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Existence of positivity of the solutions for higher order three-point boundary value problems involving p-Laplacian

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Abstract

The present study focusses on the existence of positivity of the solutions to the higher order three-point boundary value problems involving p-Laplacian

$$[\phi_p(x^{(m)}(t))]^{(n)} = g(t, x(t)),\ t \in [0,1],$$

$$\begin{aligned} x^{(i)}(0) = 0, \text{ for } 0 \le i \le m-2,\\ x^{(m-2)}(1) - \alpha x^{(m-2)}(\xi) = 0,\\ [\phi_p(x^{(m)}(t))]^{(j)}_{\text{at } t=0} = 0,\ \text{for } 0 \le j \le n-2,\\ [\phi_p(x^{(m)}(t))]^{(n-2)}_{\text{at } t=1} - \alpha[\phi_p(x^{(m)}(t))]^{(n-2)}_{\text{at } t=\xi} = 0, \end{aligned}$$

where $m, n \ge 3$, $\xi \in (0,1)$, $\alpha \in (0, \frac{1}{\xi})$ is a parameter. The approach used by the application of Guo–Krasnosel'skii fixed point theorem to determine the existence of positivity of the solutions to the problem.

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1. Introduction

Differential equations have created a tremendous amount of interest and played a vital role in many areas of mathematical sciences. The theory of differential equations gives profound and wide mathematical support for addressing many emerging issues of present society that are challenging and multidisciplinary in the universe. In this theory, one of the significant and useful operators is one-dimensional p-Laplacian operator and is defined as $\phi_p(\tau) = |\tau|^{p-2}\tau$, where $p > 1$, $\phi_p^{-1} = \phi_q$ and $\frac{1}{p} + \frac{1}{q} = 1$. Recently, researchers have given great consideration to study p-Laplacian problems due to the wide applicability in various real time applications such as biophysics, plasma physics, image processing, rheology, glaciology, turbulent filtration in porous media, radiation of heat etc. We mention a few papers devoted to p-Laplacian problems, see [3, 4, 19, 26, 20, 10, 13] for the existence of positivity of the solutions. For applications and recent developments, we refer [7, 1, 2, 5, 9, 15, 16, 18, 28].

We consider higher order three-point boundary value problems involving p-Laplacian of the form

$$[\phi_p(x^{(m)}(t))]^{(n)} = g(t, x(t)),\ t \in [0, 1], \tag{1}$$

$$\left.\begin{aligned} x^{(i)}(0) = 0, \text{ for } 0 \le i \le m-2, \\ x^{(m-2)}(1) - \alpha x^{(m-2)}(\xi) = 0, \\ [\phi_p(x^{(m)}(t))]^{(j)}_{\text{at } t=0} = 0,\ \text{ for } 0 \le j \le n-2, \\ [\phi_p(x^{(m)}(t))]^{(n-2)}_{\text{at } t=1} - \alpha[\phi_p(x^{(m)}(t))]^{(n-2)}_{\text{at } t=\xi} = 0, \end{aligned}\right\} \tag{2}$$

where $m, n \ge 3$, $\xi \in (0, 1)$, $\alpha \in (0, \frac{1}{\xi})$ is a parameter, and the function $g : [0, 1] \times \mathbb{R}^+ \to \mathbb{R}^+$ is continuous and establish the existence of positivity of the solution by using fixed point theorem of Guo–Krasnosel'skii. If $p = 2$, we get various order three-point boundary value problems by giving different values to m and n. In the past, most researchers have focussed and demonstrated the positivity results for boundary value problems of third order three-point using various methods, see [11, 29, 33, 35, 22, 23, 25, 27, 34, 24, 39]. However, some works on positivity results have been found for n^{th}, $2n^{\text{th}}$ and $3n^{\text{th}}$ order p-Laplacian boundary value problems, see [21, 12, 32, 38, 8, 36, 37, 30, 31]. Motivated by the aforementioned papers, we then extend the results to mn^{th} order p-Laplacian problem stated in (1), (2).

