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Asymptotics of Eigenvalues of the Matrix Diffusion Operators

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Abstract: In this paper, matrix diffusion equations with boundary conditions and jump conditions on $[0,\pi]\setminus\{a\}$ are considered. Under these conditions, the asymptotic of the eigenvalues of the matrix diffusion operator is obtained, while the Rouche theorem and the Gaussian elimination method are used.

Keywords: Matrix diffusion operator, eigenvalue asymptotics, spectral data.

1. Introduction

In this paper, we purpose the diffusion equation

$$-Y'' + R(x)\,Y = \lambda^2 Y \quad x \in [0,\pi]\setminus\{a\} \tag{1}$$

with the boundary conditions

$$Y'(0) = \theta\,,\ Y(\pi) = \theta \tag{2}$$

and the jump condition

$$Y(a+0) = \alpha Y(a-0)\,,\ Y'(a+0) = \alpha^{-1} Y'(a-0) \tag{3}$$

where $Y = (y_1, y_2, ... y_m)^T$ is an m-dimensional vector function, λ is the spectral parameter, I is the $m\times m$ unit matrix. Moreover $\alpha > 0$, $\alpha \neq 1$ and $a \in (0,\pi)$. The potential matrix functions $P(x) = [p_{ij}]_{i,j=\overline{1,m}}$, $Q(x) = [q_{ij}]_{i,j=\overline{1,m}}$ and $R(x) = [r_{ij}]_{i,j=\overline{1,m}} = [2\lambda p_{ij} + q_{ij}]_{i,j=\overline{1,m}}$ are $m\times m$ matrices with entires $P(x) \in W_1^1[0,\pi]$, $Q(x) \in W_1^0[0,\pi]$. K is an orthogonal projector, $K \in \mathrm{C}^{m\times m}$, $K^{\perp} = I_m - K$. $L = L^{\dagger} \in \mathrm{C}^{m\times m}$, $L = KLK$. The space C^m is the m-vectors space. The space $\mathrm{C}^{m\times m}$ is the space of $m\times m$ matrices. If $A = [a_{jk}]$, $A^{\dagger} = [a_{kj}]$, namely, the symbol † denotes as the conjugate transpose.

We consider on spectral theory of matrix diffusion equation of the form

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$$-Y'' + (2\lambda P(x) + Q(x))Y = \lambda^2 Y$$

where $P(x) = [p_{ij}]_{i,j=\overline{1,m}}$, $Q(x) = [q_{ij}]_{i,j=\overline{1,m}}$ and $R(x) = [r_{ij}]_{i,j=\overline{1,m}} = [2\lambda p_{ij} + q_{ij}]_{i,j=\overline{1,m}}$ are $m \times m$ matrix function. Differential operators are defined as singular and regular. Titchmarsh studied spectral theory of second order singular differential operators in [1]. In 1984, the studies on the spectral theory of singular differential operators were conducted by [3], differential operators whose coefficients depend on spectral parameters; used in applications of mathematics, physics and engineering. The fundamental studies on the spectral theory of the Sturm-Liouville equations were performed in [1–11]. In particular, on the spectral theory for matrix Sturm-Liouville operators is used in the major part of the literature. Generally, Drichlet or Robin boundary conditions have been considered, since they are the simplest ones. In [11], they proved that uniqueness theorems for inverse problems of scalar quadratic pencil of the Sturm-Liouville operators. In [5], Shen and Shieh studied the multiplicity of eigenvalues of the m-dimensional vectorial SL problem

$$-y'' + Q(x)y = \lambda y\ ,\ y(0) = y(1) = \theta$$

where Q is continuous $m \times m$ Jacobi matrix-valued function defined on $0 \le x \le 1$.

In [4, 5], asymptotic formulas have been obtained for the eigenvalues of the Matrix Sturm-Liouville operators. In [3, 6, 11], inverse spectral problems have been obtained using the eigenvalues.

Matrix Sturm-Liouville operators are used frequently in many fields of engineering or physics. For example, heat conduction and reaction-diffusion systems. In [6], such operators are used in elastic theory. In [7, 8], authors studied for electromagnetic waves and nuclear structure. However, many physical systems that describe important problems change their states abruptly, have discontinuous orbits. These operators are similarly used in metric and quantum graphs [9, 10].

In this study, matrix diffusion equations with boundary conditions and jump conditions on $[0, \pi]$ are considered. Under these conditions, the asymptotic of the eigenvalues of the matrix diffusion operator is obtained, while the Rouche theorem and the Gaussian elimination method are used.

