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On some inequalities for derivatives of algebraic polynomials in unbounded regions with angles

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ABSTRACT

In this work we study Bernstein-Walsh-type estimations for the derivative of an arbitrary algebraic polynomial in regions with interior zero and exterior non zero angles.

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1. Introduction

Let $\mathbb{C}$ denote the complex plane and $\overline{\mathbb{C}}:=\mathbb{C}\cup\{\infty\}$; $G\subset\mathbb{C}$ be a bounded Jordan region with boundary $L:=\partial G$ such that $0\in G$; Let $\{z_j\}_{j=1}^{l}$ be the fixed system of distinct points on the curve L. We consider generalized Jacobi weight function $h(z)$ which is defined as follows:

$$h(z):=\prod_{j=1}^{l}\left|z-z_j\right|^{\gamma_j},\quad z\in\mathbb{C}, \tag{1}$$

where $\gamma_j>-2$, for all $j=1,2,\dots,l$.
Let $\wp_n$ denotes the class of all algebraic polynomials $P_n(z)$ of degree at most $n\in\mathbb{N}$.
Let $p>0$. For the Jordan region G, we introduce:

$$\|P_n\|_p:=\|P_n\|_{A_p(h,G)}:=\left(\iint_G h(z)|P_n(z)|^p d\sigma_z\right)^{1/p}, 0<p<\infty, \tag{2}$$

$$\|P_n\|_\infty:=\|P_n\|_{A_\infty(1,G)}:=\max_{z\in\overline{G}}|P_n(z)|, p=\infty,$$

and $A_p(1,G)\equiv A_p(G)$, where σ be the two-dimensional Lebesgue measure.
When L is rectifiable, for any $p>0$, let

$$\|P_n\|_{\mathcal{L}_p(h,L)}:=\left(\int_L h(z)|P_n(z)|^p|dz|\right)^{1/p}<\infty,\ 0<p<\infty, \tag{3}$$

$$\|P_n\|_{\mathcal{L}_\infty(1,L)}:=\max_{z\in L}|P_n(z)|, p=\infty,$$

and $\mathcal{L}_p(1,L)\equiv\mathcal{L}_p(L)$.
Let us set $\Omega:=\overline{\mathbb{C}}\setminus\overline{G}=extL$; $\Delta(w,R):=\{w:|w|>R,R>1\},\Delta:=\Delta(0,1)$ and let $w=\Phi(z)$ be the univalent conformal

mapping of Ω onto Δ such that $\Phi(\infty)=\infty$ and $\lim_{z\to\infty}\frac{\Phi(z)}{z}>0$; $\Psi:=\Phi^{-1}$. For $R>1$ we define $L_R:=\{z:|\Phi(z)|=R\}, G_R:=intL_R, \Omega_R:=extL_R$.
Well known Bernstein -Walsh Lemma [26] says that:

$$\|P_n\|_{C(\overline{G}_R)}\le R^n\|P_n\|_{C(\overline{G})}. \quad (4)$$

Analogous estimation with respect to the quasinorm (4) for $p>0$ was obtained in [19] for $h(z)\equiv 1$ (i.e., $\gamma_j=0$ for all $j=1,2,\ldots,l$) and in [8, Lemma 2.4] for $h(z)\neq 1$, defined as in (1) as following:

$$\|P_n\|_{\mathcal{L}_p(h,L_R)}\le R^{n+\frac{1+\gamma^*}{p}}\|P_n\|_{\mathcal{L}_p(h,L)},\ \gamma^*=\max\{0;\gamma_j:1\le j\le l\}. \quad (5)$$

To give a similar estimation to (5) for the $A_p(h,G)$ −norm, first of all we will give the following definition.

Definition 1. [20, p.97], [23] The Jordan arc (or curve) L is called K −*quasiconformal* ($K\ge 1$), if there is a K −quasiconformal mapping f of the region D ⊃ L such that f(L) is a line segment (or circle).
Let $F(L)$ denote the set of all sense preserving plane homeomorphisms f of the region $D\supset L$ such that $f(L)$ is a line segment (or circle) and let

$$K_L:=\inf\{K(f):f\in F(L)\},$$

where $K(f)$ is the maximal dilatation of f. Then L is a quasiconformal curve, if $K_L<\infty$, and L is a K −quasiconformal curve, if $K_L\le K$.

A curve L is called a *quasiconformal*, if it is a K −*quasiconformal* for some $K>1$.
The Bernstein-Walsh type estimates for the norm (2), for the regions with quasiconformal boundary and weight function $h(z)$, defined in (1) with $\gamma_j>-2$, for all $p>0$ as follows

$$\|P_n\|_{A_p(h,G_R)}\le c_1 {R^*}^{n+\frac{1}{p}}\|P_n\|_{A_p(h,G)}, \quad (6)$$

was found in [3] (see, also [2]), where $R^*:=1+c_2(R-1)$, $c_2>0$ and $c_1:=c_1(G,p,c_2)>0$ constants, independent from n and R. It's well known that quasiconformal curves can be non-rectifiable (see, for example, [16], [20, p.104]).
Analogous estimation was studied for $A_p(1,G)$-norm, $p>0$, for arbitrary Jordan region in [4, Theorem1.1] and for any $P_n\in\wp_n$, $R_1=1+\frac{1}{n}$ and arbitrary R, $R>R_1$, was obtain

$$\|P_n\|_{A_p(G_R)}\le cR^{n+\frac{2}{p}}\|P_n\|_{A_p(G_{R_1})},$$

where $c=\left(\frac{2}{e^p-1}\right)^{\frac{1}{p}}\left[1+O(\frac{1}{n})\right],\ n\to\infty$.
For a rectifiable quasiconformal curve L, N. Stylianopoulos [24] obtained the following estimate:

$$|P_n(z)|\le c\frac{\sqrt{n}}{d(z,L)}\|P_n\|_{A_2(G)}|\Phi(z)|^{n+1},\quad z\in\Omega, \quad (7)$$

where $d(z,L):=\inf\{|\zeta-z|:\ \zeta\in L\}$, a constant $c=c(L)>0$ depending only on L.
Analogous results of (7)-type for $|P_n(z)|$, different weight function h, unbounded region Ω were obtained in [17, p.418-428], [5], [6], [7], [8], [9], [10], [11], [15], [22] and others.
In this work, we study the pointwise estimations for the derivative $|P_n'(z)|$ in unbounded region Ω with zero angles as the following type

$$|P_n'(z)|\le c_2\eta_n(G,h,p,d(z,L),|\Phi(z)|)\|P_n\|_p\ ,z\in\Omega, \quad (8)$$

where $c_2=c_2(G,p)>0$ is a constant independent of n,Z and P_n, and $\eta_n(G,h,p,d(z,L),|\Phi(z)|)\to\infty$, $n\to\infty$, depending on the properties of the G, h and from the distance of point $z\in\Omega$ to the $\overline{G}$.