For establishing the results, assume the following condition is fulfilled in the entire paper:

(F1) α is a parameter such that $0 < \alpha\xi < 1$, where $\xi \in (0, 1)$.

The remaining portion of the paper is structured as below. With the aid of Green functions, the solution of p-Laplacian problem stated in (1) and (2) is expressed as a solution to an analogous integral equation and then some inequalities for these Green functions are established in Section 2. The existence of the positivity results of the problem (1)-(2) is established and the results are validated by an example in Section 3.

2. Preliminaries

This section contains preparatory results that are necessary to demonstrate the existence results.

We first express the solution of the following m^{th} order non-homogeneous problem of three-point

$$x^{(m)}(t) + \varphi(t) = 0,\ t \in [0, 1], \tag{3}$$

$$\left.\begin{aligned} x^{(i)}(0) = 0,\ \text{ for } 0 \le i \le m-2, \\ x^{(m-2)}(1) - \alpha x^{(m-2)}(\xi) = 0, \end{aligned}\right\} \tag{4}$$

where $\varphi(t) \in C([0, 1], R^+)$, in terms of Green's function $\overline{G}_m(t, s)$ as a solution of an analogous integral equation. By taking $y(t) = \phi_p(x^{(m)}(t))$, the solution of n^{th} order non-homogeneous problem of three-point

$$y^{(n)}(t) + \psi(t) = 0,\ t \in [0, 1], \tag{5}$$

$$\left.\begin{aligned} y^{(j)}(0) = 0,\ \text{for } 0 \leq j \leq n-2, \\ y^{(n-2)}(1) - \alpha y^{(n-2)}(\xi) = 0, \end{aligned}\right\} \tag{6}$$

where $\psi(t) \in C([0,1], R^+)$, is expressed in terms of Green's function $G_n(t,s)$ as a solution of an analogous integral equation.

Lemma 2.1. *[34] If the condition (F1) is fulfilled, then the solution of the problem stated in* (3), (4) *is*

$$x(t) = \int_0^1 \overline{G}_m(t,s)\varphi(s)ds,$$

where

$$\overline{G}_m(t,s) = H(t,s) + \frac{\alpha t^{m-1}}{(m-1)!\,(1-\xi\alpha)} G(\xi,s), \tag{7}$$

$$H(t,s) = \frac{1}{(m-1)!} \begin{cases} t^{m-1}(1-s),\ 0 \leq t \leq s \leq 1, \\ [t^{m-1}(1-s) - (t-s)^{m-1}],\ 0 \leq s \leq t \leq 1, \end{cases}$$

and

$$G(\xi,s) = \begin{cases} (1-\xi)s,\ 0 \leq s \leq \xi \leq 1, \\ (1-s)\xi,\ 0 \leq \xi \leq s \leq 1. \end{cases}$$

Lemma 2.2. *[34] If the condition (F1) is fulfilled, then the solution of the problem stated in* (5), (6) *is*

$$y(t) = \int_0^1 G_n(t,s)\psi(s)ds,$$

where

$$G_n(t,s) = K(t,s) + \frac{\alpha t^{n-1}}{(n-1)!\,(1-\xi\alpha)} G(\xi,s), \tag{8}$$

$$K(t,s) = \frac{1}{(n-1)!} \begin{cases} t^{n-1}(1-s),\ 0 \leq t \leq s \leq 1, \\ [t^{n-1}(1-s) - (t-s)^{n-1}],\ 0 \leq s \leq t \leq 1, \end{cases}$$

and

$$G(\xi,s) = \begin{cases} (1-\xi)s,\ 0 \leq s \leq \xi \leq 1, \\ (1-s)\xi,\ 0 \leq \xi \leq s \leq 1. \end{cases}$$

Using Lemmas 2.1 and 2.2, the solution of the problem stated in (1), (2) is

$$x(t) = \int_0^1 \overline{G}_m(t,s)\phi_q\left[\int_0^1 G_n(s,r)g(r,x(r))dr\right]ds. \tag{9}$$