2. Main Results

We will examine the eigenfunctions corresponding to the $Y(x)$ solutions of the L problem. Let us give the following expressions in order to obtain the asymptotic of eigenvalues. We suppose that if

$$K = \begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix}, \qquad K^{\perp} = \begin{pmatrix} 0 & 0 \\ 0 & I_{m-k} \end{pmatrix} \tag{4}$$

then $k = rank(K)$ $(1 \le k \le m-1)$ can be defined. Thus, $rank(K^{\perp}) = m - k$.

Here let's define the transformation $D = \left(D^{\dagger}\right)^{-1}$ as follows:

$$\tilde{K} = D^{\dagger} K D, \qquad \tilde{K}^{\perp} = D^{\dagger} K^{\perp} D, \qquad \tilde{R}(x) = D^{\dagger} R(x) D, \qquad \tilde{Y}(x) = D^{\dagger} Y(x) D. \tag{5}$$

Denote

$$\omega = \frac{1}{2} \int_0^{\pi} R(x)\, dx = \begin{pmatrix} \omega_{11} & \omega_{12} \\ \omega_{21} & \omega_{22} \end{pmatrix}$$

where $\omega_{11} \in C^{k \times k}$, $\omega_{22} \in C^{(m-k)\times(m-k)}$. Obviously, this matrix is Hermitian: $\omega_{11} = \omega_{11}^{\dagger}$, $\omega_{22} = \omega_{22}^{\dagger}$.

Let $\varphi(x,\lambda)$ and $\psi(x,\lambda)$ be the solutions of equation (1) that satisfy the boundary condition (2) and jump condition (3) and conditions $\varphi(0,\lambda) = 0$, $\varphi'(0,\lambda) = I_m$, $\psi(0,\lambda) = I_m$, $\psi'(0,\lambda) = 0$.

If $x \in (0,a)$,

$$\varphi(x,\lambda) = \cos \lambda x I_m + \frac{1}{\lambda} \int_0^{x} \sin \lambda (x-t) R(t) \varphi(t,\lambda)\, dt,$$

if $x \in (a,\pi)$,

$$\begin{array}{l} \varphi(x,\lambda) = \alpha^{+} e^{i\lambda x} I_m + \alpha^{-} e^{i\lambda(2a-x)} I_m \\ +\alpha^{+} \int_0^a \frac{\sin \lambda(x-t)}{\lambda} R(t) \varphi(t,\lambda)\, dt + \alpha^{-} \int_0^a \frac{\sin \lambda(x+t-2a)}{\lambda} R(t) \varphi(t,\lambda)\, dt \\ + \int_0^x \frac{\sin \lambda(x-t)}{\lambda} R(t) \varphi(t,\lambda)\, dt \end{array}$$

where $\alpha^{\pm}(x) = \frac{1}{2}\left(\alpha \pm \frac{1}{\alpha}\right)$.

Definition 2.1 *$\Delta(\lambda)$ will be called the characteristic function of the eigenvalues of the problem $(1)-(3)$.*

Eigenvalues for $(1)-(3)$ are real. The boundary value problem $(1)-(3)$ has a countable number of eigenvalues that grow unlimitedly, when that are ordered according to their absolute value. The zeros of the characteristic function $\Delta(\lambda)$ are also the eigenvalues of the $(1)-(3)$ problem (see [11]). Since the functions $\varphi(x,\lambda)$ and $\psi(x,\lambda)$ and their first order derivatives are complete, the function $\Delta(\lambda)$ is complete. Because, $\Delta(\lambda) = W(\varphi(x,\lambda), \psi(x,\lambda))$ is the Wronskian of solution matrices $\varphi(x,\lambda)$ and $\psi(x,\lambda)$.

We aim to reach the asymptotic expressions of the eigenvalues with the help of the following representations of the functions $\varphi(x,\lambda)$ and $\psi(x,\lambda)$.

$$\left. \begin{array}{l} \varphi(x,\lambda) = \cos(\lambda x) I_m + O\left(|\lambda|^{-1} e^{|\tau| x}\right) \\ \varphi(x,\lambda) = \alpha^{+} \cos(\lambda x - \beta^{+}(x)) I_m + \alpha^{-} \cos(\lambda(2a-x) - \beta^{-}(x)) I_m + O\left(|\lambda|^{-1} e^{|\tau| x}\right) \end{array} \right\} \tag{6}$$

where $\tau : Im\lambda$, $\int_{-\infty}^{\infty} O\left(|\lambda|^{-1} e^{|\tau| x}\right) d\lambda < \infty$ for $\forall \lambda$. This representations are obtained as in $[2]$.

