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$N(\kappa)-$kontakt metrik manifoldlar üzerinde Z-tensor

$N(\kappa)-$contact metric manifolds admitting Z-tensor

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Özet. Bu çalışmada, kontakt manifoldların özel bir sınıfı olan $N(\kappa)-$kontakt metrik manifoldlar üzerinde çalışılmıştır. R Riemann eğrilik tensörü, $\mathcal{P}$ projektif eğrilik tenrösü, $\mathcal{L}$ concircular eğrilik tensörü ve W_2 eğrilik tensörü $\mathcal{W}_2$ olmak üzere, $N(\kappa)-$kontakt metrik manifoldlar $R(\xi, W).\mathcal{Z} = 0$, $\mathcal{P}(\xi, W).\mathcal{Z} = 0$, $\mathcal{L}(\xi, W).\mathcal{Z} = 0$ ve $\mathcal{W}_2(\xi, W).\mathcal{Z} = 0$ yarı-simetri şartları altında incelenmiştir.

Anahtar Kelimeler: $N(\kappa)-$kontakt metrik manifold, Z-Tensör, semi-simetri.

Abstract. In this study, we work on $N(\kappa)$-contact metric manifolds which are a special kind of contact manifolds. We present some results on $N(\kappa)$-contact metric manifolds by using $\mathcal{Z}$-tensor. We classify the manifolds by using some semi-symetry conditions such as $R(\xi, W).\mathcal{Z} = 0$, $\mathcal{P}(\xi, W).\mathcal{Z} = 0$, $\mathcal{L}(\xi, W).\mathcal{Z} = 0$ and $\mathcal{W}_2(\xi, W).\mathcal{Z} = 0$, where R the Riemann curvature tensor, $\mathcal{P}$ is the projective curvature tensor, $\mathcal{L}$ is the concircular curvature tensor and $\mathcal{W}_2$ is the W_2 curvature tensor.

Keywords: $N(\kappa)-$contact metric manifold, Z-tensor, semi-symmetry.

1. Introduction

A contact manifold is a $(2n + 1)$-dimensional differentiable manifold with a contact form. Contact manifolds have many applications in mathematics and some applied areas such as mechanics, optics, thermodynamics, control theory and theoretical physics. Also, contact manifolds are special solutions of Einstein fields equation with some certain conditions. $N(\kappa)$-contact metric manifolds are a special class of contact manifolds. A $N(\kappa)$-contact metric manifold is an almost contact manifold with nullity distribution. This kind of manifolds was firstly studied in [1].

In the Riemannian geometry, we use curvature tensor to examine the geometric properties of given manifolds. Especially, manifolds with structures could be classified by using certain conditions on the curvature tensors. Curvature tensors on $N(\kappa)$-contact metric manifolds have been studied in [2, 3, 4, 5, 6, 7]. On the other hand, $N(\kappa)$-contact metric manifolds have been studied under some certain semi-symmetry conditions with related to some special curvature tensors. In [9] Balir et al. classify $N(\kappa)$-contact metric manifolds under certain conditions with using concircular curvature tensor. Yıldız , De and Ghosh examined some flatness and semi-symmetry condition of $N(\kappa)$-contact metric manifolds by using concircular curvature tensor [14]. Also, the presented author and Altin [7] worked on $N(\kappa)$-contact metric manifolds with concircular curvature tensor.

In 2012, Mantica and Molinari [10] defined $\mathcal{Z}-$tensor as a $(0, 2)-$type curvature tensor on a Riemann manifold M by following;

$$\mathcal{Z}(W_1, W_2) = Ric(W_1, W_2) + \psi g(W_1, W_2) \tag{1}$$

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for all $W_1, W_2 \in \Gamma(TM)$ and an arbitrary function ψ. $\mathcal{Z}$- tensor has many important properties and applications. It is a general notion of the Einstein gravitational tensor in General Relativity. As we know a Riemannian manifold is said to be Ricci semi-symmetric if $R.Ric = 0$. It is point out that an Einstein manifolds is Ricci semi-symmetric, but the converse is not true in generally. Under some special conditions such as Ricci semi-symmetry, we can classify contact manifolds Einstein or $\eta-$Einstein. We recall a Riemannian manifold as $\mathcal{Z}$-semi-symmetric if $R.\mathcal{Z} = 0$. Thus, also by using $\mathcal{Z}$-tensor we can classify the contact manifolds.

