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Research Article

Estimation of $Pr(X < Y)$ for exponential power records

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Abstract

In this study, we tackle the problem of estimation of stress-strength reliability $R = Pr(X < Y)$ based on upper record values for exponential power distribution. We use the maximum likelihood and Bayes methods to estimate R. The Tierney-Kadane approximation is used to compute the Bayes estimation of R since the Bayes estimator can not be obtained analytically. We also derive asymptotic confidence interval based on the asymptotic distribution of the maximum likelihood estimator of R. We consider a Monte Carlo simulation study in order to compare the performances of the maximum likelihood estimators and Bayes estimators according to mean square error criteria. Finally, a real data application is presented.

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1. Introduction

Lower record values are defined as the observation has a lower value than all the observations obtained before it while upper record values are described as the observation has a upper value than all the observations obtained before it. The record is a common concept in daily life. The record values are values that are immediately noticed, memorized and not forgotten in a large number of data. The records can be encountered in many areas such as sports science and natural sciences. For instance, in weather news, the presenter emphasizes the highest and lowest temperatures of the day. These temperature values are lower and upper records respectively. On the other hand, the athletes who broke records in the Olympic games are often mentioned throughout the tournament, even after years if the record they broke is not repeated. While the player with the lowest time to complete the race wins the gold medal, the player who completes the race with the lowest time

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among the gold medal winners during all the Olympics also breaks the record.The upper record values are defined mathematically as follows:

Let X_i, i≥1 be a sequence of independent and identically distributed (iid) continuous random variables with cumulative distribution function (cdf) $F(x)$ and probability density function (pdf) $f(x)$. The first record time is defined as $U(1) = 1$ and m-th upper record time is defined as

$$U(m) = \min\left\{j > U(m-1) : X_j > X_{U(m-1)}\right\}, \tag{1.1}$$

where $X_{U(m)}$ is m-th upper record value.The first study about record values was done by [12]. Then, increasing the interest of many authors led to there are various studies about record values in literature. For more information about records see [1,4–6,8,20,26,35,39].

The record values are significant in data analysis obtained from many real life area such as engineering, meteorology, agriculture, hydrology, sports, medical science and life-tests. Many products may break under stress in reliability studies. For instance, an elevator fails when exposed to too much load, or an electronic component breaks when exposed to too high temperature. However, the exact breaking stress or breaking point differs even between the same parts. Therefore, the measurements can be made sequentially and only values greater (or smaller) than all previous values are recorded in such experiments. The record values may be useful in such experiments. Since the number of measurements made is considerably smaller than the whole sample size. Also, these record values can be significant when measurements of these experiments are costly and the whole sample is destroyed [37]. Stress-strength model defines the life of a component (or a system) having Y strength and exposed to X stress and $R = P(X < Y)$ is defined as stress-strength reliability. The stress-strength models have extensive applications in many areas such as medicine, biology, engineering and agriculture. The R can be written as follows:

$$R = P(X < Y) = \int_0^\infty F_X(y)\, f_Y(y)\, dy, \tag{1.2}$$

where $f_Y(.)$ is pdf of Y, and $F_X(.)$ is the cdf of X.

Recently, the estimation the R is very popular in the literature and many authors have studied the problem of estimation R under various assumptions on X and Y for various distributions. For some references and more applications of the R, see [2,15,19,22,29]. On the other hand, some authors have investigated the estimation of R based on record data. Baklizi [10] obtained the maximum likelihood and Bayesian estimation of R based on lower record values for the one-parameter generalized exponential distribution. Baklizi [9] considered the interval estimation of R based on record values for the two-parameter exponential distribution. Nadar and Kızılaslan [28] obtained the maximum likelihood estimator (MLE) and Bayes estimators of R based on upper record values from Kumaraswamy distribution. Tarvirdizade and Kazemzadeh [36] studied the estimation of R based on record data from the Burr Type X distribution. Asgharzadeh et al. [3] extended the results of [10] and considered the estimation of R based on the two-parameter generalized exponential records. Kızılaslan and Nadar [23] studied about the estimation of R based on upper records for Burr Type XII distribution.