We know that due to the Euclidean norm, if s is the eigenvalue of $A^{\dagger} A$, then $\|A\| = \sqrt{s}$.

Theorem 2.2 *The problem* $(1)-(3)$ *has a countable number of eigenvalues* $\{\lambda_{rs}\}_{s=\overline{1,m}}$ $(r \in \mathrm{N})$, *that grow unlimitedly, which* $\lambda_{r(i+1),s(i+1)} \geq \lambda_{ri,si}$*; where* $\left(r_{(i+1)}, s_{(i+1)}\right) > (r_i, s_i)$*. Moreover, eigenvalues can also be shown asymptotically as the following:*

$$\lambda_{rs} = \left(r + \frac{1}{\pi}\right) + \frac{z_s}{\pi\,(r-1/2)} + \frac{\varsigma_{rs}}{r}, \ s = \overline{1,p}, \tag{7}$$

$$\lambda_{rs} = r + \frac{z_s}{\pi r} + \frac{\varsigma_{rs}}{r}, \ s = \overline{p+1,m} \tag{8}$$

where $(\omega_{11} - D)$ *and* ω_{22} *are Hermitian matrices,* (z_s) *in equations* (7) *and* (8) *are their eigenvalues, respectively,* $(\varsigma_{nk}) \in l_2$.

In the scalar case, applying the Rouche theorem, we came to the conclusion that there is a sufficiently large n in counter $\left\{\lambda : |\lambda| = \left|\lambda_n^0\right| + \frac{1}{2\alpha},\ n = 0, 1, ...\right\}$, the characteristic function has number of zeros counting their multiplicities. Thus, $\det(K)$, $\det(K+L)$ scalar cases are also evaluated in the same way the matrix functions K and L.

Lemma 2.3 *The problem* $(1)-(3)$ *has a countable number of eigenvalues* $\{\lambda_{rs}\}_{s=\overline{1,m}}$ $(r \in \mathrm{N})$, *that grow unlimitedly, and eigenvalues have asymptotically as the following:*

$$\begin{aligned} &\lambda_{rs} = r - \tfrac{1}{\pi} + \eta_{rs} \ \ , \ s = \overline{1,p} \\ &\lambda_{rs} = r + \eta_{rs}\,, \quad s = \overline{p+1,m} \end{aligned} \tag{9}$$

where $\eta_{rs} = O\left(r^{-1}\right)$, $r \to \infty$.

Proof Under initial conditions the function $\Delta(\lambda) = W(\varphi(x,\lambda), \psi(x,\lambda))$ can be expressed as $W(\varphi(\pi,\lambda))$. Using (6) for the eigenvalues of the problem $(1)-(3)$ as well as the zeros of $W(\varphi(\pi,\lambda))$

$$\begin{aligned} &W(\varphi,\psi)|_{x=\pi} = W(\lambda) = \\ &K\left(\alpha^+ \cos(\lambda\pi - \beta^+(\pi))\, O\left(\tfrac{e^{|\tau|\pi}}{\lambda}\right)\right) \\ &-K^{\perp}\left(\alpha^- \tfrac{1}{\lambda} \sin(\lambda(2a-\pi) - \beta^-(\pi))\, O\left(\tfrac{e^{|\tau|\pi}}{\lambda^2}\right)\right), \ \ |\lambda| \to \infty. \end{aligned} \tag{10}$$

Let $K(\alpha^+ \cos(\lambda\pi - \beta^+(\pi))) - K^{\perp}\left(\alpha^- \frac{1}{\lambda}\sin(\lambda(2a-\pi) - \beta^-(\pi))\right) = X(\lambda)$.

In this case, the roots of $\det(X(\lambda))$ can be written as

$$\lambda_{rs}^0 = r - \frac{1}{\pi}, \ \ s = 1, ..., p \qquad\qquad \lambda_{rs}^0 = r, \ \ s = p+1, ..., m. \tag{11}$$

Denote $\Gamma_\delta = \{\lambda : |\lambda - \lambda_0| \geq \delta, \delta > 0\}$ where δ is sufficiently small number as in [3].