In this study, we worked on $\mathcal{Z}-$tensor on $N(\kappa)$-contact metric manifolds under certain curvature conditions. After present some fundamental facts on $N(\kappa)$-contact metric manifolds and curvature tensors in Section 2, we give the main results of the paper in Section 3. Also, we examine an example to verify our results

2. Preliminaries

In this section we give some fundamental facts on $N(\kappa)$-contact metric manifolds and curvature tensors. For more details on contact manifolds, we refer to reader [1, 9, 11].

Definition 1. *An almost contact metric manifold is a $(2n+1)-$dimensional differentiable manifold with a structure (φ, ξ, η, g) such as*

$$\varphi^2(W_1) = -W_1 + \eta(W_1)\xi, \quad \eta(\xi) = 1, \quad \varphi(\xi) = 0, \quad \eta(\varphi(W_1)) = 0$$

for any vector fields $W_1 \in \Gamma(M)$, where φ is a $(1,1)-$tensor field, ξ is a vector field and η is a $1-$form on M [11].

The $h = \frac{1}{2}\mathcal{L}_\xi\varphi$ is an important operator for contact manifolds. Also on an almost contact metric manifold, we have $\nabla_{W_1}\xi = -\varphi W_1 - \varphi hW$ for any $W_1 \in \Gamma(TM)$. Two important classes of contact manifolds are K-contact and Sasakian manifolds. If ξ is Killing vector field on M then , M is said to be $K-$contact . On the other hand , M is called normal contact metric manifold if $N_\varphi + 2d\eta \otimes \xi = 0$, where N_φ is the Nijenhuis tensor of φ. A normal contact metric manifold is called Sasakian. Thus we can state on a $K-$contact and Sasakian manifold $h = 0$.

2.1. A short review on $N(\kappa)$-contact metric manifolds

In 1995, the notion of (κ, μ)-manifolds were defined in [1]. Nullity distribution of an almost contact metric manifold M that is defined by

$$N(\kappa, \mu) : p \to N_p(\kappa, \mu)$$
$$N_p(\kappa, \mu) = [W_3 \in \Gamma(T_pM) : R(W_1, W_2)W_3 = (\kappa I + \mu h)\{g(W_2, W_3)W_1 - g(W_1, W_3)W_2\}]$$

for all $W_1, W_2 \in \Gamma(TM)$ where κ, μ are constants and R is the Riemannian curvature tensor of M. If $\mu = 0$, the (κ, μ)-nullity distribution reduces to κ-nullity distribution. κ-nullity distribution of an almost contact metric manifold $(M, \varphi, \xi, \eta, g)$ is defined as

$$N(\kappa) : p \to N_p(\kappa) = [W_3 \in \Gamma(T_pM) : R(W_1, W_2)W_3 = \kappa\{g(W_2, W_3)W_1 - g(W_1, W_3)W_2\}]$$

for any $W_1,\ W_2 \in \Gamma(T_pM)$ and $\kappa \in \mathbb{R}$. If ξ belongs to $\kappa-$nullity distribution then M is called $N(\kappa)-$contact metric manifold. Thus on a $N(\kappa)$ contact metric manifold we get

$$R(W_1, W_2)\xi = \kappa[\eta(W_2)W_1 - \eta(W_1)W_2]. \tag{2}$$

A $N(\kappa)-$contact metric manifold is Sasakian if and only if $\kappa = 1$. Also, if $\kappa = 0$, then the manifold is locally isometric to the product $E^{n+1}(0) \times S^n(4)$ for $n > 1$ and flat for $n = 1$ [9].

The Riemannian curvature tensor R and the Ricci curvature tensor Ric of a $N(\kappa)-$contact metric manifold has following properties:

$$R(W_1, \xi)W_2 = \kappa[\eta(W_2)W_1 - g(W_1, W_2)\xi] \tag{3}$$
$$Ric(W_1, W_2) = 2(n-1)g(W_1, W_2) + 2(n-1)g(hW_1, W_2) + 2(n\kappa - (n-1))\eta(W_1)\eta(W_2) \tag{4}$$
$$Ric(\varphi W_1, \varphi W_2) = Ric(W_1, W_2) - 2n\kappa\eta(W_1)\eta(W_2) - 4(n-1)g(hW_1, W_2) \tag{5}$$
$$Ric(W_1, \xi) = 2\kappa n\eta(W_1) \tag{6}$$

and the scalar curvature is given by $scal = 2n(2n + \kappa - 2)$ [1, 9].