The exponential power (EP) distribution was proposed as a lifetime model by [31]. The pdf and cdf of EP distribution are given by

$$f(x;\alpha,\beta) = \frac{\beta}{\alpha}\left(\frac{x}{\alpha}\right)^{\beta-1} e^{\left(\frac{x}{\alpha}\right)^\beta} e^{\left[1-e^{\left(\frac{x}{\alpha}\right)^\beta}\right]}, \quad x > 0,\ \alpha > 0,\ \beta > 0 \tag{1.3}$$

and

$$F(x;\alpha,\beta) = 1 - e^{\left[1-e^{\left(\frac{x}{\alpha}\right)^\beta}\right]}, \quad x > 0,\ \alpha > 0,\ \beta > 0 \tag{1.4}$$

where α and β are the scale and shape parameters, respectively. Smith and Bain [31] mentioned that the hazard function of EP distribution may be U-shaped. Therefore, the EP distribution may be more flexible in data modeling than similar distributions. EP distribution has been discussed and extended by some authors, see for example, [11,13,25, 30]. Akdam et al. [7] studied the estimation of R for the EP distribution under progressive type II censoring. Zhi [40] examined the MLEs of parameters based on upper records for EP distribution. Tans et al. [32] suggested a new extension of EP distribution.

In this paper, we consider the problem of estimation $R = P(X < Y)$, under the assumption that $X \sim EP(\alpha_1, \beta_1)$ and $Y \sim EP(\alpha_2, \beta_2)$, and X and Y are independently distributed. Thus, R can be written as follows:

$$\begin{aligned} R &= \int_0^\infty \frac{\beta_2}{\alpha_2}\left(\frac{y}{\alpha_2}\right)^{\beta_2-1} \exp\left[\left(\frac{y}{\alpha_2}\right)^{\beta_2}\right] \exp\left[1-e^{\left(\frac{y}{\alpha_2}\right)^{\beta_2}}\right] \\ &\quad \times \left[1-\exp\left(1-e^{\left(\frac{y}{\alpha 1}\right)^{\beta_1}}\right)\right] dy. \end{aligned} \tag{1.5}$$

The integral in (1.5) can be computed by numerical methods. The motivation of this paper is the lack of study about estimation of $R = P(X < Y)$ for the EP records. We provide a new example to relevant literature on estimation of R based on upper records. We obtain the MLE and approximate Bayesian estimators of R based on upper record values for EP distribution. It is used Tierney Kadane approximation for approximate Bayes estimation of R. We also compute the asymptotic confidence interval based on EP records.

The paper is organized as follows: In Section 2, MLE and approximate Bayesian estimator under squared loss function based on upper record values are given. It is also presented asymptotic confidence interval of R in this section. In Section 3, Monte Carlo simulations are carried out to compare the performances of these estimators in terms of bias and mean square error (MSE). In Section 4, a real data application is given. Finally, conclusion is presented in Section 5.

2. Estimation of R based on upper records

2.1. Maximum likelihood estimation of R

In order to compute the MLE of R, we first obtain the MLEs of $\alpha_1, \beta_1, \alpha_2$ and β_2. Let $X_{U(1)}, X_{U(2)}, ..., X_{U(m)}$ are the first m upper records coming from $EP(\alpha_1, \beta_1)$. Also, let $Y_{U(1)}, Y_{U(2)}, ..., Y_{U(k)}$ are upper record values taken from $EP(\alpha_2, \beta_2)$ and are independent from X-sequence. Then, the likelihood function based on the observed record data is given (see [8]) as follows:

$$\begin{aligned} L(\alpha_1, \beta_1, \alpha_2, \beta_2 | \mathbf{x}, \mathbf{y}) &= f\left(x_{U(m)} | \alpha_1, \beta_1\right) f\left(y_{U(k)} | \alpha_2, \beta_2\right) \\ &\quad \times \prod_{i=1}^{m-1} \frac{f\left(x_{U(i)} : \alpha_1, \beta_1\right)}{1 - F\left(x_{U(i)} : \alpha_1, \beta_1\right)} \prod_{j=1}^{k-1} \frac{f\left(y_{U(j)} : \alpha_2, \beta_2\right)}{1 - F\left(y_{U(j)} : \alpha_2, \beta_2\right)} \\ &= \left(\frac{\beta_1}{\alpha_1}\right)^m e^{\left[1-e^{\left(\frac{x_{U(m)}}{\alpha_1}\right)^{\beta_1}}\right]} \times \left(\frac{\beta_2}{\alpha_2}\right)^k e^{\left[1-e^{\left(\frac{y_{U(k)}}{\alpha_2}\right)^{\beta_2}}\right]} \\ &\quad \times \prod_{i=1}^{m} \left(\frac{x_{U(i)}}{\alpha_1}\right)^{\beta_1-1} e^{\left(\frac{x_{U(i)}}{\alpha_1}\right)^{\beta_1}} \prod_{j=1}^{k} \left(\frac{y_{U(j)}}{\alpha_2}\right)^{\beta_2-1} e^{\left(\frac{y_{U(j)}}{\alpha_2}\right)^{\beta_2}} \end{aligned} \tag{2.1}$$

where, $\mathbf{x} = (x_1, x_2, ..., x_m)$ and $\mathbf{y} = (y_1, y_2, ..., y_n)$. From (6), the log-likelihood function is

$$\begin{aligned}
\ell\left(\alpha_1, \beta_1, \alpha_2, \beta_2 | \mathbf{x}, \mathbf{y}\right) &= \log\left(L\left(\alpha_1, \beta_1, \alpha_2, \beta_2 | \underline{x}, \underline{y}\right)\right) \\
&= m\left(\log\beta_1 - \log\alpha_1\right) + 1 - e^{\left(\frac{x_{U(m)}}{\alpha_1}\right)^{\beta_1}} \\
&\quad + (\beta_1 - 1)\sum_{i=1}^{m}\log\left(\frac{x_{U(i)}}{\alpha_1}\right) + \sum_{i=1}^{m}\left(\frac{x_{U(i)}}{\alpha_1}\right)^{\beta_1} \\
&\quad + k\left(\log\beta_2 - \log\alpha_2\right) + 1 - e^{\left(\frac{y_{U(k)}}{\alpha_2}\right)^{\beta_2}} \\
&\quad + (\beta_2 - 1)\sum_{j=1}^{k}\log\left(\frac{y_{U(j)}}{\alpha_2}\right) + \sum_{j=1}^{k}\left(\frac{y_{U(j)}}{\alpha_2}\right)^{\beta_2}. \qquad (2.2)
\end{aligned}$$

By differentiating partially the log-likelihood function $\ell\left(\alpha_1, \beta_1, \alpha_2, \beta_2 | \underline{x}, \underline{y}\right)$ with respect to α_1, β_1, α_2 and β_2 and then equalizing them to zero, we we obtain the likelihood equations as

