap K^{\mathrm{ab}})) & \mathrm{K}_n^{\mathrm{top}}(K)/\mathrm{N}_{T'\cap K^{\mathrm{ab}}/K}^{\mathrm{top}}(\mathrm{K}_n^{\mathrm{top}}(T'\cap K^{\mathrm{ab}})) \arrow[l, "c_{T\cap K^{\mathrm{ab}}}^{T'\cap K^{\mathrm{ab}}}"]
\end{tikzcd}$$

is commutative, where the vertical arrows $\mathrm{N}_{L/K_*}^{\mathrm{top}}$ are the induced morphisms from $\mathrm{N}_{L/K}^{\mathrm{top}}$, for each finite extension T and T' of L inside L^{ab} satisfying $T \subseteq T'$.

Recall that, local abelian n-dimensional K-theoretic class field theory for K establishes a unique natural algebraic and topological [38] isomorphism

$$\mathsf{Rec}_K : \widehat{\mathrm{K}}_n^{\mathrm{top}}(K) \xrightarrow{\sim} G_K^{\mathrm{ab}},$$

called the *local abelian n-dimensional Kato-Parshin reciprocity law of K*, which, among other things, has the following properties :

(1) For every abelian extension L/K, the surjective homomorphism

$$\mathsf{Rec}_{L/K} : \widehat{\mathrm{K}}_n^{\mathrm{top}}(K) \xrightarrow{\mathsf{Rec}_K} G_K^{\mathrm{ab}} \xrightarrow{\mathrm{res}_L} \mathrm{Gal}(L/K)$$

has kernel

$$\mathrm{Ker}(\mathsf{Rec}_{L/K}) = \widehat{\mathrm{N}}_{L/K}^{\mathrm{top}}(\widehat{\mathrm{K}}_n^{\mathrm{top}}(L)) = \bigcap_{K \underset{\text{finite}}{\subseteq} F \subset L} \widehat{\mathrm{N}}_{F/K}^{\mathrm{top}}(\widehat{\mathrm{K}}_n^{\mathrm{top}}(F)) =: \mathfrak{N}_{L/K}^{\mathrm{top}},$$

and induces a topological group isomorphism

$$\mathsf{Rec}_{L/K_*} : \widehat{\mathrm{K}}_n^{\mathrm{top}}(K)/\mathfrak{N}_{L/K}^{\mathrm{top}} \xrightarrow{\sim} \mathrm{Gal}(L/K)$$

called the *local abelian n-dimensional Kato-Parshin reciprocity law of L/K*;

(2) (Existence theorem). For each abelian extension L/K, the mapping

$$L/K \mapsto \mathfrak{N}_{L/K}^{\mathrm{top}}$$

defines a bijective correspondence

$$\{L/K : \mathrm{abelian}\} \rightleftarrows \{\mathfrak{N} : \mathfrak{N} \underset{\text{“closed”}}{\leq} \widehat{\mathrm{K}}_n^{\mathrm{top}}(K)\}.$$

For Kato's approach to the existence theorem, look at [25];

(3) (Functoriality). For any finite extension L/K,

$$\mathsf{Rec}_L(x) \mid_{K^{\mathrm{ab}}} = \mathsf{Rec}_K\left(\widehat{\mathrm{N}}_{L/K}^{\mathrm{top}}(x)\right),$$

for every $x \in \widehat{\mathrm{K}}_n^{\mathrm{top}}(L)$, and

$$\mathsf{Rec}_L\left(\widehat{j_{L/K}^{\mathrm{top}}}(x)\right) = V_{L/K}\left(\mathsf{Rec}_K(x)\right),$$

for every $x \in \widehat{\mathrm{K}}_n^{\mathrm{top}}(K)$. That is, the following squares

$$\begin{tikzcd}
\widehat{\mathrm{K}}_n^{\mathrm{top}}(L) \arrow[r, "\mathsf{Rec}_L"] \arrow[d, "\widehat{\mathrm{N}}_{L/K}^{\mathrm{top}}"'] & G_L^{\mathrm{ab}} \arrow[d, "\mathrm{res}_{K^{\mathrm{ab}}}"] \\
\widehat{\mathrm{K}}_n^{\mathrm{top}}(K) \arrow[r, "\mathsf{Rec}_K"] & G_K^{\mathrm{ab}}
\end{tikzcd}
\qquad
\begin{tikzcd}
\widehat{\mathrm{K}}_n^{\mathrm{top}}(L) \arrow[r, "\mathsf{Rec}_L"] & G_L^{\mathrm{ab}} \\
\widehat{\mathrm{K}}_n^{\mathrm{top}}(K) \arrow[r, "\mathsf{Rec}_K"] \arrow[u, "\widehat{j_{L/K}^{\mathrm{top}}}"] & G_K^{\mathrm{ab}} \arrow[u, "V_{L/K}:\mathrm{Verlagerung}"']
\end{tikzcd}$$

are commutative;