2. Definitions and main results

Throughout this paper, $c, c_0, c_1, c_2, \dots$ are positive and $\varepsilon_0, \varepsilon_1, \varepsilon_2, \dots$ are sufficiently small positive constants (generally, different in different relations), which depends on G in general and, on parameters inessential for the argument, otherwise, the dependence will be explicitly stated. For any $k \geq 0$ and $m > k$, notation $i = \overline{k, m}$ means $i = k, k+1, \dots, m$.
Let $z = z(s),\ s \in [0, mesL]$ denote the natural representation of L .

Definition 2. We say that $L \in C_\theta,\ if\ L$ has a continuous tangent $\theta(z) := \theta(z(s))$ at every point $z(s)$. Then we write $G \in C_\theta \Leftrightarrow \partial G \in C_\theta$.

According to the "three-point" criterion [13, p.100], every piecewise smooth curve (without any cusps) is quasiconformal. Moreover, according to [23], we have the following:

Corollary 3. *If $G \in C_\theta$, then ∂G is $(1+\varepsilon)$ −quasiconformal for arbitrary small $\varepsilon > 0$.*

Now we give the definitions of regions with a piecewise smooth curve, which we present our main result and some notation that will be used later in the text.
Definition 4. [5] We say that a Jordan region $G \in C_\theta(\lambda_1, \dots, \lambda_l),\ 0 < \lambda_j \leq 2,\ j = \overline{1, l}$, if $L = \partial G$ consists of the union of finite smooth arcs $\{L_j\}_{j=1}^{l}$, such that they have exterior (with respect to $\overline{G}$) angles $\lambda_j \pi,\ 0 < \lambda_j \leq 2$, at the corner points $\{z_j\}_{j=1}^{l} \in L$, where two arcs meet.

Without loss of generality, we assume that these points on the curve $L = \partial G$ are located in the positive direction such that, G has exterior $\lambda_j \pi,\ 0 < \lambda_j < 2,\ j = \overline{0, l_1}$, angle at the points $\{z_j\}_{j=1}^{l_1},\ l_1 \leq l$, and interior zero angle (i.e. $\lambda_j = 2$ −interior cusps) at the points $\{z_j\}_{j=l_1+1}^{l}$.

It is clear from Definition 4, the each region $G \in C_\theta(\lambda_1, \dots, \lambda_l),\ 0 < \lambda_j \leq 2,\ j = \overline{1, l}$, may have exterior nonzero $\lambda_j \pi,\ 0 < \lambda_j < 2$, angles at the points $\{z_j\}_{j=1}^{l_1} \in L$, and interior zero angles ($\lambda_j = 2$) at the the points $\{z_j\}_{j=l_1+1}^{l} \in L$. If $l_1 = l = 0$, then the region G doesn't have such angles, and in this case we will write: $G \in C_\theta$; if $l_1 = l \geq 1$, then G has only $\lambda_i \pi,\ 0 < \lambda_i < 2$, $i = \overline{1, l_1}$, exterior nonzero angles, and in this case we will write: $G \in C_\theta(\lambda_i)$; if $l_1 = 0$ and $l \geq 1$, then G has only interior zero angles, and in this case we will write: $G \in C_\theta(2)$.

Throughout this work, we will assume that the points $\{z_j\}_{j=1}^{l} \in L$ defined in (1) and Definition 4 are identical and $w_j := \Phi(z_j)$. For simplicity of exposition and in order to avoid cumbersome calculations, without loss of generality, we will take $l_1 = 1, l = 2$. Then, after this assumption, in the future we will have region $G \in C_\theta(\lambda_1, 2),\ 0 < \lambda_1 < 2$, such that at the point $z_1 \in L$ region G have exterior nonzero $\lambda_1 \pi,\ 0 < \lambda_1 < 2$, and at the point $z_2 \in L$ - interior zero angle. Note that, the notation "$G \in C_\theta(\lambda_1, \lambda_2)$, $0 < \lambda_1, \lambda_2 < 2$" means that the region G has two exterior nonzero $\lambda_j \pi, 0 < \lambda_j < 2, j = 1,2$, angles at the point $z_j \in L$.
For $0 < \delta_j < \delta_0 := \frac{1}{4}\min\{|z_1 - z_2| : j = 1,2\}$, $\delta := \min_{1 \leq j \leq 2} \delta_j$, let

$$\Omega(z_j,\ \delta_j) := \Omega \cap \{z : |z - z_j| \leq \delta_j\}; \qquad (9)$$
$$\Omega(\delta) := \cup_{j=1}^{2} \Omega(z_j,\ \delta), \widehat{\Omega} := \Omega \setminus \Omega(\delta).$$

In this work, we study problem of (8) type in regions with piecewise smooth boundary without exterior cusps and generalized Jacobi weight function $h(z)$, as defined in (1).
Now, we start to formulate the new results.

Theorem 5. *Let $p > 1$; $G \in C_\theta(\lambda_1, 2)$, for some $0 < \lambda_1 < 2$; $h(z)$ be defined as in (1). Then, for any $P_n \in \wp_n,\ n \in \mathbb{N},\ \gamma_j > -2$ and arbitrary small $\varepsilon > 0$*

$$|P_n'(z)| \le c_1 \left[\frac{|\Phi(z)|^{n+1}}{d(z,L_{R_1})} G_{n,1}(z) + \frac{|\Phi(z)|^{2(n+1)}}{d^{2/p}(z,L_{1+1/n})} B_{n,1}(z) E_{n,1}\right] \|P_n\|_p$$

holds, where $c_1 = c_1(G, \gamma_i, p, \varepsilon) > 0$;