Lemma 2.3. *[34] If the condition (F1) is fulfilled, then $\overline{G}_m(t,s)$ in* (7) *fulfills the subsequent conditions:*

(i) $\overline{G}_m(t,s) \geq 0$, *for all* $t \in [0,1]$ *and* $s \in [0,1]$,

(ii) $\overline{G}_m(t,s) \leq \overline{G}_m(1,s)$, *for all* $t \in [0,1]$ *and* $s \in [0,1]$,

(iii) $\min\limits_{t\in[\xi,1]} \overline{G}_m(t,s) \geq \xi^{m-1}\overline{G}_m(1,s)$, *for all* $s \in [0,1]$.

Lemma 2.4. *[34] If the condition (F1) is fulfilled, then $G_n(t,s)$ in* (8) *fulfills the subsequent conditions:*

(i) $G_n(t,s) \geq 0$, *for all* $t \in [0,1]$ *and* $s \in [0,1]$,

(ii) $G_n(t,s) \le G_n(1,s)$, *for all* $t \in [0,1]$ *and* $s \in [0,1]$,

(iii) $\min_{t\in[\xi,1]} G_n(t,s) \ge \xi^{n-1} G_n(1,s)$, *for all* $s \in [0,1]$.

The fixed point theorem of Guo–Krasnosel'skii mentioned below is often used as the fundamental tool to establish the positivity results of the problem stated in (1), (2).

Theorem 2.5. *[6, 14, 17] Let* X *be a Banach Space and the set* $\kappa \subseteq \mathsf{X}$ *be a cone. Suppose the sets* Ω_1 *and* Ω_2 *are any two open subsets of* X *such that* $0 \in \Omega_1$ *and* $\overline{\Omega}_1 \subset \Omega_2$. *If* $\mathsf{F} : \kappa \cap (\overline{\Omega}_2 \backslash \Omega_1) \to \kappa$ *is a completely continuous operator such that, either*

(*i*) $\|\mathsf{F}x\| \le \|x\|,\ x \in \kappa \cap \partial\Omega_1$ *and* $\|\mathsf{F}x\| \ge \|x\|,\ x \in \kappa \cap \partial\Omega_2$, *or*

(*ii*) $\|\mathsf{F}x\| \ge \|x\|,\ x \in \kappa \cap \partial\Omega_1$ *and* $\|\mathsf{F}x\| \le \|x\|,\ x \in \kappa \cap \partial\Omega_2$ *holds.*

Then the operator F *has a fixed point in* $\kappa \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

3. Existence of Positivity of the Solutions

This section presents the existence of positivity results of the problem stated in (1), (2).

For our construction, let us take $\mathsf{X} = \{x : x \in \mathsf{C}[0,1]\}$, a Banach space with the standard norm $\|x\| = \max_{t\in[0,1]} |x(t)|$. Consider a set κ as

$$\kappa = \{x \in \mathsf{X} : x(t) \ge 0 \text{ for } t \in [0,1] \text{ and } \min_{t\in[\xi,1]} x(t) \ge \mathcal{M}\|x\|\},$$

where

$$\mathcal{M} = \min\{\xi^{m-1}, \xi^{n-1}\}. \tag{10}$$

Then, it is obvious that the set κ is a cone in X. To establish the results, we use the operator $\mathsf{F} : \kappa \to \mathsf{X}$ by defining as

$$\mathsf{F}x(t) = \int_0^1 \overline{G}_m(t,s)\phi_q\left[\int_0^1 G_n(s,r) g(r, x(r)) dr\right] ds. \tag{11}$$

Define the non-negative extended real numbers g_0, g^0, g_∞ and g^∞ by

$$g_0 = \lim_{x\to 0^+} \min_{t\in[0,1]} \frac{g(t,x)}{\phi_p(x)},\ g^0 = \lim_{x\to 0^+} \max_{t\in[0,1]} \frac{g(t,x)}{\phi_p(x)},$$

$$g_\infty = \lim_{x\to\infty} \min_{t\in[0,1]} \frac{g(t,x)}{\phi_p(x)} \text{ and } g^\infty = \lim_{x\to\infty} \max_{t\in[0,1]} \frac{g(t,x)}{\phi_p(x)},$$

and assume that the above are exist. The case $g^0 = 0$ and $g_\infty = \infty$ represents superlinear and the case $g_0 = \infty$ and $g^\infty = 0$ represents the sublinear.