In [14], an example of $N(\kappa)-$contact metric manifolds was given by using Boeckx invariant.

Example 1. *As we mentioned above* (κ,μ)*-spaces contains the Sasakian manifolds and non-Sasakian manifolds. The classifications of non -Sasakian* $(\kappa,\mu)-$*spaces were presented by Boeckx* [13] *with an invariant* $I_M = \frac{1-\frac{\mu}{2}}{\sqrt{1-\kappa}}$. *Blair et al.* [11] *showed that* $(\kappa,\mu)-$*nullity distribution is invariant under* $\bar{\kappa} = \frac{\kappa+a^2-1}{a}$, $\bar{\mu} = \frac{\mu+2c-2}{a}$. *By consider the tangent sphere bundle of an* $(n+1)$*-dimensional manifold of constant curvature c, as the resulting D-homothetic deformation we have* $\kappa = c(2-c)$, $\mu = -2c$. *Thus, we could obtain such examples from the standard contact metric structure on the tangent sphere bundle of a manifold of constant curvature c where we have* $I_M = \frac{1+c}{|1-c|}$.

Let take $\overline{\kappa} = 1 - \frac{1}{n}$ *and* $\overline{\mu} = 0$. *Then, we obtain* $I_M = \sqrt{n}$. *Also, from the equalities* $1 - \frac{1}{n} = \frac{k+a^2-1}{a^2}$, $0 = \frac{\mu+2a-2}{a}$ *for a and c, we get* $c = \frac{\sqrt{n}\pm 1}{n-1}$, $a = 1 + c$. *Thus we obtain* $N(1-\frac{1}{n})$*-contact metric manifold.*

2.2. Curvature tensors on $N(\kappa)$-contact metric manifolds

A Euclidean space is a manifold with zero Riemannian curvature tensor i.e. it is a flat space. The flatness of a Riemannian manifold is measured with the being zero of the Riemannian curvature tensor of the manifold. If a Riemannian manifold is flat it is understood that the manifold is locally Euclidean. Except for Rieman curvature tensor we have many other curvature tensors such as projective, concircular and W_2-curvature tensor. These tensors have similar symmetry properties to Riemann curvature tensor and we also examined the flatness of the manifold by using them.

Let M be a $(2n+1)$-dimensional $N(\kappa)$-contact metric manifold. Projective $(\mathcal{P})$, concircular $(\mathcal{L})$ and W_2-curvature $(\mathcal{W}_2)$ on M are defined as follow:

$$\mathcal{P}(W_1,W_2)W_3 = R(W_1,W_2)W_3 - \frac{1}{2n}\left(Ric(W_2,W_3)W_1 - Ric(W_1,W_3)W_2\right) \tag{7}$$

$$\mathcal{L}(W_1,W_2)W_3 = R(W_1,W_2)W_3 - \frac{1}{2n(2n+1)}\left(g(W_2,W_3)W_1 - g(W_1,W_3)W_2\right) \tag{8}$$

$$\begin{aligned}\mathcal{W}_2(W_1,W_2,W_3,W_4) = {} & R(W_1,W_2,W_3,W_4) \\ & - \frac{1}{2n}\left(Ric(W_2,W_4)g(W_1,W_3) - Ric(W_1,W_4)g(W_2,W_3)\right)\end{aligned} \tag{9}$$

for all $W_1, W_2, W_3, W_4 \in \Gamma(TM)$ and arbitrary vector fields ψ.

Projective curvature tensor and concircular curvature tensor on $N(\kappa)$-contact metric manifolds have been studied in [7] and [9, 14, 7], respectively.

From the definition of curvature tensors and by using curvature properties (2),(3) and (6) we obtain the following lemma.