$$G_{n,1}(z) := \begin{cases} n^{\frac{\gamma_1+2}{p}\tilde{\lambda}_1}, & \gamma_1 \ge \frac{2}{\tilde{\lambda}_1}(\gamma_2+2)-2, \gamma_2 \ge \frac{\tilde{\lambda}_1}{2\lambda_1}-2, & z \in \Omega(\delta), \\ n^{\frac{\gamma_2+2}{p}2}, & \frac{1}{\lambda_1}-2 \le \gamma_1 < \frac{2}{\tilde{\lambda}_1}(\gamma_2+2)-2, \gamma_2 \ge \frac{\tilde{\lambda}_1}{2\lambda_1}-2, & z \in \Omega(\delta), \\ n^{\frac{\gamma_2+2}{p}2}, & \gamma_1 < \frac{1}{\lambda_1}-2, \gamma_2 \ge \frac{\tilde{\lambda}_1}{2\lambda_1}-2, & z \in \Omega(\delta), \\ 1, & \gamma_1 < \frac{1}{\lambda_1}-2, \gamma_2 < -\frac{3}{2}, & z \in \Omega(\delta), \\ n^{\frac{2}{p}+\varepsilon}, & for\ all\ \lambda_1, \gamma_1, \gamma_2, & z \in \widehat{\Omega}(\delta), \end{cases}$$

$$E_{n;1} := \begin{cases} n^{\frac{\tilde{\gamma}\cdot\hat{\lambda}}{p}}, & if\ \tilde{\gamma}\cdot\hat{\lambda} \ge 1, \\ n^{\frac{1}{p}}, & if\ \tilde{\gamma}\cdot\hat{\lambda} < 1, \end{cases} \quad B_{n,1}(z) := \begin{cases} n^{\hat{\lambda}}, & z \in \Omega(\delta), \\ n^{1+\varepsilon}, & z \in \widehat{\Omega}(\delta), \end{cases};$$

$$\hat{\lambda} := \begin{cases} \max\{1;\lambda\}+\varepsilon, & if\ 0<\lambda<2, \\ 2, & if\ \lambda = 2, \end{cases}; \quad \tilde{\gamma} := \begin{cases} \tilde{\gamma}_1, & if\ 0<\lambda<2, \\ \tilde{\gamma}_2, & if\ \lambda = 2, \end{cases}$$

$$\tilde{\gamma}_i := \max\{0;\gamma_i\}, \qquad i = 1,2; \quad \tilde{\lambda}_1 := \max\{1;\lambda_1\}+\varepsilon.$$

Theorem 6. *Let* $p > 1$; $G \in C_\theta(\lambda_1, \lambda_2)$, *for some* $0 < \lambda_j < 2, j = 1,2$; $h(z)$ *be defined as in (1). Then, for any* $P_n \in \wp_n$, $n \in \mathbb{N}$, $\gamma_j > -2$ *and arbitrary small* $\varepsilon > 0$

$$|P_n'(z)| \le c_2 \|P_n\|_p \left[\frac{|\Phi(z)|^{n+1}}{d(z,L_{R_1})} G_{n,2}(z) + \frac{|\Phi(z)|^{2(n+1)}}{d^{2/p}(z,L_{1+1/n})} B_{n,2}(z) E_{n,2}\right]$$

holds, where $c_2 = c_2(G, \gamma_i, p, \varepsilon) > 0$;

$$G_{n,2}(z) := \begin{cases} n^{\frac{\gamma_1+2}{p}\tilde{\lambda}_1}, & \gamma_1 \ge \frac{1}{\lambda_1}-2, \gamma_2 < \frac{1}{\lambda_2}-2, & z \in \Omega(\delta), \\ n^{\frac{\gamma_2+2}{p}\tilde{\lambda}_2}, & \gamma_1 < \frac{1}{\lambda_1}-2, \gamma_2 \ge \frac{1}{\lambda_2}-2, & z \in \Omega(\delta), \\ 1, & \gamma_1 < \frac{1}{\lambda_1}-2, \gamma_2 < \frac{1}{\lambda_2}-2, & z \in \Omega(\delta), \\ n^{\frac{\gamma_1+2}{p}\tilde{\lambda}_1}, & \gamma_1 \ge \frac{\lambda_2}{\lambda_1}(\gamma_2+2)-2, \gamma_2 \ge \frac{1}{\lambda_2}-2, & z \in \Omega(\delta), \\ n^{\frac{\gamma_2+2}{p}\tilde{\lambda}_2}, & \frac{1}{\lambda_1}-2 \le \gamma_1 < \frac{\lambda_2}{\lambda_1}(\gamma_2+2)-2, \gamma_2 \ge \frac{1}{\lambda_2}-2, & z \in \Omega(\delta), \\ n^{\frac{2}{p}+\varepsilon}, & for\ all\ \lambda_1, \gamma_1, & z \in \widehat{\Omega}(\delta), \end{cases}$$

$$E_{n,2} := \begin{cases} n^{\frac{\tilde{\gamma}\cdot\tilde{\lambda}}{p}}, & if\ \tilde{\gamma}\cdot\tilde{\lambda} \ge 1, \\ n^{\frac{1}{p}}, & if\ \tilde{\gamma}\cdot\tilde{\lambda} < 1, \end{cases}; \quad B_{n,2}(z) := \begin{cases} n^{\tilde{\lambda}}, & z \in \Omega(\delta), \\ n^{1+\varepsilon}, & z \in \widehat{\Omega}(\delta), \end{cases};$$

$$\hat{\lambda} := \begin{cases} \max\{1;\lambda\}+\varepsilon, & if\ 0<\lambda<2, \\ 2, & if\ \lambda = 2, \end{cases}; \quad \tilde{\gamma} := \begin{cases} \tilde{\gamma}_1, & if\ 0<\lambda<2, \\ \tilde{\gamma}_2, & if\ \lambda = 2, \end{cases}$$

$$\tilde{\gamma}_i := \max\{0;\gamma_i\}, \quad \tilde{\lambda}_i := \max\{1;\lambda_i\}+\varepsilon, i = 1,2.$$

Analogously, we also can give a theorem for the regions such as $G \in C_\theta(2,2)$.