(4) The square

$$\begin{tikzcd} \widehat{\mathrm{K}}_n^{\mathrm{top}}(K) \arrow[r, "\mathsf{Rec}_K"] \arrow[d, "\widehat{(\partial_{n-1}^n)}^{\mathrm{top}}"'] & G_K^{\mathrm{ab}} \arrow[d] \\ \widehat{\mathrm{K}}_{n-1}^{\mathrm{top}}(K_{n-1}) \arrow[r, "\mathsf{Rec}_{K_{n-1}}"] & G_{K_{n-1}}^{\mathrm{ab}} \end{tikzcd}$$

is commutative, where the left-vertical arrow

$$\widehat{(\partial_{n-1}^n)}^{\mathrm{top}} : \widehat{\mathrm{K}}_n^{\mathrm{top}}(K) \to \widehat{\mathrm{K}}_{n-1}^{\mathrm{top}}(K_{n-1})$$

is defined by the commutativity of the diagram

$$\begin{tikzcd} \mathrm{K}_n^{\mathrm{top}}(K)/\mathrm{N}_{E'/K}^{\mathrm{top}}(\mathrm{K}_n^{\mathrm{top}}(E')) \arrow[r, "(\partial_{n-1}^n)_*^{\mathrm{top}}"] \arrow[d, "c_E^{E'}"'] & \mathrm{K}_{n-1}^{\mathrm{top}}(K_{n-1})/\mathrm{N}_{E'_{n-1}/K_{n-1}}^{\mathrm{top}}(\mathrm{K}_n^{\mathrm{top}}(E'_{n-1})) \arrow[d, "c_{E_{n-1}}^{E'_{n-1}}"] \\ \mathrm{K}_n^{\mathrm{top}}(K)/\mathrm{N}_{E/K}^{\mathrm{top}}(\mathrm{K}_n^{\mathrm{top}}(E)) \arrow[r, "(\partial_{n-1}^n)_*^{\mathrm{top}}"] & \mathrm{K}_{n-1}^{\mathrm{top}}(K_{n-1})/\mathrm{N}_{E_{n-1}/K_{n-1}}^{\mathrm{top}}(\mathrm{K}_n^{\mathrm{top}}(E_{n-1})) \end{tikzcd}$$

where E and E' are finite extensions of K inside K^{ab} satisfying $E \subsetneq E'$;

(5) (Ramification theoretic properties) Let $\chi \in H^1(K)$; that is, let $\chi : G_K^{\mathrm{ab}} \to \mathbb{Q}/\mathbb{Z}$ be a character of G_K^{ab}, and let L_χ be the finite extension of K in K^{ab} such that $\mathrm{Ker}(\chi) = \mathrm{Gal}(K^{\mathrm{ab}}/L_\chi)$. The Kato-Swan conductor $\mathrm{KSw}(\chi)$ of the character $\chi : G_K^{\mathrm{ab}} \to \mathbb{Q}/\mathbb{Z}$, defined in Section 6 as the smallest integer $f \geq 0$ satisfying $U_K^{f+1} \subseteq N_{L_\chi/K} L_\chi^\times$, is furthermore the smallest integer $f \geq 0$ such that

$$U_{\overline{v}_K}^{(i_1,\cdots,i_n)} \mathrm{K}_n^{\mathrm{top}}(K) \subseteq \mathfrak{N}_{L_\chi/K}^{\mathrm{top}},$$

whenever $i_n > f$.

For details about local abelian K-theoretic class field theory, look at [7–9], [22], and [42, 44, 45].