We also consider the following conditions are fulfilled in this paper:

(F2) $0 < \int_0^1 \overline{G}_m(t,s)ds < \infty$ and $0 < \int_0^1 G_n(t,s)ds < \infty$, and

(F3) the function $g(t,x)$ is a non-decreasing for the second variable x.

Lemma 3.1. *If* $\mathsf{F} : \kappa \to \mathsf{X}$ *is given in* (11) *then* F *is a self map on the cone* κ.

Proof. By using the Lemmas 2.3, 2.4 and the condition (F2), $\mathsf{F}x(t) \geq 0$ for $x \in \kappa$ and $t \in [0,1]$. Then, by Lemma 2.3 and for $x \in \kappa$, we obtain

$$\begin{aligned}\mathsf{F}x(t) &= \int_0^1 \overline{G}_m(t,s)\phi_q\Big(\int_0^1 G_n(s,r)g\big(r,x(r)\big)dr\Big)ds\\ &\leq \int_0^1 \overline{G}_m(1,s)\phi_q\Big(\int_0^1 G_n(s,r)g\big(r,x(r)\big)dr\Big)ds\end{aligned}$$

so that

$$\|\mathsf{F}x\| \leq \int_0^1 \overline{G}_m(1,s)\phi_q\Big(\int_0^1 G_n(s,r)g\big(r,x(r)\big)dr\Big)ds. \tag{12}$$

Then, by Lemma 2.3 and (10), for $x \in \kappa$ that

$$\begin{aligned}\min_{t\in[\xi,1]} \mathsf{F}x(t) &= \min_{t\in[\xi,1]}\left\{\int_0^1 \overline{G}_m(t,s)\phi_q\Big(\int_0^1 G_n(s,r)g\big(r,x(r)\big)dr\Big)ds\right\}\\ &\geq \xi^{m-1}\int_0^1 \overline{G}_m(1,s)\phi_q\Big(\int_0^1 G_n(s,r)g\big(r,x(r)\big)dr\Big)ds\\ &\geq \mathcal{M}\int_0^1 \overline{G}_m(1,s)\phi_q\Big(\int_0^1 G_n(s,r)g\big(r,x(r)\big)dr\Big)ds\\ &\geq \mathcal{M}\|Fx\|.\end{aligned}$$

Hence, the operator F is a self map on a cone κ. □

Moreover by applying Arzela-Ascoli theorem, F is a completely continuous operator. The existence of positivity results of the p-Laplacian problem stated in (1), (2) for both the superlinear case as well as the sublinear case is now established.

Theorem 3.2. *Suppose the assumptions (F1), (F2) and (F3) are fulfilled. If the conditions $g^0 = 0$ and $g_\infty = \infty$ hold, then the nonlinear p-Laplacian problem stated in* (1), (2) *has at least one positive solution and it lies in the cone κ.*

Proof. From the definition of $g^0 = 0$, there exist $\rho_1 > 0$ and $\mathcal{H}_1 > 0$ such that

$$g(t,x) \leq \rho_1\phi_p(x), \text{ for } 0 < x \leq \mathcal{H}_1,$$

where ρ_1 satisfies

$$(\rho_1)^{q-1}\int_0^1 \overline{G}_m(1,s)\phi_q\Big(\int_0^1 G_n(1,r)dr\Big)ds \leq 1. \tag{13}$$