$$\begin{aligned}
\frac{\partial\ell}{\partial\alpha_1} &= -\frac{m}{\alpha_1} + \frac{\beta_1}{\alpha_1}\left(\frac{x_{U(m)}}{\alpha_1}\right)^{\beta_1} e^{\left(\frac{x_{U(m)}}{\alpha_1}\right)^{\beta_1}} \\
&- \frac{m(\beta_1 - 1)}{\alpha_1} - \frac{\beta_1}{\alpha_1}\sum_{i=1}^{m}\left(\frac{x_{U(i)}}{\alpha_1}\right)^{\beta_1} = 0, \qquad (2.3)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial\ell}{\partial\beta_1} &= \frac{m}{\beta_1} - \left(\frac{x_{U(m)}}{\alpha_1}\right)^{\beta_1}\log\left(\frac{x_{U(m)}}{\alpha_1}\right) e^{\left(\frac{x_{U(m)}}{\alpha_1}\right)^{\beta_1}} \\
&+ \sum_{i=1}^{m}\left(\frac{x_{U(i)}}{\alpha_1}\right) + \sum_{i=1}^{m}\left(\frac{x_{U(i)}}{\alpha_1}\right)^{\beta_1}\log\left(\frac{x_{U(i)}}{\alpha_1}\right) = 0, \qquad (2.4)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial\ell}{\partial\alpha_2} &= -\frac{k}{\alpha_2} + \frac{\beta_2}{\alpha_2}\left(\frac{y_{U(k)}}{\alpha_2}\right)^{\beta_2} e^{\left(\frac{y_{U(k)}}{\alpha_2}\right)^{\beta_2}} \\
&- \frac{k(\beta_2 - 1)}{\alpha_2} - \frac{\beta_2}{\alpha_2}\sum_{j=1}^{k}\left(\frac{y_{U(j)}}{\alpha_2}\right)^{\beta_2} = 0, \qquad (2.5)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial\ell}{\partial\beta_2} &= \frac{k}{\beta_2} - \left(\frac{y_{U(k)}}{\alpha_2}\right)^{\beta_2}\log\left(\frac{y_{U(k)}}{\alpha_2}\right) e^{\left(\frac{y_{U(k)}}{\alpha_2}\right)^{\beta_2}} \\
&+ \sum_{j=1}^{k}\left(\frac{y_{U(j)}}{\alpha_2}\right) + \sum_{j=1}^{k}\left(\frac{y_{U(j)}}{\alpha_2}\right)^{\beta_2}\log\left(\frac{y_{U(j)}}{\alpha_2}\right) = 0. \qquad (2.6)
\end{aligned}$$

Since the likelihood equations do not have explicit forms, the MLEs of α_1, β_1, α_2 and β_2 can be obtained using numerical methods such as Newton Raphson and Nelder-Mead. By using Eq. (1.5) and invariant property of MLE, we can compute the MLE of R as

$$\begin{aligned}
R &= \int_0^\infty \frac{\widehat{\beta}_2}{\widehat{\alpha}_2}\left(\frac{y}{\widehat{\alpha}_2}\right)^{\widehat{\beta}_2 - 1}\exp\left[\left(\frac{y}{\widehat{\alpha}_2}\right)^{\widehat{\beta}_2}\right]\exp\left[1 - e^{\left(\frac{y}{\widehat{\alpha}_2}\right)^{\widehat{\beta}_2}}\right] \\
&\times\left[1 - \exp\left(1 - e^{\left(\frac{y}{\widehat{\alpha}_1}\right)^{\widehat{\beta}_1}}\right)\right] dy. \qquad (2.7)
\end{aligned}$$

2.2. Asymptotic confidence interval

In this subsection, we obtain the asymptotic variances and covariances of the MLEs $\hat{\alpha}_1$,$\hat{\beta}_1$,$\hat{\alpha}_2$ and $\hat{\beta}_2$ by entries of the inverse of the observed Fisher information matrix