3. Some auxiliary results

Lemma 1. [1] *Let L be a K −quasiconformal curve, $z_1 \in L$, $z_2, z_3 \in \Omega \cap \{z: |z - z_1| \prec d(z_1, L_{R_0})\}$; $w_j = \Phi(z_j)$, $(z_2, z_3 \in G \cap \{z: |z - z_1| \prec d(z_1, L_{R_0})\}; w_j = \varphi(z_j))$, $j = 1,2,3$. Then*

a) *The statements* $|z_1 - z_2| \prec |z_1 - z_3|$ and $|w_1 - w_2| \prec |w_1 - w_3|$ *are equivalent.*

So are $|z_1 - z_2| \asymp |z_1 - z_3|$ *and* $|w_1 - w_2| \asymp |w_1 - w_3|$.

b) *If* $|z_1 - z_2| \prec |z_1 - z_3|$, *then*

$$\left|\frac{w_1 - w_3}{w_1 - w_2}\right|^{K^2} \prec \left|\frac{z_1 - z_3}{z_1 - z_2}\right| \prec \left|\frac{w_1 - w_3}{w_1 - w_2}\right|^{K^{-2}},$$

where $\varepsilon < 1$, $c > 1$, $R_0 > 1$ are constants, depending on G.

Corollary 7. *Under the assumptions of Lemma 1, if $z_3 \in L_{R_0}$, then*

$$|w_1 - w_2|^{K^2} \prec |z_1 - z_2| \prec |w_1 - w_2|^{K^{-2}}.$$

Corollary 8. *If $L \in C_\theta$, then*

$$|w_1 - w_2|^{1+\varepsilon} \prec |z_1 - z_2| \prec |w_1 - w_2|^{1-\varepsilon},$$

for all $\varepsilon > 0$.

The following lemma is a consequence of the results given in [18], [21], [27] and of estimate for the $|\Psi'|$ (see, for example, [14, Th.2.8]):

$$|\Psi'(\tau)| \asymp \frac{d(\Psi(\tau), L)}{|\tau| - 1}. \tag{10}$$

Let $w_j := \Phi(z_j), \varphi_j := \arg w_j$. Without loss of generality, we will assume that $\varphi_l < 2\pi$. Additionally to the notations (9), for $\eta_j = \min_{t \in \partial\Phi(\Omega(z_j, \delta_j))} |t - w_j| > 0$ and $\eta := \min\{\eta_j, j = \overline{1, l}\}$ let us set: $\Delta_j(\eta_j) := \{t: |t - w_j| \leq \eta_j\} \subset \Phi(\Omega(z_j, \delta_j))$, $\Delta(\eta) := \bigcup_{j=1}^{l} \Delta_j(\eta)$, $\widehat{\Delta}_j = \Delta \backslash \Delta(\eta_j)$; $\widehat{\Delta}(\eta) := \Delta \backslash \Delta(\eta)$; $\Delta_1' := \Delta_1'(1)$, $\Delta_1'(\rho) := \left\{t = R \cdot e^{i\theta}: R \geq \rho > 1, \frac{\varphi_0 + \varphi_1}{2} \leq \theta < \frac{\varphi_1 + \varphi_2}{2}\right\}$, $\Delta_j' := \Delta_j'(1)$, $\Delta_j'(\rho) := \left\{t = R \cdot e^{i\theta}: R \geq \rho > 1, \frac{\varphi_{j-1} + \varphi_j}{2} \leq \theta < \frac{\varphi_j + \varphi_0}{2}\right\}$, $j = 2,3\ldots, l$, where $\varphi_0 = 2\pi - \varphi_l$; $\Omega_j := \Psi(\Delta_j')$, $L_{R_1}^j := L_{R_1} \cap \Omega_j$. Clearly, $\Omega = \bigcup_{j=1}^{l} \Omega_j$.

The following lemma is a consequence of the results given in [27] and [18].

Lemma 2. *Let $G \in C_\theta(\lambda, \ldots, \lambda_l)$, $0 < \lambda_j < 2$, $j = 1,2,\ldots, l$,. Then*

i) *for any $w \in \Delta_j$, $|w - w_j|^{\lambda_j + \varepsilon} \prec |\Psi(w) - \Psi(w_j)| \prec |w - w_j|^{\lambda_j - \varepsilon}$, $|w - w_j|^{\lambda_j - 1 + \varepsilon} \prec |\Psi'(w)| \prec |w - w_j|^{\lambda_j - 1 - \varepsilon}$,*

ii) *for any $w \in \overline{\Delta} \backslash \Delta_j$, $(|w| - 1)^{1+\varepsilon} \prec d(\Psi(w), L)| \prec (|w| - 1)^{1-\varepsilon}$, $(|w| - 1)^{\varepsilon} \prec |\Psi'(w)| \prec (|w| - 1)^{-\varepsilon}$.*

Let $\{z_j\}_{j=1}^{l}$ be a fixed system of distinct points on curve L ordered in the positive direction and the weight function $h(z)$ be defined as in (1).

Lemma 3. [6] *Let L is a K −quasiconformal curve; $R = 1 + \frac{c}{n}$. Then, for any fixed $\varepsilon \in (0,1)$ there exist a level curve $L_{1+\varepsilon(R-1)}$ such that the following holds for any polynomial $P_n(z) \in \wp_n$, $n \in \mathbb{N}$:*

$$\|P_n\|_{\mathcal{L}_p\left(\frac{h}{|\Phi'|}, L_{1+\varepsilon(R-1)}\right)} \prec n^{\frac{1}{p}} \|P_n\|_p, \; p > 0. \tag{11}$$

Lemma 4. [6] *Let L be a K −quasiconformal curve; $h(z)$ be defined as in (1). Then, for arbitrary $P_n(z) \in \wp_n$, any $R > 1$ and $n = 1,2,\ldots,$ we have*

$$\|P_n\|_{A_p(h,G_R)} \prec \tilde{R}^{n+\frac{1}{p}}\|P_n\|_{A_p(h,G)},\ p>0, \tag{12}$$

where $\tilde{R}=1+c(R-1)$ *and* c *is independent from* n *and* R.