Let $x \in \kappa$ and $\|x\| = \mathcal{H}_1$. Then, for $t \in [0,1]$ and by Lemmas 2.3, 2.4, we get

$$\begin{aligned}Fx(t) &= \int_0^1 \overline{G}_m(t,s)\phi_q\Big(\int_0^1 G_n(s,r)g\big(r,x(r)\big)dr\Big)ds\\ &\leq \int_0^1 \overline{G}_m(1,s)\phi_q\Big(\int_0^1 G_n(1,r)\rho_1\phi_p(x)dr\Big)ds\\ &\leq (\rho_1)^{q-1}\int_0^1 \overline{G}_m(1,s)\phi_q\Big(\int_0^1 G_n(1,r)dr\Big)ds\|x\|\\ &\leq \|x\|.\end{aligned}$$

Hence, $\|\mathsf{F}x\| \leq \|x\|$. Now, if we are setting

$$\Omega_1 = \{x \in \mathsf{X} : \|x\| < \mathcal{H}_1\}$$

then

$$\|\mathsf{F}x\| \leq \|x\|, \text{ for } x \in \kappa \cap \partial\Omega_1. \tag{14}$$

Next, since $g_\infty = \infty$, there exist $\rho_2 > 0$ and $\bar{\mathcal{H}}_2 > 0$ such that

$$g(t, x(t)) \geq \rho_2 \phi_p(x), \text{ for } x \geq \bar{\mathcal{H}}_2,$$

where ρ_2 satisfies

$$(\rho_2)^{q-1}\mathcal{M}^2 \int_{s\in[\xi,1]} \overline{G}_m(1,s)\phi_q\Big(\mathcal{M}\int_{r\in[\xi,1]} G_n(1,r)dr\Big)ds \geq 1. \tag{15}$$

Let $\mathcal{H}_2 = \max\left\{2\mathcal{H}_1, \frac{\bar{\mathcal{H}}_2}{\mathcal{M}}\right\}$. Choose $x \in \kappa$ and $\|x\| = \mathcal{H}_2$. Then

$$\min_{t\in[\xi,1]} x(t) \geq \mathcal{M}\|x\| \geq \bar{\mathcal{H}}_2.$$

By the Lemmas 2.3, 2.4 and (10), and for $t \in [0, 1]$, we obtain

$$\begin{aligned}
\mathsf{F}x(t) &= \int_0^1 \overline{G}_m(t,s)\phi_q\Big(\int_0^1 G_n(s,r)g(r,x(r))dr\Big)ds \\
&\geq \min_{t\in[\xi,1]}\left\{\int_0^1 \overline{G}_m(t,s)\phi_q\Big(\int_0^1 G_n(s,r)g\big(r,x(r)\big)dr\Big)ds\right\} \\
&\geq \mathcal{M}\int_0^1 \overline{G}_m(1,s)\phi_q\Big(\int_0^1 G_n(s,r)g(r,x(r))dr\Big)ds \\
&\geq \mathcal{M}\int_{s\in[\xi,1]} \overline{G}_m(1,s)\phi_q\Big(\mathcal{M}\int_{r\in[\xi,1]} G_n(1,r)\rho_2\phi_p(x)dr\Big)ds \\
&\geq \mathcal{M}(\rho_2)^{q-1}\int_{s\in[\xi,1]} \overline{G}_m(1,s)\phi_q\Big(\mathcal{M}\int_{r\in[\xi,1]} G_n(1,r)dr\Big)\mathcal{M}\|x\|ds \\
&\geq (\rho_2)^{q-1}\mathcal{M}^2\int_{s\in[\xi,1]} \overline{G}_m(1,s)\phi_q\Big(\mathcal{M}\int_{r\in[\xi,1]} G_n(1,r)dr\Big)\|x\|ds \\
&\geq \|x\|.
\end{aligned}$$