Lemma 1. *On a* $N(\kappa)$*-contact metric manifold we have*

$$\mathcal{P}(\xi,W_2)W_3 = -\frac{1}{2n}\left(Ric(W_2,W_3) + \kappa g(W_2,W_3)\right)\xi \tag{10}$$

$$\mathcal{L}(\xi,W_2)W_3 = \left(\kappa - \frac{1}{2n(2n+1)}\right)\left(g(W_2,W_3)\xi - \eta(W_3)W_2\right) \tag{11}$$

$$\mathcal{Z}(\xi,W_2) = (\psi + 2\kappa n)\,\eta(W_2), \quad \mathcal{Z}(\xi,\xi) = \psi + 2\kappa n \tag{12}$$

for all $W_2, W_3 \in \Gamma(TM)$.

3. Main Results

For a $(1,3)-$type curvature tensor $\mathcal{T}$, $R(W_1,W_2).\mathcal{T}$ is defined by

$$R(W_1,W_2).\mathcal{T} = \nabla_{W_1}\nabla_{W_2}\mathcal{T} - \nabla_{W_2}\nabla_{W_1}\mathcal{T} - \nabla_{[W_1,W_2]}\mathcal{T}.$$

The operation of "$\cdot$" acts like as a derivation on curvature tensor and it is obtained as follow ;

$$\begin{aligned}(R(W_1,W_2).\mathcal{T})(W_3,W_4)W_5 = {} & R(W_1,W_2)\mathcal{T}(W_3,W_4)W_5 - \mathcal{T}(R(W_1,W_2)W_3,W_4)W_5 \\ & - \mathcal{T}(W_3,R(W_1,W_2)W_4)W_5 - \mathcal{T}(W_3,W_4)R(W_1,W_2)W_5.\end{aligned} \tag{13}$$

A Riemann manifold is called locally semi-symmetric if $R(W_1,W_2).R = 0$. In general, we recall a Riemannian manifold as locally $\mathcal{T}$-semi-symmetric if $R(W_1,W_2).\mathcal{T} = 0$. Semi-symmetry of Riemannian manifolds is an important notion in the Riemannian geometry. The another semi-symmetry notion is Ricci semi-symmetry. For $(0,2)$-type tensor field $\mathcal{S}$ on M, we have

$$(\mathcal{T}(W_1,W_2)\cdot\mathcal{S})(W_3,W_4)=\mathcal{S}(\mathcal{T}(W_1,W_2)W_3,W_4)-\mathcal{S}(W_3,\mathcal{T}(W_1,W_2)W_4).$$

If $R(W_1,W_2)\cdot Ric=0$, then the manifold is said to be Ricci semi-symmetric. In this section, we take $\mathcal{T}$ as Riemann, projective, concircular and W_2-tensor and $\mathcal{S}=\mathcal{Z}$ for to examine $N(\kappa)-$contact metric manifolds under certain semi-symmetry conditions.

3.1. $N(\kappa)$-contact metric manifolds satisfying $R(\xi,W).\mathcal{Z}=0$

Theorem 1. *A $N(\kappa)$-contact metric manifold M satisfying $R(\xi,W).\mathcal{Z}=0$ is flat, or in addition if ξ is Killing then M is locally isometric to Example 1.*

Proof. Let M be a $N(\kappa)$-contact metric manifold and $R(\xi,W).\mathcal{Z}=0$ on M, for all $W\in\Gamma(TM)$. Thus, from (13) we have

$$(R(\xi,W_2)\cdot\mathcal{Z})(W_3,\xi)=\mathcal{Z}(R(\xi,W_2)W_3,\xi)-\mathcal{Z}(W_3,R(\xi,W_2)\xi)=0$$

for all $W_2,W_3\in\Gamma(TM)$. By using (1), (3) and from (12) we obtain

$$\kappa\left[(2\kappa n-2n+2)(g(W_2,W_3)+(2\kappa n-2)g(hW_2,W_3)-(2\kappa n-2(n-1))\eta(W_2)\eta(W_3)\right]=0.$$

We have two cases:

- In first case $\kappa=0$ which means M is flat.
- In second case we have,

$$[(2\kappa n-2n+2)(g(W_2,W_3)+(2\kappa n-2)g(hW_2,W_3)-(2\kappa n-2(n-1))\eta(W_2)\eta(W_3)]=0.$$

Suppose that ξ is Killing vector field. Thus $h=0$ and so, we get $(2\kappa n-2n+2)(g(\varphi W_2,\varphi W_3)=0$ which implies $\kappa=1-\frac{1}{n}$. This shows us M is locally isometric to Example 1.