Lemma 5. *Let* $G\in C_\theta(\lambda_1,\dots,\lambda_l)$, $0<\lambda_j\le 2, j=\overline{1,l}$. *Then, for arbitrary* $P_n(z)\in \wp_n$ *and any* $p>0$, *we have:*

$$\|P_n\|_{A_p(h,G_{1+c/n})} \prec \|P_n\|_{A_p(h,G)}. \tag{13}$$

4. Proof of Theorems

4.1. Proof of Theorems 5 and 6.

Proof. We will prove both theorems simultaneously. Suppose that $G\in C_\theta(\lambda;2)$ $(C_\theta(\lambda_1;\lambda_2))$ for some $0<\lambda<2$; $h(z)$ be defined as in (1). For $z\in\Omega$, we define:

$$T_n(z):=\frac{P_n(z)}{\Phi^{n+1}(z)}\ . \tag{14}$$

Then

$$T_n'(z)=\frac{P_n'(z)}{\Phi^{n+1}(z)}+P_n(z)\left(\frac{1}{\Phi^{n+1}(z)}\right)',z\in\Omega.$$

For any $R>1$ and $R_1:=1+\frac{R-1}{2}$, Cauchy integral representation for the region Ω_{R_1} gives

$$\begin{aligned}T_n'(z)&=-\frac{1}{2\pi i}\int_{L_{R_1}}T_n(\zeta)\frac{d\zeta}{(\zeta-z)^2}\\&=-\frac{1}{2\pi i}\int_{L_{R_1}}\frac{P_n(\zeta)}{\Phi^{n+1}(\zeta)}\frac{d\zeta}{(\zeta-z)^2},\ z\in\Omega_{R_1},\end{aligned} \tag{15}$$

and

$$\left(\frac{1}{\Phi^{n+1}(z)}\right)'=-\frac{1}{2\pi i}\int_{L_{R_1}}\frac{1}{\Phi^{n+1}(\zeta)}\frac{d\zeta}{(\zeta-z)^2}\ ,\ z\in\Omega_{R_1}.$$

Then, from (15), we get

$$P_n'(z)=\Phi^{n+1}(z)\left[T_n'(z)-P_n(z)\left(\frac{1}{\Phi^{n+1}(z)}\right)'\right]$$

$$=\Phi^{n+1}(z)\left[-\frac{1}{2\pi i}\int_{L_{R_1}}\frac{P_n(\zeta)}{\Phi^{n+1}(\zeta)}\frac{d\zeta}{(\zeta-z)^2}+\frac{P_n(z)}{2\pi i}\int_{L_{R_1}}\frac{1}{\Phi^{n+1}(\zeta)}\frac{d\zeta}{(\zeta-z)^2}\right],\ z\in\Omega_{R_1}.$$

Therefore,

$$|P_n'(z)|\le\frac{|\Phi^{n+1}(z)|}{2\pi}\left[\int_{L_{R_1}}\left|\frac{P_n(\zeta)}{\Phi^{n+1}(\zeta)}\right|\frac{|d\zeta|}{|\zeta-z|^2}+|P_n(z)|\int_{L_{R_1}}\left|\frac{1}{\Phi^{n+1}(\zeta)}\right|\frac{|d\zeta|}{|\zeta-z|^2}\right].$$

Since $|\Phi(\zeta)|>1$, for $\zeta\in L_{R_1}$, then, we have:

$$\begin{aligned}|P_n'(z)|&<\frac{|\Phi(z)|^{n+1}}{2\pi}\left[\int_{L_{R_1}}|P_n(\zeta)|\frac{|d\zeta|}{|\zeta-z|^2}+|P_n(z)|\int_{L_{R_1}}\frac{|d\zeta|}{|\zeta-z|^2}\right]\\&\le\frac{|\Phi(z)|^{n+1}}{2\pi}\left[\frac{1}{d(z,L_{R_1})}\int_{L_{R_1}}|P_n(\zeta)|\frac{|d\zeta|}{|\zeta-z|}+|P_n(z)|\int_{L_{R_1}}\frac{|d\zeta|}{|\zeta-z|^2}\right].\end{aligned} \tag{16}$$

Denote by

$$A_n(z):=\int_{L_{R_1}}|P_n(\zeta)|\frac{|d\zeta|}{|\zeta-z|};B_n(z):=\int_{L_{R_1}}\frac{|d\zeta|}{|\zeta-z|^2}, \tag{17}$$

and will be estimate these integrals separately.
To estimate $A_n(z)$, first of all replacing the variable $\tau=\Phi(\zeta)$ and multiplying the numerator and denominator of the integrant by $\prod_{j=1}^{2}\left|\Psi(\tau)-\Psi(w_j)\right|^{\frac{\gamma_j}{p}}\ |\Psi'(\tau)|^{\frac{2}{p}}$ and applying the Hölder inequality, we obtain:

$$A_n(z)=\int_{L_{R_1}}|P_n(\zeta)|\frac{|d\zeta|}{|\zeta-z|} \tag{18}$$

$$=\sum_{i=1}^{3}\int_{F_{R_1}^i}\frac{\prod_{j=1}^{2}|\Psi(\tau)-\Psi(w_j)|^{\frac{\gamma_j}{p}}\left|P_n(\Psi(\tau))(\Psi'(\tau))^{\frac{2}{p}}\right||\Psi'(\tau)|^{1-\frac{2}{p}}}{\prod_{j=1}^{2}|\Psi(\tau)-\Psi(w_j)|^{\frac{\gamma_j}{p}}|\Psi(\tau)-\Psi(w)|}|d\tau|$$

$$\leq\sum_{i=1}^{3}\left(\int_{F_{R_1}^i}\prod_{j=1}^{2}\left|\Psi(\tau)-\Psi(w_j)\right|^{\gamma_j}|P_n(\Psi(\tau))|^p|\Psi'(\tau)|^2|d\tau|\right)^{\frac{1}{p}}$$

$$\times\left(\int_{F_{R_1}^i}\left(\frac{|\Psi'(\tau)|^{1-\frac{2}{p}}}{\prod_{j=1}^{2}|\Psi(\tau)-\Psi(w_j)|^{\frac{\gamma_j}{p}}|\Psi(\tau)-\Psi(w)|}\right)^q|d\tau|\right)^{\frac{1}{q}}$$

$$=:\sum_{i=1}^{3}A_n^i(z),$$

where $F_{R_1}^j:=\Phi(L_{R_1}^1)=\Delta_j'\ \cap\{\tau:|\tau|=R_1\}, j=1,2;\ F_{R_1}^3:=\Phi(L_{R_1}^1)\backslash(F_{R_1}^1\cup F_{R_1}^{21})$ and

$$A_n^i(z):=\left(\int\limits_{F_{R_1}^i}\left|f_{n,p}(\tau)\right|^p|d\tau|\right)^{\frac{1}{p}}\left(\int\limits_{F_{R_1}^i}\frac{|\Psi'(\tau)|^{2-q}}{\prod_{j=1}^{2}\left|\Psi(\tau)-\Psi(w_j)\right|^{\gamma_j(q-1)}|\Psi(\tau)-\Psi(w)|^q}|d\tau|\right)^{\frac{1}{q}}$$

$$=:J_{n,1}^i\cdot J_{n,2}^i(z),$$

$$f_{n,p}(\tau):=h^{\frac{1}{p}}(\Psi(\tau))P_n(\Psi(\tau))(\Psi'(\tau))^{\frac{2}{p}},\ |\tau|=R_1.$$