Therefore, $\|\mathsf{F}x\| \geq \|x\|$. So, if we take

$$\Omega_2 = \{x \in \mathsf{X} : \|x\| < \mathcal{H}_2\}$$

then

$$\|\mathsf{F}x\| \geq \|x\| \text{ for } x \in \kappa \cap \partial\Omega_2. \tag{16}$$

By Theorem 2.5 to (14) and (16), it follows that the operator F has a fixed point $x \in \kappa \cap (\Omega_2 \setminus \bar{\Omega}_1)$ and that fixed point x is the positive solution of the p-Laplacian problem (1)-(2). □

Theorem 3.3. *Suppose the assumptions (F1), (F2) and (F3) are fulfilled. If the conditions $g_0 = \infty$ and $g^\infty = 0$ hold, then the nonlinear p-Laplacian problem stated in* (1), (2) *has at least one positive solution and it lies in the cone κ.*

Proof. From the definition of $g_0 = \infty$, there exist $\bar{\rho}_1 > 0$ and $\mathcal{R}_1 > 0$ such that

$$g(t, x) \geq \bar{\rho}_1\phi_p(x), \text{ for } 0 < x \leq \mathcal{R}_1,$$

where $\bar{\rho}_1 \geq \rho_2$ and ρ_2 is given in (15).

Let $x \in \kappa$ and $\|x\| = \mathcal{R}_1$. Then, by Lemmas 2.3, 2.4 and (10), and for $t \in [0, 1]$, we obtain

$$\begin{aligned}
\mathsf{F}x(t) &= \int_0^1 \overline{G}_m(t,s)\phi_q\Big(\int_0^1 G_n(s,r)g\big(r,x(r)\big)dr\Big)ds \\
&\geq \min_{t\in[\xi,1]}\left\{\int_0^1 \overline{G}_m(t,s)\phi_q\Big(\int_0^1 G_n(s,r)g\big(r,x(r)\big)dr\Big)ds\right\} \\
&\geq \mathcal{M}\int_{s\in[\xi,1]} \overline{G}_m(1,s)\phi_q\Big(\int_{r\in[\xi,1]} G_n(s,r)\bar{\rho}_1\phi_p(x)dr\Big)ds \\
&\geq \mathcal{M}\int_{s\in[\xi,1]} \overline{G}_m(1,s)\phi_q\Big(\mathcal{M}\int_{r\in[\xi,1]} G_n(1,r)\bar{\rho}_1\phi_p(x)dr\Big)ds \\
&\geq \mathcal{M}(\bar{\rho}_1)^{q-1}\int_{s\in[\xi,1]} \overline{G}_m(1,s)\phi_q\Big(\mathcal{M}\int_{r\in[\xi,1]} G_n(1,r)dr\Big)\mathcal{M}\|x\|ds \\
&= (\bar{\rho}_1)^{q-1}\mathcal{M}^2\int_{s\in[\xi,1]} \overline{G}_m(1,s)\phi_q\Big(\mathcal{M}\int_{r\in[\xi,1]} G_n(1,r)dr\Big)\|x\|ds \\
&\geq \|x\|.
\end{aligned}$$

Therefore, $\|\mathsf{F}x\| \geq \|x\|$. Now, if we take

$$\Omega_3 = \{x \in \mathsf{X} : \|x\| < \mathcal{R}_1\}$$

then

$$\|\mathsf{F}x\| \geq \|x\|, \text{ for } x \in \kappa \cap \partial\Omega_3. \tag{17}$$

Next, since $g^\infty = 0$, there exist $\bar{\rho}_2 > 0$ and $\bar{\mathcal{R}}_2 > 0$ such that

$$g(t, x(t)) \leq \bar{\rho}_2\phi_p(x), \text{ for } x \geq \bar{\mathcal{R}}_2,$$

where $\bar{\rho}_2 \leq \rho_1$ and ρ_1 is given in (13).