□

3.2. $N(\kappa)$-contact metric manifolds satisfying $\mathcal{P}(\xi,W).\mathcal{Z}=0$

Theorem 2. *A $N(\kappa)$-contact metric manifold M satisfying $\mathcal{P}(\xi,W).\mathcal{Z}=0$ if and only if $\psi+2\kappa n=0$.*

Proof. Let M be a $N(\kappa)$-contact metric manifold and suppose that it is satisfied $\mathcal{P}(\xi,W).\mathcal{Z}=0$, for all $W\in\Gamma(TM)$. Then, from (13) we have

$$(\mathcal{P}(\xi,W_2)\cdot\mathcal{Z})(W_3,W_4)=\mathcal{Z}(\mathcal{P}(\xi,W_2)W_3,W_4)-\mathcal{Z}(W_3,\mathcal{P}(\xi,W_2)W_4)=0.$$

By using (7), (10) and (12) we get

$$\begin{aligned}(\psi+2kn)[(&\frac{1}{2n}Ric(W_2,W_3)-\kappa g(W_2,W_3))\eta(W_4)\\&+(-\frac{1}{2n}Ric(W_2,W_4)+\kappa g(W_2,W_4))\eta(W_4)]=0.\end{aligned} \tag{14}$$

Thus, $\mathcal{P}(\xi,W).\mathcal{Z}=0$ if and only $\psi+2kn=0$. □

Also, from (14) we obtain following corollary:

Corollary 1. *If a $N(\kappa)$-contact metric manifold M is Einstein, then we have $\mathcal{P}(\xi,W).\mathcal{Z}=0$.*

3.3. $N(\kappa)$-contact metric manifolds satisfying $\mathcal{L}(\xi,W).\mathcal{Z}=0$

Theorem 3. *Let M be a $N(\kappa)$-contact metric manifold which is satisfied $\mathcal{L}(\xi,W).\mathcal{Z}=0$ and $\kappa\neq\frac{scal}{2n(2n+1)}$. If ξ is Killing, then M is locally isometric to Example 1.*

Proof. Let M be a $N(\kappa)$-contact metric manifold and $\mathcal{L}(\xi,W).\mathcal{Z}=0$ is satisfied on M, for all $W\in\Gamma(TM)$. Then, from (13) we have

$$(\mathcal{L}(\xi,W_2)\cdot\mathcal{Z})(W_3,W_4)=\mathcal{Z}(\mathcal{L}(\xi,W_2)W_3,W_4)-\mathcal{Z}(W_3,\mathcal{L}(\xi,W_2)W_4)=0.$$

Thus, with consider (1), (8), (3) and (12) we obtain

$$\begin{aligned}(\kappa-\tfrac{scal}{2n(2n+1)})[(2\kappa n-2n+2)(g(W_2,W_3)\eta(W_4)+g(W_2,W_4)\eta(W_4))+(2(n-1))(g(hW_2,W_3)\eta(W_4)\\+g(hW_2,W_4)\eta(W_4))]=0.\end{aligned}$$

Since $\kappa \neq \frac{scal}{2n(2n+1)}$, we get

$$(2\kappa n - 2n + 2)(g(W_2, W_3)\eta(W_4) + g(W_2, W_4)\eta(W_4))$$
$$+(2(n-1))(g(hW_2, W_3)\eta(W_4) + g(hW_2, W_4)\eta(W_4)) = 0.$$

Thus, if ξ is a Killing vector filed $h = 0$, an so we get $\kappa = 1 - \frac{1}{n}$ which provide that M is locally isometric to Example 1. □

3.4. $N(\kappa)$-contact metric manifolds satisfying $\mathcal{W}_2(\xi, W).\mathcal{Z} = 0$

Theorem 4. *Let M be a $N(\kappa)$-contact metric manifold which is satisfied $\mathcal{W}_2(\xi, W).\mathcal{Z} = 0$ for all, $W \in \Gamma(TM)$. If ξ is Killing then $\psi = -2n + 2$.*