There are four main approaches to construct the local abelian n-dimensional K-theoretic class field theory:

- The explicit approach of Fesenko [7–9] is based on extending the local abelian Hasse reciprocity law construction of Neukirch-Iwasawa [39, 40] and on extending the local norm residue symbol construction of Hazewinkel [17] to the setting of n-dimensional local fields;
- Kato's approach [22, 25] is cohomological and extends Tate's construction of the local abelian Hasse reciprocity law [48];
- Koya on the other hand [30–32], using Lichtenbaum's complexes $\mathbb{Z}(i)$ [34], generalizes class formation approach of local abelian class field theory to construct the local abelian 2-dimensional class field theory, which is extended and streamlined by Spiess [47] to the n-dimensional setting;
- The final approach, due to Parshin [42, 44, 45], which is the genesis of the whole program, generalizes Kawada-Satake construction of local abelian class field theory [29] to construct the local abelian n-dimensional class field theory in positive characteristic.

In this work we shall review Fesenko's explicit approach, where as stated above, the idea is to generalize the classical Neukirch-Iwasawa and Hazewinkel methods to higher-dimensional local fields, which will be recalled next with extra care following closely [10, 11, 13]. The explicit approach also has the advantage of extending local abelian n-dimensional

K-theoretic class field theory to the non-abelian setting; namely, constructing the local non-abelian n-dimensional K-theoretic class field theory [20].

As first recollection, Fesenko's extension of Neukirch-Iwasawa method to n-dimensional local fields can be very briefly summarized as follows: Let L denote a finite Galois extension of the n-dimensional local field K in a fixed K^{sep}. As usual, let $L^{\mathrm{pur}} = LK^{\mathrm{pur}}$. For any $\sigma \in \mathrm{Gal}(L/K)$, let $\widetilde{\sigma}$ be any element of $\mathrm{Gal}(L^{\mathrm{pur}}/K)$ such that:

- $\widetilde{\sigma}\mid_L = \sigma$;
- $\widetilde{\sigma}\mid_{K^{\mathrm{pur}}} = \varphi_K^i$ for some $i \in \mathbb{Z}$.

The n-dimensional Neukirch-Iwasawa map

$$\mathscr{N}_{L/K} : \mathrm{Gal}(L/K) \to \mathrm{K}_n^{\mathrm{top}}(K)/\mathrm{N}_{L/K}^{\mathrm{top}}(\mathrm{K}_n^{\mathrm{top}}(L))$$

of L/K is then defined by

$$\mathscr{N}_{L/K} : \sigma \mapsto \mathrm{N}_{\Sigma/K}^{\mathrm{top}}(\Pi_{\mathrm{K}_n^{\mathrm{top}}(\Sigma)}) \pmod{\mathrm{N}_{L/K}^{\mathrm{top}}(\mathrm{K}_n^{\mathrm{top}}(L))},$$

where Σ denotes the fixed field of $\widetilde{\sigma}$ and $\Pi_{\mathrm{K}_n^{\mathrm{top}}(\Sigma)}$ any prime element of $\mathrm{K}_n^{\mathrm{top}}(\Sigma)$. This map does not depend on the choice of lifting $\widetilde{\sigma}$ of σ to L^{pur} and to the choice of prime element $\Pi_{\mathrm{K}_n^{\mathrm{top}}(\Sigma)}$ of $\mathrm{K}_n^{\mathrm{top}}(\Sigma)$. Moreover, the n-dimensional Neukirch-Iwasawa map $\mathscr{N}_{L/K}$: $\mathrm{Gal}(L/K) \to \mathrm{K}_n^{\mathrm{top}}(K)/\mathrm{N}_{L/K}^{\mathrm{top}}(\mathrm{K}_n^{\mathrm{top}}(L))$ of L/K induces a topological group homomorphism

$$\mathscr{N}_{L/K}^{\mathrm{ab}} : \mathrm{Gal}(L/K)^{\mathrm{ab}} \to \mathrm{K}_n^{\mathrm{top}}(K)/\mathrm{N}_{L/K}^{\mathrm{top}}(\mathrm{K}_n^{\mathrm{top}}(L)),$$

which is actually the inverse of the local abelian n-dimensional reciprocity law of L/K.