Applying to Lemmas 3, 4 and 5, we get:

$$J_{n,1}^i\prec n^{\frac{1}{p}}\|P_n\|_p,\ i=1,2,3. \tag{19}$$

For the estimation of the integral $J_{n,2}^i(z)$, for $i=1,2,3$, and $j=1,2$, we set:

$$E_{R_1}^{11}(w_j)=\left\{\tau:\tau\in F_{R_1}^j,\ \left|\tau-w_j\right|<c_j(R_1-1)\right\},$$
$$E_{R_1}^{12}(w_j):=\left\{\tau:\tau\in F_{R_1}^j, c_j(R_1-1)\ \leq\left|\tau-w_j\right|<\eta\right\},$$
$$E_{R_1}^{13}(w_j):=\left\{\tau:\tau\in\Phi(L_{R_1}^j),\left|\tau-w_j\right|\geq\eta\right\},$$

where $0<c_j<\eta$ is chosen so that $\left\{\tau:\left|\tau-w_j\right|<c_j(R_1-1)\right\}\cap\Delta\neq\varnothing$ and $\Phi(L_{R_1}^j)=\bigcup_{k=1}^{3}E_{R_1}^{1k}(w_j)$. Taking into consideration these notations, (19) can be written as:

$$\sum_{i=1}^{3}J_{n,2}^i(z)=:J_2(z)=\sum_{i=1}^{3}\sum_{j=1}^{2}J_2(E_{R_1}^{1i}(w_j),z) \tag{20}$$
$$=:\sum_{i=1}^{3}\sum_{j=1}^{2}J_{2,j}^i(z)$$

and, consequently,

$$A_n(z)\prec n^{\frac{1}{p}}\|P_n\|_p\cdot\sum_{j=1}^{2}\sum_{i=1}^{3}J_{2,j}^i(z)=:\sum_{j=1}^{2}\sum_{i=1}^{3}A_{n,i}^j(z), \tag{21}$$

where

$$A_{n,i}^j(z):=n^{\frac{1}{p}}\|P_n\|_p\cdot J_{2,j}^i(z),\ i=1,2,3;j=1,2. \tag{22}$$

$$\left(J_{2,j}^{i}(z)\right)^{q} := \int_{E_{R_1}^{1i}(w_j)} \frac{|\Psi'(\tau)|^{2-q}|d\tau|}{\prod_{j=1}^{2}|\Psi(\tau)-\Psi(w_j)|^{\gamma_j(q-1)}|\Psi(\tau)-\Psi(w)|^{q}}$$

$$\asymp \sum_{j=1}^{2} \int_{E_{R_1}^{1i}(w_j)} \frac{|\Psi'(\tau)|^{2-q}|d\tau|}{|\Psi(\tau)-\Psi(w_j)|^{\gamma_j(q-1)}|\Psi(\tau)-\Psi(w)|^{q}},\ i = 1,2,3,$$

since the points w_1 and w_2 are isolated.

Therefore, we need to estimate the quantity (21). In case of $j = 1$, for any $p > 1, 0 < \lambda_1 < 2, \gamma > -2$, and for all sufficiently small $\varepsilon > 0$, in [12] is proved following estimate:

$$\sum_{i=1}^{3} A_{n,i}^{1}(z) \prec \|P_n\|_p \cdot \begin{cases} n^{\frac{\gamma_1+2}{p}\tilde{\lambda}_1}, & 0 < \lambda_1 < 2,\ \gamma_1 \geq \frac{1}{\lambda_1} - 2, \quad z \in \Omega(\delta), \\ n^{\frac{2}{p}+\varepsilon}, & \text{for all } \lambda_1, \gamma_1, \quad z \in \widehat{\Omega}(\delta), \\ 1, & \text{otherwise}, \end{cases} \tag{23}$$

where $\tilde{\lambda}_1 := \max\{1; \lambda_1\} + \varepsilon$.
Similarly to the case $j = 1$, for the case $j = 2$, we obtain:

$$\sum_{i=1}^{3} A_{n,i}^{2}(z) \prec \|P_n\|_p \cdot \begin{cases} n^{\frac{\gamma_2+2}{p}2+\varepsilon}, & \gamma_2 \geq -\frac{3}{2}, \quad z \in \Omega(\delta), \\ n^{\frac{2}{p}+\varepsilon}, & \gamma_1 > -2, \quad z \in \widehat{\Omega}(\delta), \\ 1, & \text{otherwise} \end{cases} \tag{24}$$

Combining (23) and (24), for the region $G \in C_\theta(\lambda_1, 2)$, any $p > 1, \gamma_1 > -2, 0 < \lambda_1 < 2$, and for all sufficiently small $\varepsilon > 0$, we obtain:

$$A_n(z) = \sum_{k=1}^{3} A_{n,k}^{1} \prec \|P_n\|_p \times \tag{25}$$

$$\times \begin{cases} n^{\frac{\gamma_1+2}{p}\tilde{\lambda}_1}, & \gamma_1 \geq \frac{1}{\lambda_1} - 2, \quad z \in \Omega(\delta), \\ n^{\frac{2}{p}+\varepsilon}, & \text{for all } \lambda_1, \gamma_1, \quad z \in \widehat{\Omega}(\delta), \\ 1, & \text{otherwise} \end{cases} + \begin{cases} n^{\frac{\gamma_2+2}{p}2+\varepsilon}, & \gamma_2 \geq -\frac{3}{2}, \quad z \in \Omega(\delta), \\ n^{\frac{2}{p}+\varepsilon}, & \gamma_1 > -2, \quad z \in \widehat{\Omega}(\delta), \\ 1, & \text{otherwise} \end{cases}$$