Set

$$g^*(t,x) = \sup_{0\leq s\leq x} g(t,s).$$

Then, it is evident that the real-valued function g^* is a non-decreasing, $g \leq g^*$ and

$$\lim_{x\to\infty}\frac{g^*(t,x)}{x} = 0.$$

It follows that there exists $\mathcal{R}_2 > \max\{2\mathcal{R}_1, \bar{\mathcal{R}}_2\}$ such that

$$g^*(t,x) \leq g^*(t,\mathcal{R}_2), \text{ for } 0 < x \leq \mathcal{R}_2.$$

Choose $x \in \kappa$ with $\|x\| = \mathcal{R}_2$. Then, we get

$$\begin{aligned}
\mathsf{F}x(t) &= \int_0^1 \overline{G}_m(t,s)\phi_q\Big(\int_0^1 G_n(s,r)g(r,x(r))dr\Big)ds \\
&\leq \int_0^1 \overline{G}_m(1,s)\phi_q\Big(\int_0^1 G_n(s,r)g(r,\mathcal{J}_2)dr\Big)ds \\
&\leq \int_0^1 \overline{G}_m(1,s)\phi_q\Big(\int_0^1 G_n(s,r)\bar{\rho}_2\phi_p(\mathcal{J}_2)dr\Big)ds \\
&\leq (\bar{\rho}_2)^{q-1}\int_0^1 \overline{G}_m(1,s)\phi_q\Big(\int_0^1 G_n(1,r)dr\Big)ds\mathcal{R}_2 \\
&\leq \mathcal{R}_2 = \|x\|.
\end{aligned}$$

Hence, $\|\mathsf{F}x\| \leq \|x\|$. Now, if we take

$$\Omega_4 = \{x \in \mathsf{X} : \|x\| < \mathcal{R}_2\}$$

then

$$\|\mathsf{F}x\| \leq \|x\|, \text{ for } x \in \kappa \cap \partial\Omega_4. \tag{18}$$

Applying by Theorem 2.5 to (17) and (18), we have that the operator F has a fixed point $x \in \kappa \cap (\Omega_4 \setminus \bar{\Omega}_3)$ and that fixed point x is the positive solution of the p-Laplacian problem (1)-(2). □

We consider the example to validate the established results.

Example 3.4. Let us take $m = 4$, $n = 3$ and $\xi = \frac{1}{3}$. Now, consider a nonlinear p-Laplacian problem

$$[\phi_p(x^{(4)}(t))]^{(3)} = g(t, x(t)), \ t \in [0, 1], \tag{19}$$

satisfying

$$\left.\begin{aligned} x(0) = 0, \ x^{(1)}(0) = 0, \ x^{(2)}(0) = 0, \ x^{(2)}(1) - 2x^{(2)}(\frac{1}{3}) = 0, \\ [\phi_p(x^{(4)}(t))]_{\text{at } t=0} = 0, \ [\phi_p(x^{(4)}(t))]'_{\text{at } t=0} = 0, \\ [\phi_p(x^{(4)}(t))]'_{\text{at } t=1} - 2[\phi_p(x^{(4)}(t))]'_{\text{at } t=\xi=\frac{1}{3}} = 0. \end{aligned}\right\} \tag{20}$$

Let us take $p = 2$ for simplicity. By algebraic computations, we get

$$\mathcal{M} = 0.1111.$$

(a) If we take $g(t, x(t)) = x^2(1 - e^{-2t})$, then $g^0 = 0$ and $g_\infty = \infty$. So, all the claims in the Theorem 3.2 are fulfilled. Therefore, the boundary value problem (19)-(20) has at least one positive solution.

(b) If we take $g(t, x(t)) = \frac{\sqrt{(t^2+1)}}{t^2x^2}$, then $g_0 = \infty$ and $g^\infty = 0$. So, all the claims in the Theorem 3.3 are fulfilled. Therefore, the boundary value problem (19)-(20) has at least one positive solution.

4. Conclusion

In this paper, we established the existence of positive solutions to the higher order three-point boundary value problem involving p-Laplacian operator by an application of the Guo–Krasnosel'skii fixed point theorem for operators on a cone in a Banach space.

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