As second recollection, Fesenko's generalization of Hazewinkel's method to n-dimensional local fields can be sketched as follows: First assume that the n-dimensional local field K is of positive characteristic, which is the easier case, as the Galois descent for $\mathrm{K}_*^{\mathrm{top}}$-groups holds. Let L denote a finite Galois extension of K in a fixed K^{sep}, and let K^{pur} denote the maximal purely unramified extension of K in K^{sep}. Assume further that L/K is linearly disjoint with K^{pur}/K; that is, the extension L/K is totally ramified by (2.1). Recall that, the n^{th} topological Milnor K-group $\mathrm{K}_n^{\mathrm{top}}(K^{\mathrm{pur}})$ of K^{pur} is defined by the direct limit

$$\mathrm{K}_n^{\mathrm{top}}(K^{\mathrm{pur}}) = \varinjlim_{K'} \mathrm{K}_n^{\mathrm{top}}(K'),$$

where K' runs over all finite extensions of K in K^{pur}, with respect to the connecting morphisms

$$j_{K''/K'}^{\mathrm{top}} : \mathrm{K}_n^{\mathrm{top}}(K') \to \mathrm{K}_n^{\mathrm{top}}(K'')$$

defined for any two finite extensions K' and K'' of K inside K^{pur} satisfying $K' \subseteq K''$. Introduce the group $\mathrm{K}_n^{\mathrm{top}}(L^{\mathrm{pur}})$ similary and define a subgroup $V(L/K)$ of $\mathrm{K}_n^{\mathrm{top}}(L^{\mathrm{pur}})$ by

$$V(L/K) = \left\langle \sigma(\alpha) - \alpha \mid \sigma \in \mathrm{Gal}(L^{\mathrm{pur}}/K^{\mathrm{pur}}), \alpha \in V_{K_n}\mathrm{K}_n^{\mathrm{top}}(L^{\mathrm{pur}}) \right\rangle.$$

Then $V(L/K) \subseteq \mathrm{Ker}(\mathrm{N}_{L^{\mathrm{pur}}/K^{\mathrm{pur}}}^{\mathrm{top}})$ and the norm map $\mathrm{N}_{L^{\mathrm{pur}}/K^{\mathrm{pur}}}^{\mathrm{top}} : \mathrm{K}_n^{\mathrm{top}}(L^{\mathrm{pur}}) \to \mathrm{K}_n^{\mathrm{top}}(K^{\mathrm{pur}})$, which is surjective, induces a morphism

$$\mathrm{N}_{L^{\mathrm{pur}}/K^{\mathrm{pur}}\,*}^{\mathrm{top}} : \mathrm{K}_n^{\mathrm{top}}(L^{\mathrm{pur}})/V(L/K) \to \mathrm{K}_n^{\mathrm{top}}(K^{\mathrm{pur}})$$

sitting in the short exact sequence

$$1 \to \mathrm{Gal}(L^{\mathrm{pur}}/K^{\mathrm{pur}}) \xrightarrow{c} \mathrm{K}_n^{\mathrm{top}}(L^{\mathrm{pur}})/V(L/K) \xrightarrow{\mathrm{N}_{L^{\mathrm{pur}}/K^{\mathrm{pur}}\,*}^{\mathrm{top}}} \mathrm{K}_n^{\mathrm{top}}(K^{\mathrm{pur}}) \to 0, \tag{7.1}$$

where the arrow

$$c : \mathrm{Gal}(L^{\mathrm{pur}}/K^{\mathrm{pur}}) \to \mathrm{K}_n^{\mathrm{top}}(L^{\mathrm{pur}})/V(L/K)$$

is defined by

$$c(\sigma) = \sigma(\Pi_{\mathrm{K}_n^{\mathrm{top}}(L^{\mathrm{pur}})}) - \Pi_{\mathrm{K}_n^{\mathrm{top}}(L^{\mathrm{pur}})} \pmod{V(L/K)},$$