Proof. Let M be a $N(\kappa)$-contact metric manifold and suppose that it is satisfied $\mathcal{W}_2(\xi, W).\mathcal{Z} = 0$, for all $W \in \Gamma(TM)$. Then, from (13) we have

$$(\mathcal{W}_2(\xi, W_2) \cdot \mathcal{Z})(W_3, W_4) = \mathcal{Z}(\mathcal{W}_2(\xi, W_2)W_3, W_4) - \mathcal{Z}(W_3, \mathcal{W}_2(\xi, W_2)W_4) = 0.$$

Thus, by using (1), we obtain

$$(\psi + 2n - 2)\left[\mathcal{W}_2(\xi, W_2, W_3, W_4) + \mathcal{W}_2(\xi, W_2, W_4, W_3)\right]$$
$$+ 2(n-1)\left[g(h\mathcal{W}_2(\xi, W_2)W_3, W_4) + g(hW_3,\ \mathcal{W}_2(\xi, W_2)W_4\right]$$
$$+ (2\kappa n - n + 1)\left(\mathcal{W}_2(\xi, W_2, W_3, \xi)\,\eta(W_4) + \mathcal{W}_2(\xi, W_2, W_4, \xi)\,\eta(W_3)\right) = 0.$$

Suppose that ξ is Killing vector field and from (9) since $\mathcal{W}_2(\xi, W_2, W_3, \xi) = 0$, we get

$$(\psi + 2n - 2)\left(\frac{1}{2n}[(Ric(W_2, W_4) - 2n\kappa g(W_2, W_4))\eta(W_3) + (Ric(W_2, W_3) - 2n\kappa g(W_2, W_3))\eta(W_4)]\right) = 0. \tag{15}$$

Thus we obtain $\mathcal{W}_2(\xi, W).\mathcal{Z} = 0$ if and only if $\psi = -2n + 2$. □

From (15) we state the following result:

Corollary 2. *If a $N(\kappa)$-contact metric manifold M is Einstein with Killing vector field ξ, then we have $\mathcal{W}_2(\xi, W).\mathcal{Z} = 0$.*

Example 2. *Let $M = \{(x_1, x_2, x_3) \in \mathbb{R}^3\}$ be a subset of $\mathbb{R}^3$, where x_1, x_2, x_3 are standard coordinates in $\mathbb{R}^3$. Let take E_1, E_2, E_3 , 3-vector fields in $\mathbb{R}^3$ satisfying*

$$[E_1, E_2] = (1 - \lambda)E_3,\ [E_2, E_3] = 2E_1,\ [E_3, E_1] = (1 - \lambda)E_2$$

and take a Riemann metric as

$$g(E_1, E_3) = g(E_2, E_3) = g(E_1, E_2) = 0, g(E_1, E_1) = g(E_2, E_2) = 1,\ \eta(U) = g(U, E_1)$$

where λ is a real constant and U is an arbitrary vector field on M. Let define a $(1,1)-$tensor field φ by

$$\varphi E_1 = 0,\ \varphi E_2 = E_3,\ \varphi E_3 = -E_2.$$

Then, by using the linearity of ϕ and g we have $\eta(E_1) = 1$, $\varphi^2(U) = -U + \eta(U)E_1$ and $g(\varphi W_1, \varphi W_2) = g(W_1, W_2) - \eta(W_1)\eta(W_2)$ for any $W_1, W_2 \in \Gamma(TM)$. Moreover, h is given by

$$hE_1 = 0,\ hE_2 = \lambda E_2, hE_3 = -\lambda E_3.$$

In [12], *it is showed that (M, φ, η, g) is a $N(1 - \lambda^2)$-contact metric manifold. In* [14] *the authors computed the Ricci curvatures as follow;*

$$Ric(E_1, E_1) = 2(1 - \lambda^2), \quad Ric(E_2, E_2) = 0, \quad Ric(E_3, E_3) = 0$$

thus we get

$$\mathcal{Z}(E_1, E_1) = 2(1 - \lambda^2) + \psi, \quad \mathcal{Z}(E_2, E_2) = \psi, \quad \mathcal{Z}(E_3, E_3) = \psi.$$