$$\left|\alpha^{+}\cos\left(\lambda\pi-\beta^{+}\left(\pi\right)\right)\right|\geq ce^{\left(|\tau|\pi\right)},\quad \left|\alpha^{-}\frac{1}{\lambda}\sin\left(\lambda\left(2a-\pi\right)-\beta^{-}\left(\pi\right)\right)\right|\geq ce^{\left(|\tau|\pi\right)},\quad \lambda\in\Gamma_{\delta},$$

$$\begin{array}{l} X^{-1}\left(\lambda\right)\left(W\left(\lambda\right)-X\left(\lambda\right)\right)= \\ K\left(\alpha^{+}\cos\left(\lambda\pi-\beta^{+}\left(\pi\right)\right)\right)^{-1}O\left(\frac{e^{|\tau|\pi}}{\lambda}\right) \\ +K^{\perp}\left(\alpha^{-}\frac{1}{\lambda}\sin\left(\lambda\left(2a-\pi\right)-\beta^{-}\left(\pi\right)\right)\right)^{-1}O\left(\frac{e^{|\tau|\pi}}{\lambda^{2}}\right)=O\left(\frac{1}{\lambda}\right). \end{array}$$

Thus, we get $\left\|X^{-1}\left(\lambda\right)\left(W\left(\lambda\right)-X\left(\lambda\right)\right)\right\|<1$.

Applying Rouche's theorem we conclude that for sufficiently large δ inside the contour Γ_{δ} the functions $\det\left(X\left(\lambda\right)\right)$ and $\det\left(W\left(\lambda\right)\right)$ have the same number of zeros counting their multiplicities.

Thus, $\eta_{rs}=o\left(1\right)$ as $r\to\infty$ and $\eta_{rs}=O\left(r^{-1}\right)$ as $r\to\infty$ for $s=\overline{1,m}$.

For $s=\overline{1,p}$, $\lambda_{rs}=\lambda$ and using the expression (10);

$$\begin{array}{l} W\left(\lambda\right)=W\left(\lambda_{rs}\right)= \\ K\left(\alpha^{+}\cos\left(\lambda\pi-\beta^{+}\left(\pi\right)\right)O\left(\frac{e^{|\tau|\pi}}{\lambda}\right)\right) \\ -K^{\perp}\left(\alpha^{-}\frac{1}{\lambda}\sin\left(\lambda\left(2a-\pi\right)-\beta^{-}\left(\pi\right)\right)O\left(\frac{e^{|\tau|\pi}}{\lambda^{2}}\right)\right) \\ =\left(-1\right)^{n}\left(K^{\perp}\left(\frac{\cos\eta_{rs}\pi}{\lambda}\right)+K\left(\sin\eta_{rs}\pi+O\left(r^{-1}\right)\right)\right),\ r\in\mathrm{N}. \end{array}$$

As a result, $\frac{(-1)^{rm}}{n^{m-p}}S_{rs}\left(\sin\eta_{rs}\pi\right)=\det\left(W\left(\lambda_{nk}\right)\right)=0, der\left(S_{rs}\right)=p$.

□

Proof [Proof of Theorem 2.2] We will use Lemma 2.3 to prove the Theorem 2.2. Let's give the proof of the (8) asymptotic expression. Let's examine the zeros of the $W\left(\lambda\right)$ function by writing (7) more clearly.

Let's define the k-plane, such that $r+\frac{k}{\pi r}:=\lambda_{r}\left(k\right)$ is provided on the circle $|k|\leq n$ for $n>0$.

Let

$$\sum_{r=1}^{\infty}\left\|P_{r}\left(k\right)\right\|^{2}\leq T,\quad |k|\leq n$$

where the sequence $\left(P_{r}\left(k\right)\right)_{n\in\mathrm{N}}$ of matrix functions depend on T but does not depends on k. Thus,

$$\begin{array}{l} \sin\left(\lambda_{r}\left(k\right)\pi\right)=\frac{(-1)^{r}k}{r}\left(1+O\left(r^{-2}\right)\right), \\ \cos\left(\lambda_{r}\left(k\right)\pi\right)=\left(-1\right)^{r}\left(1+O\left(\left(r^{-2}\right)\right)\right),\ r\to\infty. \end{array} \tag{12}$$

We obtain the following asymptotic expression using (6) and (12).

$$W\left(\lambda_{r}^{2}\left(k\right)\right)=\left(-1\right)^{r}\left(K\left(I_{m}+\frac{P_{r}\left(k\right)}{r}\right)-K^{\perp}\left(\frac{kI_{m}-\omega}{r^{2}}+\frac{P_{r}\left(k\right)}{r^{2}}\right)\right). \tag{13}$$