for every $\sigma \in \mathrm{Gal}(L^{\mathrm{pur}}/K^{\mathrm{pur}})$, which is independent of the choice of $\Pi_{\mathrm{K}_n^{\mathrm{top}}(L^{\mathrm{pur}})}$. Now, for $\varepsilon \in \mathrm{Ker}(\nu_{\mathrm{K}_n^{\mathrm{top}}(K)})$ there exists $\eta_\varepsilon \in \mathrm{K}_n^{\mathrm{top}}(L^{\mathrm{pur}})$ such that $\varepsilon = \mathrm{N}^{\mathrm{top}}_{L^{\mathrm{pur}}/K^{\mathrm{pur}}}(\eta_\varepsilon)$. Let $\varphi : L^{\mathrm{pur}} \to L^{\mathrm{pur}}$ denote a lifting of the Frobenius automorphism $\varphi_K : K^{\mathrm{pur}} \to K^{\mathrm{pur}}$ of K^{pur} to L^{pur}. Then, $\varphi(\eta_\varepsilon) - \eta_\varepsilon \pmod{V(L/K)} \in \mathrm{Ker}({\mathrm{N}^{\mathrm{top}}_{L^{\mathrm{pur}}/K^{\mathrm{pur}}}}_*)$ and as the sequence (7.1) is exact, there exists $\widetilde{\sigma}_\varepsilon \in \mathrm{Gal}(L^{\mathrm{pur}}/K^{\mathrm{pur}})$ so that

$$c(\widetilde{\sigma}_\varepsilon) = \widetilde{\sigma}_\varepsilon(\Pi_{\mathrm{K}_n^{\mathrm{top}}(L^{\mathrm{pur}})}) - \Pi_{\mathrm{K}_n^{\mathrm{top}}(L^{\mathrm{pur}})} \pmod{V(L/K)} = \varphi(\eta_\varepsilon) - \eta_\varepsilon \pmod{V(L/K)}.$$

Then, there exists a unique and well-defined continuous homomorphism

$$\mathscr{H}_{L/K} : \mathrm{K}_n^{\mathrm{top}}(K)/\mathrm{N}^{\mathrm{top}}_{L/K}(\mathrm{K}_n^{\mathrm{top}}(L)) \to \mathrm{Gal}(L/K)^{\mathrm{ab}}$$

satisfying

$$\mathscr{H}_{L/K} : \varepsilon \pmod{\mathrm{N}^{\mathrm{top}}_{L/K}(\mathrm{K}_n^{\mathrm{top}}(L))} \mapsto \widetilde{\sigma}_\varepsilon^{-1} \mid_{L\cap K^{\mathrm{ab}}},$$

for all $\varepsilon \in \mathrm{Ker}(\nu_{\mathrm{K}_n^{\mathrm{top}}(K)})$, called the n-dimensional Hazewinkel map of L/K, where L/K is a finite Galois extension linearly disjoint with K^{pur}/K.

Let L/K denote a finite Galois extension which is linearly disjoint with K^{pur}/K, where $\mathrm{char}(K) > 0$. It turns out that, the n-dimensional Neukirch-Iwasawa map of L/K and the n-dimensional Hazewinkel map of L/K are inverses of each other; that is,

$$\mathscr{H}_{L/K} \circ \mathscr{N}^{\mathrm{ab}}_{L/K} = \mathrm{Id}_{\mathrm{Gal}(L/K)^{\mathrm{ab}}} \text{ and } \mathscr{N}^{\mathrm{ab}}_{L/K} \circ \mathscr{H}_{L/K} = \mathrm{Id}_{\mathrm{K}_n^{\mathrm{top}}(K)/\mathrm{N}^{\mathrm{top}}_{L/K}(\mathrm{K}_n^{\mathrm{top}}(L))}.$$

In case $\mathrm{char}(K) = 0$, unfortunately the construction sketched for the positive characteristic case does not work for p-extensions L over K in general. However, there is a method to overcome this difficulty. In fact, there is a special class of p-extensions L over K, called strong Artin-Schreier trees [10,11,13], where the construction outlined for char. > 0 works perfectly well. In fact, we have the short exact sequence (7.1) for strong Artin-Schreier trees. That is, if L/K is a strong Artin-Schreier tree, then the following sequence

$$1 \to \mathrm{Gal}(L/K) \xrightarrow{c} V_{K_n}\mathrm{K}_n^{\mathrm{top}}(L^{\mathrm{pur}})/V(L/K) \xrightarrow{{\mathrm{N}^{\mathrm{top}}_{L^{\mathrm{pur}}/K^{\mathrm{pur}}}}_*} V_{K_n}\mathrm{K}_n^{\mathrm{top}}(K^{\mathrm{pur}}) \to 0, \tag{7.2}$$

is exact. Therefore, for a finite strong Artin-Schreier tree L/K linearly disjoint with K^{pur}/K; that is the extension L/K is totally ramified by (2.1), there exists a unique and well-defined continuous homomorphism