$$
\begin{aligned}
I^{-1}\left(\hat{\Theta}\right) &= \left.\begin{pmatrix}
-\frac{\partial^2\ell(\Theta|\underline{x})}{\partial\alpha_1^2} & -\frac{\partial^2\ell(\Theta|\mathbf{x})}{\partial\alpha_1\partial\alpha_2} & -\frac{\partial^2\ell(\Theta|\mathbf{x})}{\partial\alpha_1\partial\beta_1} & -\frac{\partial^2\ell(\Theta|\mathbf{x})}{\partial\alpha_1\partial\beta_2} \\
-\frac{\partial^2\ell(\Theta|\mathbf{x})}{\partial\alpha_2\partial\alpha_1} & -\frac{\partial^2\ell(\Theta|\mathbf{x})}{\partial\alpha_2^2} & -\frac{\partial^2\ell(\Theta|\mathbf{x})}{\partial\alpha_2\partial\beta_1} & -\frac{\partial^2\ell(\Theta|\mathbf{x})}{\partial\alpha_2\partial\beta_2} \\
-\frac{\partial^2\ell(\Theta|\underline{x})}{\partial\beta_1\partial\alpha_1} & -\frac{\partial^2\ell(\Theta|\mathbf{x})}{\partial\beta_1\partial\alpha_2} & -\frac{\partial^2\ell(\Theta|\mathbf{x})}{\partial\beta_1^2} & -\frac{\partial^2\ell(\Theta|\mathbf{x})}{\partial\beta_1\partial\beta_2} \\
-\frac{\partial^2\ell(\Theta|\mathbf{x})}{\partial\beta_2\partial\alpha_1} & -\frac{\partial^2\ell(\Theta|\mathbf{x})}{\partial\beta_2\partial\alpha_2} & -\frac{\partial^2\ell(\Theta|\mathbf{x})}{\partial\beta_2\partial\beta_1} & -\frac{\partial^2\ell(\Theta|\mathbf{x})}{\partial\beta_2^2}
\end{pmatrix}^{-1}\right|_{\left(\underline{\theta}=\widehat{\underline{\theta}}\right)} \\
&= \begin{pmatrix}
Var\left(\hat{\alpha}_1\right) & Cov\left(\hat{\alpha}_1,\hat{\alpha}_2\right) & Cov\left(\hat{\alpha}_1,\hat{\beta}_1\right) & Cov\left(\hat{\alpha}_1,\hat{\beta}_2\right) \\
Cov\left(\hat{\alpha}_2,\hat{\alpha}_1\right) & Var\left(\hat{\alpha}_2\right) & Cov\left(\hat{\alpha}_2,\hat{\beta}_1\right) & Cov\left(\hat{\alpha}_2,\hat{\beta}_2\right) \\
Cov\left(\hat{\beta}_1,\hat{\alpha}_1\right) & Cov\left(\hat{\beta}_1,\hat{\alpha}_2\right) & Var\left(\hat{\beta}_1\right) & Cov\left(\hat{\beta}_1,\hat{\beta}_2\right) \\
Cov\left(\hat{\beta}_2,\hat{\alpha}_1\right) & Cov\left(\hat{\beta}_2,\hat{\alpha}_2\right) & Cov\left(\hat{\beta}_2,\hat{\beta}_1\right) & Var\left(\hat{\beta}_2\right)
\end{pmatrix} \\
&= \begin{pmatrix}
I_{11} & I_{12} & I_{13} & I_{14} \\
I_{21} & I_{22} & I_{23} & I_{24} \\
I_{31} & I_{32} & I_{33} & I_{34} \\
I_{41} & I_{42} & I_{43} & I_{44}
\end{pmatrix}^{-1},
\end{aligned}
$$

where $\Theta = (\alpha_1, \beta_1, \alpha_2, \beta_2)$ and $I(\Theta) = (I_{ij}(\Theta))$ for $i; j = 1, 2, 3, 4$ is the observed Fisher information matrix. It can be shown that

$$
I_{11} = -\frac{m}{\alpha_1^2} - \frac{e^{\left(\frac{x_{U(m)}}{\alpha_1}\right)^{\beta_1}}\beta_1^2 x_{U(m)}^2}{\alpha_1^4} - \frac{2e^{\left(\frac{x_{U(m)}}{\alpha_1}\right)^{\beta_1}}\beta_1 x_{U(m)}}{\alpha_1^3} + \frac{(\beta_1-1)\,m}{\alpha_1^2} + \sum_{i=1}^{m}\left(\frac{(\beta_1+1)\left(\frac{x_{U(i)}}{\alpha_1}\right)^{\beta_1}\beta_1}{\alpha_1^2}\right),
$$

$$
I_{22} = \frac{m}{\beta_1^2} - e^{\left(\frac{x_{U(m)}}{\alpha_1}\right)^{\beta_1}} \log\left(e^{\left(\frac{x_{U(m)}}{\alpha_1}\right)}\right)^2 + \sum_{i=1}^{m}\left(\left(\frac{x_{U(i)}}{\alpha_1}\right)^{\beta_1} + \log\left(\frac{x_{U(i)}}{\alpha_1}\right)^2\right),
$$