Suppose that M satisfies $R(E_1, W).\mathcal{Z} = 0$. Then, we have

$$(1 - \lambda^2)[g(W_2, W_3)g(E_1, W_4) - \eta(W_3)g(E_1, W_2)g(E_2, W_4) + g(W_2, W_4)g(E_1, W_3)$$
$$- \eta(W_4)g(E_1, W_2)g(E_1, W_3)] = 0.$$

From this equaiton we have $\lambda = 1$ *or* $\eta(W_4)g(\varphi W_2, \varphi W_3) + \eta(W_3)g(\varphi W_2, \varphi W_4) = 0$. *Since second equality do not satisfy for arbitrary vector fields* W_2, W_3, W_4 *we have* $\lambda = 1$, *thus* M *is flat. This verify the Theorem 1.*

Suppose that, we have $\mathcal{P}(E_1, W).\mathcal{Z} = 0$. *Then, we get*

$$\mathcal{Z}(\mathcal{P}(E_1, W_2)W_3, W_4) = (2(1-\lambda^2) + \psi)\eta(W_4)\left(Ric(W_2, W_3) - (1-\lambda^2)g(W_2, W_3)\right)$$

and

$$\mathcal{Z}(W_3, \mathcal{P}(E_1, W_2)W_4) = (2(1-\lambda^2) + \psi)\eta(W_3)\left(-Ric(W_2, W_4) - (1-\lambda^2)g(W_2, W_4)\right).$$

From (13), we obtain

$$\begin{aligned}(2(1-\lambda^2) + \psi)(\eta(W_3)(-Ric(W_2, W_4) - (1-\lambda^2)g(W_2, W_4))\\ -\eta(W_3)(-Ric(W_2, W_4) - (1-\lambda^2)g(W_2, W_4))) = 0.\end{aligned}$$

Since the manifold is not Einstein this equality satisfies only $2(1-\lambda^2) + \psi = 0$ *and from the fact that* $n = 1$, *the Theorem 2. is verified.*

References

[1] Blair D.E., Koufogiorgos T., Papantoniou B.J., Contact metric manifolds satisfying a nullity condition, Israel Journal of Mathematics, 91(1-3), 189-214, (1995).

[2] Majhi P., and De U.C., Classifications of $N(k)-$contact metric manifolds satisfying certain curvature conditions, Acta Mathematica Universitatis Comenianae, 84(1), 167-178, (2015).

[3] De U.C., Certain results on $N(k)-$contact metric manifolds, Tamkang Journal of Mathematics, 49(3), 205-220, (2018).

[4] Ingalahalli G., Anil S., and Bagewadi C., Certain Results on $N(k)-$Contact Metric Manifold, Asian Journal of Mathematics and Computer Research, 123-130, (2019).

[5] Özgur C., and Sular S., On $N(k)-$contact metric manifolds satisfying certain conditions, SUT J. Math, 44(1), 89-99, (2008).

[6] Yıldız A., De U. C., Murathan C., Arslan K., On the Weyl projective curvature tensor of an $N(k)-$contact metric manifold, Mathematica Panonoica, 21(1), 129-142, (2010).

[7] Ünal İ., and Altin M., $N(\kappa)-$ contact Metric Manifolds with Generalized Tanaka-Webster Connection, arXiv preprint arXiv:2004.02536, (2020).

[8] Altın M., Projective Curvature Tensor on $N(k)$ contact metric manifold with respect to semi-symmetric non-metric connection, Fundamental Journal of Mathematics and Applications, 3(2), 94-100, (2020).

[9] Blair J-S. Kim and Tripathi M.M., On the concircular curvature tensor of a contact metric manifold, J. Korean Math. Soc., 42, 883-892, (2005).

[10] Mantica C. A. and Molonari L. G., Weakly Z symmetric manifolds, Acta Math. Hunger., 135, 80–96, (2012).

[11] Blair D. E., Riemannian Geometry of Contact and Symplectic Manifolds, 2nd edn. Birkhäuser,Boston (2010).

[12] Blair D. E., Two remarks on contact metric structures, Tohoku Mathematical Journal, Second Series, 29(3), 319-324, (1977).

[13] Boeckx E., A full classification of contact metric (k,μ)-spaces, Illinois J. Math., 44(1), 212-219, (2000).

[14] De U.C., Yildiz A., and Ghosh S., On a class of $N(k)-$contact metric manifolds, Math. Reports, 16, 66, (2014).