$$\mathscr{H}_{L/K} : V_{K_n}\mathrm{K}_n^{\mathrm{top}}(K)/\mathrm{N}^{\mathrm{top}}_{L/K}(V_{K_n}\mathrm{K}_n^{\mathrm{top}}(L)) \to \mathrm{Gal}(L/K)^{\mathrm{ab}},$$

the n-dimensional Hazewinkel map of L/K, constructed as in the char. > 0 case, which further satisfies

$$\mathscr{H}_{L/K} \circ \mathscr{N}^{\mathrm{ab}}_{L/K} = \mathrm{Id}_{\mathrm{Gal}(L/K)^{\mathrm{ab}}}. \tag{7.3}$$

Therefore, if L/K is a finite strong Artin-Schreier tree linearly disjoint with K^{pur}/K, then the continuous homomorphism

$$\mathscr{H}_{L/K} : V_{K_n}\mathrm{K}_n^{\mathrm{top}}(K)/\mathrm{N}^{\mathrm{top}}_{L/K}(V_{K_n}\mathrm{K}_n^{\mathrm{top}}(L)) \to \mathrm{Gal}(L/K)^{\mathrm{ab}}$$

is a surjection, and the continuous homomorphism

$$\mathscr{N}^{\mathrm{ab}}_{L/K} : \mathrm{Gal}(L/K)^{\mathrm{ab}} \to \mathrm{K}_n^{\mathrm{top}}(K)/\mathrm{N}^{\mathrm{top}}_{L/K}(\mathrm{K}_n^{\mathrm{top}}(L))$$

is an injection. Now, the class of all strong Artin-Schreier trees over K is "dense" in the class of all p-extensions of K in the sense that, for any totally ramified finite Galois p-extension L/K, there exists a totally ramified finite p-extension Q_L/K such that LQ_L/Q_L is a strong Artin-Schreier tree and $L^{\mathrm{pur}} \cap Q_L^{\mathrm{pur}} = K^{\mathrm{pur}}$. So let L/K be a totally ramified finite Galois p-extension. Then, $L^{\mathrm{pur}} \cap Q_L^{\mathrm{pur}} = K^{\mathrm{pur}}$ implies that $L \cap Q_L = K$, so the Galois extension L/K and the p-extension Q_L/K are linearly disjoint. Therefore, the restriction map $\mathrm{Res}_L^{LQ_L} : \mathrm{Gal}(LQ_L/Q_K) \xrightarrow{\sim} \mathrm{Gal}(L/K)$ is an isomorphism of profinite groups, and the following square

$$\begin{tikzcd}
\mathrm{Gal}(LQ_L/Q_L)^{\mathrm{ab}} \arrow[r, "\mathscr{N}^{\mathrm{ab}}_{LQ_L/Q_L}"] \arrow[d, "\mathrm{Res}_L^{LQ_L}"', "\wr"] & \mathrm{K}_n^{\mathrm{top}}(Q_L)/\mathrm{N}^{\mathrm{top}}_{LQ_L/Q_L}(\mathrm{K}_n^{\mathrm{top}}(LQ_L)) \arrow[d, "\mathrm{N}^{\mathrm{top}}_{Q_L/K}"] \\
\mathrm{Gal}(L/K)^{\mathrm{ab}} \arrow[r, "\mathscr{N}^{\mathrm{ab}}_{L/K}"'] & \mathrm{K}_n^{\mathrm{top}}(K)/\mathrm{N}^{\mathrm{top}}_{L/K}(\mathrm{K}_n^{\mathrm{top}}(L))
\end{tikzcd}$$

is commutative. Therefore,

$$\mathscr{N}^{\mathrm{ab}}_{L/K} : \mathrm{Gal}(L/K)^{\mathrm{ab}} \to \mathrm{K}_n^{\mathrm{top}}(K)/\mathrm{N}^{\mathrm{top}}_{L/K}(\mathrm{K}_n^{\mathrm{top}}(L))$$

is an injective homomorphism of topological groups, since the Neukirch-Iwasawa map

$$\mathscr{N}^{\mathrm{ab}}_{LQ_L/Q_L} : \mathrm{Gal}(LQ_L/Q_L)^{\mathrm{ab}} \to \mathrm{K}_n^{\mathrm{top}}(Q_L)/\mathrm{N}^{\mathrm{top}}_{LQ_L/Q_L}(\mathrm{K}_n^{\mathrm{top}}(LQ_L))$$