$$\prec \|P_n\|_p \cdot \begin{cases} n^{\frac{\gamma_1+2}{p}\tilde{\lambda}_1}, & \gamma_1 \geq \frac{2}{\tilde{\lambda}_1}(\gamma_2+2) - 2, \gamma_2 \geq \frac{\tilde{\lambda}_1}{2\lambda_1} - 2, & z \in \Omega(\delta), \\ n^{\frac{\gamma_2+2}{p}2}, & \frac{1}{\lambda_1} - 2 \leq \gamma_1 < \frac{2}{\tilde{\lambda}_1}(\gamma_2+2) - 2, \gamma_2 \geq \frac{\tilde{\lambda}_1}{2\lambda_1} - 2, & z \in \Omega(\delta), \\ n^{\frac{\gamma_2+2}{p}2}, & \gamma_1 < \frac{1}{\lambda_1} - 2, \gamma_2 \geq \frac{\tilde{\lambda}_1}{2\lambda_1} - 2, & z \in \Omega(\delta), \\ 1, & \gamma_1 < \frac{1}{\lambda_1} - 2, \gamma_2 < -\frac{3}{2}, & z \in \Omega(\delta), \\ n^{\frac{2}{p}+\varepsilon}, & \text{for all } \lambda_1, \gamma_1, \gamma_2, & z \in \widehat{\Omega}(\delta), \end{cases}$$

$$=: \|P_n\|_p \cdot G_{n,1}(z)$$

If the angle at point z_2 is equals $\lambda_2\pi$ such that $0 < \lambda_1, \lambda_2 < 2$, , then, analogously to (25), for the region $G \in C_\theta(\lambda_1, \lambda_2)$ all $0 < \lambda_1, \lambda_2 < 2$,we have:

$$A_n(z) = \sum_{k=1}^{3} A_{n,k}^{1} \prec \|P_n\|_p \times \tag{26}$$

$$\begin{cases} n^{\frac{\gamma_1+2}{p}\tilde{\lambda}_1}, & \gamma_1 \geq \frac{1}{\lambda_1}-2, & z \in \Omega(\delta), \\ n^{\frac{2}{p}+\varepsilon}, & \text{for all } \lambda_1,\gamma_1, & z \in \widehat{\Omega}(\delta), \\ 1, & \text{otherwise} \end{cases} + \begin{cases} n^{\frac{\gamma_2+2}{p}\tilde{\lambda}_2}, & \gamma_2 \geq \frac{1}{\lambda_2}-2, & z \in \Omega(\delta), \\ n^{\frac{2}{p}+\varepsilon}, & \text{for all } \lambda_2,\gamma_2, & z \in \widehat{\Omega}(\delta), \\ 1, & \text{otherwise} \end{cases}$$

$$\prec \|P_n\|_p \times \begin{cases} n^{\frac{\gamma_1+2}{p}\tilde{\lambda}_1}, & \gamma_1 \geq \frac{1}{\lambda_1}-2, \gamma_2 < \frac{1}{\lambda_2}-2, & z \in \Omega(\delta), \\ n^{\frac{\gamma_2+2}{p}\tilde{\lambda}_2}, & \gamma_1 < \frac{1}{\lambda_1}-2, \gamma_2 \geq \frac{1}{\lambda_2}-2, & z \in \Omega(\delta), \\ 1, & \gamma_1 < \frac{1}{\lambda_1}-2, \gamma_2 < \frac{1}{\lambda_2}-2, & z \in \Omega(\delta), \\ n^{\frac{\gamma_1+2}{p}\tilde{\lambda}_1}, & \gamma_1 \geq \frac{\lambda_2}{\lambda_1}(\gamma_2+2)-2, \gamma_2 \geq \frac{1}{\lambda_2}-2, & z \in \Omega(\delta), \\ n^{\frac{\gamma_2+2}{p}\tilde{\lambda}_2}, & \frac{1}{\lambda_1}-2 \leq \gamma_1 < \frac{\lambda_2}{\lambda_1}(\gamma_2+2)-2, \gamma_2 \geq \frac{1}{\lambda_2}-2, & z \in \Omega(\delta), \\ n^{\frac{2}{p}+\varepsilon}, & \text{for all } \lambda_1,\gamma_1, & z \in \widehat{\Omega}(\delta), \end{cases}$$
$$=: \|P_n\|_p \times G_{n,2}(z).$$

Now, let us estimate $B_n(z)$. Let $G \in C_\theta(\lambda_1, 2)$. By replacing the variable $\tau = \Phi(\zeta)$ and according to (10) and Lemma 2, we obtain:

$$B_n(z) = \int_{L_{R_1}} \frac{|d\zeta|}{|\zeta-z|^2} = \int_{|\tau|=R_1} \frac{|\Psi'(\tau)||d\tau|}{|\Psi(\tau)-\Psi(w)|^2} |\Psi'(\tau)| \asymp \frac{d(\Psi(\tau),L)}{|\tau|-1}. \tag{27}$$
$$= \int_{\{|\tau|=R_1\}\cap\Delta_1} \frac{|\tau-w_1|^{\lambda_1-1-\varepsilon}|d\tau|}{|\tau-w|^{2(\lambda_1-\varepsilon)}} + \int_{\{|\tau|=R_1\}\cap\Delta_2} \frac{d(\Psi(\tau),L)|d\tau|}{(|\tau|-1)|\Psi(\tau)-\Psi(w)|^2}$$
$$+ \int_{\{|\tau|=R_1\}\cap\ (\widehat{\Delta}_1\cup\widehat{\Delta}_2)} \frac{(|\tau|-1)^{-\varepsilon}|d\tau|}{|\tau-w|^{2(1-\varepsilon)}}$$
$$=: B_n^1(z) + B_n^2(z) + B_n^3(z).$$