Using equation (13)

$$H_{r}\left(k\right):=\left(-1\right)^{r}\left(K+r^{2}K^{\perp}\right)W\left(\lambda_{r}^{2}\left(k\right)\right) \tag{14}$$

can be written. Clearly,

$$H_r(k) = K\left(I_m + \frac{H_r(k)}{r}\right) - K^{\perp}\left(kI_m - \omega + H_r(k)\right). \tag{15}$$

Let's write the form below from equation (15).

$$H(k) := K - K^{\perp}\left(kI_m - \omega\right). \tag{16}$$

If we write the $H(k)$ expression in matrix form we obtain

$$H(k) = \begin{pmatrix} I_p & 0 \\ \omega_{21} & -\left(kI_{m-p} - \omega_{22}\right) \end{pmatrix}. \tag{17}$$

Hence, $\det H(z) = (-1)^{m-p} \det\left(kI_{m-p} - \omega_{22}\right)$.

Real $\{k_s\}_{s=p+1}^{m}$ are eigenvalues of matrix $\omega_{22} = \omega_{22}^{\dagger}$ and also zeros of $\det H(z)$.

Define the region

$G_\delta := \{z \in \mathrm{C} : |k| \le r, |k - k_s| \ge \delta, s = p+1, ..., m\}, \delta > 0$ and $|k_s - k_l| < \delta$ for all $l \ne k$, where $\delta > 0$ is so small that $|z_k - z_l| < \delta$ for $l \ne k$, $l, k = p+1, ..., m$.

Using equation (17)

$$H^{-1}(k) = \begin{pmatrix} I_p & 0 \\ \left(kI_{m-p} - \omega_{22}\right)^{-1}\omega_{21} & -\left(kI_{m-p} - \omega_{22}\right)^{-1} \end{pmatrix} \tag{18}$$

can be written. As a result, $\left\|H^{-1}(k)\right\| \le T$ for $k \in G_\delta$. Equations (15) and (16) imply $H_r(k) - H(k) = P_r(k), r \in \mathrm{N}$. Thus, $\|H_r(k)\| \cdot \|H_r(k) - H(k)\| < 1$ for $k \in G_\delta$.

The $\{k_s\}_{s=p+1}^{m}$, which are the zeros of the function $\det(H(k))$, are $(m-p)$ and have the following asymptotic expression:

$$k_{rs} = k_s + \upsilon_{rs}, \quad \upsilon_{rs} = o(1), \ r \to \infty, \quad s = p+1, ..., m. \tag{19}$$

Let $D = \left(D^{\dagger}\right)^{-1} \in \mathrm{C}^{(m-p)x(m-p)}$ and $\tilde{D} \in \mathrm{C}^{mxm}$, such that

$$D\omega_{22}D^{\dagger} = D := diag\{k_s\}_{s=p+1}^{m}, \quad \tilde{D} := \begin{pmatrix} I_p & 0 \\ 0 & D \end{pmatrix}. \tag{20}$$

Using (15) we obtain

$$\tilde{D}H_r(k)\tilde{D}^{\dagger} = \begin{pmatrix} I_p & 0 \\ D\omega_{21} & -\left(zI_{m-p} - D\right) \end{pmatrix} + K_r(k).$$

Thus,

$$J_r(k) = \begin{pmatrix} I_p & 0 \\ 0 & -\left(kI_{m-p} - D\right) \end{pmatrix} + K_r(k), \ s \in \{p+1, ..., m\}. \tag{21}$$

We get $J_r(k)$ by Gaussian method. Obviously, the zeros of the $\det(H(k))$ function are multiplicity. Let this multiplicity be m_k where $m_k = \{t : k_t = k_s, p+1 \leq t \leq m\}$. Using (19) and (21), $\det(J_r(k_{rs})) = R_{rs}(\vartheta_{rs})$, where $der(R_{rs}) = m_k$. In that case, the zeros of $\det(J_r(k))$ and $W\left(\lambda_r^2(k)\right)$ are equal. We proved the equation (8).

Similarly to obtain the equation (7), the proof is made by taking for $\lambda_r(k)\, \tilde{\lambda}_r(k) := r - \frac{1}{2} + \frac{k}{\pi(r-1/2)}$ and

$$\tilde{H}_r(k) := (-1)^r \left(r - \frac{1}{2}\right) W\left(\lambda_r^2(k)\right) = K(kI_m - \omega + P_r(k)) + T^{\perp}\left(I_m + \frac{P_r(k)}{r}\right)$$

for $H_r(k)$. □

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