of LQ_L/Q_L is an injective arrow by equality (7.3) as LQ_L/Q_L is a finite strong Artin-Schreier tree linearly disjoint with Q_L^{pur}/Q_L ††. The surjectivity of the n-dimensional Neukirch-Iwasawa map

$$\mathscr{N}^{\mathrm{ab}}_{L/K} : \mathrm{Gal}(L/K)^{\mathrm{ab}} \to \mathrm{K}_n^{\mathrm{top}}(K)/\mathrm{N}^{\mathrm{top}}_{L/K}(\mathrm{K}_n^{\mathrm{top}}(L))$$

of L/K follows via induction on the degree $[L:K]$.

Now, for a finite Galois p-extension L/K which is linearly disjoint with K^{pur}/K, where $\mathrm{char}(K) = 0$, the n-dimensional Hazewinkel map

$$\mathscr{H}_{L/K} : \mathrm{K}_n^{\mathrm{top}}(K)/\mathrm{N}^{\mathrm{top}}_{L/K}(\mathrm{K}_n^{\mathrm{top}}(L)) \to \mathrm{Gal}(L/K)^{\mathrm{ab}},$$

of L/K is then defined as the inverse of the n-dimensional Neukirch-Iwasawa map

$$\mathscr{N}^{\mathrm{ab}}_{L/K} : \mathrm{Gal}(L/K)^{\mathrm{ab}} \to \mathrm{K}_n^{\mathrm{top}}(K)/\mathrm{N}^{\mathrm{top}}_{L/K}(\mathrm{K}_n^{\mathrm{top}}(L))$$

of L/K.

This completes the review of Fesenko's constructive local abelian higher-dimensional class field theory following [10, 11, 13].

††It suffices to prove that $LQ_L \cap Q_L^{\mathrm{pur}} = LQ_L \cap K^{\mathrm{pur}}Q_L = Q_L$. Let $q \in Q_L$ be a primitive element over K; namely, let $Q_L = K(q)$. Let $B = \{1, q, \cdots, q^{s-1}\}$ be a basis of the K-vector space Q_L. As $L \cap Q_L = K$, the extensions Q_L/K and L/K are linearly disjoint. Therefore, B is a basis of the L-vector space LQ_L. Likewise, B is a basis of the K^{pur}-vector space $K^{\mathrm{pur}}Q_L = Q_L^{\mathrm{pur}}$ since the extension Q_L/K is totally ramified. Now, let $a \in LQ_L \cap K^{\mathrm{pur}}Q_L$. Then there exists unique $\lambda_0, \cdots, \lambda_{s-1} \in L$ and there exists unique $\kappa_0, \cdots, \kappa_{s-1} \in K^{\mathrm{pur}}$ such that

$$a = \lambda_0 + \lambda_1 q + \cdots + \lambda_{s-1}q^{s-1} = \kappa_0 + \kappa_1 q + \cdots + \kappa_{s-1}q^{s-1}.$$

Therefore,

$$(\lambda_0 - \kappa_0) + (\lambda_1 - \kappa_1)q + \cdots + (\lambda_{s-1} - \kappa_{s-1})q^{s-1} = 0.$$

Now, as Q_L/K and K^{pur}/K are linearly disjoint, it follows that $L^{\mathrm{pur}} \cap Q_L^{\mathrm{pur}} = K^{\mathrm{pur}} \Rightarrow L^{\mathrm{pur}} \cap Q_L = K$. Thus, L^{pur}/K and Q_L/K are linearly disjoint, which implies that the K-basis B of Q_L is also an L^{pur}-basis of $(LQ_L)^{\mathrm{pur}} = K^{\mathrm{pur}}LQ_L$. Therefore,

$$\lambda_0 - \kappa_0 = \cdots = \lambda_{s-1} - \kappa_{s-1} = 0 \Rightarrow \lambda_0 = \kappa_0; \cdots; \lambda_{s-1} = \kappa_{s-1}.$$

The extension L/K is totally ramified. Therefore, $\lambda_0, \cdots, \lambda_{s-1} \in K$ and $a = \lambda_0 + \lambda_1 q + \cdots + \lambda_{s-1}q^{s-1} \in Q_L$, which completes the proof.

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