Let us set:
$F_1 := \{\{|\tau| = R_1\} \cap \Delta_1 : |\tau - w_1| \geq |\tau - w|\}$, $F_2 := \{\{|\tau| = R_1\} \cap \Delta_2 : |\tau - w_2| \geq |\tau - w|\}$,
$F_3 := (\{|\tau| = R_1\}) \backslash (F_1 \cup F_2)$.
Under this notations we have:

$$B_n^1(z) = \int_{F_1} \frac{|\tau-w_1|^{\lambda_1-1-\varepsilon}|d\tau|}{|\tau-w|^{2(\lambda_1+\varepsilon)}} + \int_{F_2} \frac{|\tau-w_1|^{\lambda_1-1-\varepsilon}|d\tau|}{|\tau-w|^{2(\lambda_1+\varepsilon)}}$$
$$\prec \begin{cases} \left(\frac{1}{n}\right)^{\lambda_1-1-\varepsilon} \int_{F_1} \frac{|d\tau|}{|\tau-w|^{2(\lambda_1-\varepsilon)}} + \int_{F_2} \frac{|d\tau|}{|\tau-w|^{\lambda_1+1+\varepsilon}}, & \text{if } \lambda_1 \geq 1, \\ \int_{F_1} \frac{|d\tau|}{|\tau-w|^{2(\lambda_1+\varepsilon)-\lambda_1+1-\varepsilon}} + \left(\frac{1}{n}\right)^{\lambda_1-1-\varepsilon} \int_{F_2} \frac{|d\tau|}{|\tau-w|^{2(\lambda_1+\varepsilon)}}, & \text{if } \lambda_1 < 1, \end{cases}$$
$$\prec \begin{cases} n^{\lambda_1+\varepsilon}, & \text{if } \lambda_1 \geq 1, \\ n^{\lambda_1+\varepsilon}, & \text{if } \lambda_1 < 1, \end{cases} \forall \varepsilon > 0;$$
$$B_n^2(z) = \int_{\{|\tau|=R_1\}\cap(\widehat{\Delta}_1\cup\widehat{\Delta}_2)} \frac{d(\Psi(\tau),L)|d\tau|}{(|\tau|-1)|\Psi(\tau)-\Psi(w)|^2} \prec \begin{cases} n^2, & z \in \Omega(\delta), \\ n^{1+\varepsilon}, & z \in \widehat{\Omega}(\delta), \end{cases} \forall \varepsilon > 0.$$

$$B_n^3(z) = \int_{\{|\tau|=R_1\}\cap\ \widehat{\Delta}_1} \frac{(|\tau|-1)^{-\varepsilon}|d\tau|}{|\tau-w|^{2(1-\varepsilon)}} \prec n^{1+\varepsilon}, \forall \varepsilon > 0.$$

So, from (27), we have:

$$B_n(z) \prec B_{n,1}(z) := \begin{cases} n^{\tilde{\lambda}}, & z \in \Omega(\delta), \\ n^{1+\varepsilon}, & z \in \widehat{\Omega}(\delta), \end{cases} \forall \varepsilon > 0. \tag{28}$$

Similarly, for the region $G \in C_\theta(\lambda_1, \lambda_2)$, we obtain:

$$B_n(z) \prec B_{n,2}(z) := \begin{cases} n^{\tilde{\lambda}}, & z \in \Omega(\delta), \\ n^{1+\varepsilon}, & z \in \hat{\Omega}(\delta), \end{cases}, \forall \varepsilon > 0. \quad (29)$$

Now, combining (16), (17), (25), (28) and (29) for the region , any $p > 1$, $\gamma_1 > -2, 0 < \lambda_1 < 2$, and for all sufficiently small $\varepsilon > 0$,we obtain:

$$|P_n'(z)| \prec |\Phi(z)|^{n+1} \left[\frac{1}{d(z, L_{R_1})} \int_{L_{R_1}} |P_n(\zeta)| \frac{|d\zeta|}{|\zeta - z|} + |P_n(z)| \int_{L_{R_1}} \frac{|d\zeta|}{|\zeta - z|^2} \right]$$

$$\prec |\Phi(z)|^{n+1} \|P_n\|_p \left[\frac{1}{d(z, L_{R_1})} G_{n,1}(z) + |P_n(z)| B_{n,1}(z) \right], \text{if } G \in C_\theta(\lambda_1, 2),$$

$$|P_n'(z)| \prec |\Phi(z)|^{n+1} \|P_n\|_p \left[\frac{1}{d(z, L_{R_1})} G_{n,2}(z) + |P_n(z)| B_{n,2}(z) \right], \text{if } G \in C_\theta(\lambda_1, \lambda_2),$$

Now, using estimates for $|P_n(z)|$ ([25, Theorem 1 and Corollary 1]) for the cases $G \in C_\theta(\lambda_1, 2)$ and $G \in C_\theta(\lambda_1, \lambda_2)$, we get:

$$|P_n'(z)| \prec \|P_n\|_p \left[\frac{|\Phi(z)|^{n+1}}{d(z, L_{R_1})} G_{n,1}(z) + \frac{|\Phi(z)|^{2(n+1)}}{d^{\frac{2}{p}}\left(z, L_{1+\frac{1}{n}}\right)} E_{n,1} B_{n,1}(z) \right], \text{if } G \in C_\theta(\lambda_1, 2),$$

and

$$|P_n'(z)| \prec \|P_n\|_p \left[\frac{|\Phi(z)|^{n+1}}{d(z, L_{R_1})} G_{n,2}(z) + \frac{|\Phi(z)|^{2(n+1)}}{d^{\frac{2}{p}}\left(z, L_{1+\frac{1}{n}}\right)} E_{n,2} B_{n,2}(z) \right], \text{if } G \in C_\theta(\lambda_1, \lambda_2),$$

where

$$E_{n;1} := \begin{cases} n^{\frac{\tilde{\gamma} \cdot \hat{\lambda}}{p}}, & \text{if } \tilde{\gamma} \cdot \hat{\lambda} \geq 1, \\ n^{\frac{1}{p}}, & \text{if } \tilde{\gamma} \cdot \hat{\lambda} < 1, \end{cases}; \qquad E_{n,2} := \begin{cases} n^{\frac{\tilde{\gamma} \cdot \tilde{\lambda}}{p}}, & \text{if } \tilde{\gamma} \cdot \tilde{\lambda} \geq 1, \\ n^{\frac{1}{p}}, & \text{if } \tilde{\gamma} \cdot \tilde{\lambda} < 1, \end{cases}$$

$$\hat{\lambda} := \begin{cases} \max\{1; \lambda\} + \varepsilon, & \text{if } 0 < \lambda < 2, \\ 2, & \text{if } \lambda = 2, \end{cases}; \quad \tilde{\gamma} := \begin{cases} \tilde{\gamma}_1, & \text{if } 0 < \lambda < 2, \\ \tilde{\gamma}_2, & \text{if } \lambda = 2, \end{cases}$$

$$\tilde{\gamma}_i := \max\{0; \gamma_i\}, \ i = 1,2; \ \tilde{\lambda} := \max\{1; \lambda_1, \lambda_2\} + \varepsilon.$$

Therefore, we complete the proof of Theorems 5 and 6.

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