$$
I_{33} = -\frac{k}{\alpha_2^2} - \frac{e^{\left(\frac{y_{U(k)}}{\alpha_2}\right)^{\beta_2}}\beta_2^2 y_{U(k)}^2}{\alpha_2^4} - \frac{2e^{\left(\frac{y_{U(k)}}{\alpha_2}\right)^{\beta_2}}\beta_2 y_{U(k)}}{\alpha_2^3} + \frac{(\beta_2-1)\,k}{\alpha_2^2} + \sum_{j=1}^{k}\left(\frac{(\beta_2+1)\left(\frac{y_{U(j)}}{\alpha_2}\right)^{\beta_2}\beta_2}{\alpha_2^2}\right),
$$

$$
I_{44} = -\frac{k}{\beta_2^2} - e^{\left(\frac{y_{U(k)}}{\alpha_2}\right)^{\beta_2}} \log\left(e^{\left(\frac{y_{U(k)}}{\alpha_2}\right)}\right)^2 + \sum_{j=1}^{k}\left(\left(\frac{y_{U(j)}}{\alpha_2}\right)^{\beta_2} + \log\left(\frac{y_{U(j)}}{\alpha_2}\right)^2\right),
$$

$$
I_{12} = I_{21} = I_{14} = I_{41} = I_{34} = I_{43} = 0,
$$

$$
I_{13} = \frac{e^{\left(\frac{x_{U(m)}}{\alpha_1}\right)^{\beta_1}}\left(\frac{x_{U(m)}}{\alpha_1}\right)^{\beta_1}\beta_1 x_{U(m)}}{\alpha_1^2} + \frac{e^{\left(\frac{x_{U(m)}}{\alpha_1}\right)^{\beta_1}} x_{U(m)}}{\alpha_1^2} - \frac{m}{\alpha_1} - \sum_{i=1}^{m}\left(\frac{\left(\frac{x_{U(i)}}{\alpha_1}\right)^{\beta_1}\left(\beta_1\log\left(\frac{x_{U(i)}}{\alpha_1}\right)-1\right)}{\alpha_1}\right),
$$

$$
I_{24} = \frac{e^{\left(\frac{y_{U(k)}}{\alpha_2}\right)^{\beta_2}}\left(\frac{y_{U(k)}}{\alpha_2}\right)^{\beta_2}\beta_2 y_{U(k)}}{\alpha_2^2} + \frac{e^{\left(\frac{y_{U(k)}}{\alpha_2}\right)^{\beta_2}} y_{U(k)}}{\alpha_2^2} - \frac{k}{\alpha_2} - \sum_{j=1}^{k}\left(\frac{\left(\frac{y_{U(j)}}{\alpha_2}\right)^{\beta_2}\left(\beta_2\log\left(\frac{y_{U(j)}}{\alpha_2}\right)-1\right)}{\alpha_2}\right).
$$

Now, we can obtain the variance of $Var(\hat{R})$ using delta method as $Var(\hat{R}) = \mathbf{b}' I^{-1}(\hat{\Theta})\mathbf{b}$ where $\mathbf{b}' = \left(\frac{\partial R}{\partial \alpha_1}, \frac{\partial R}{\partial \beta_1}, \frac{\partial R}{\partial \alpha_2}, \frac{\partial R}{\partial \beta_2}\right)$.

Firstly, it is needed to estimate $Var(\hat{R})$ in order to compute the confidence interval of R. By using the MLEs of $\alpha_1, \beta_1, \alpha_2$ and β_2, $Var(\hat{R})$ can be estimated. Now, the asymptotic $100\,(1-\eta)\,\%$ confidence interval of R is obtained as

$$\left(\widehat{R} - z_{1-\frac{\eta}{2}}\sqrt{Var\left(\widehat{R}\right)},\ \widehat{R} + z_{1-\frac{\eta}{2}}\sqrt{Var\left(\widehat{R}\right)}\right), \tag{2.8}$$

where z_η is 100 ηth percentile of $N(0,1)$.

2.3. Bootstrap confidence interval

In this subsection, we present bootstrap confidence interval for R. The percentile bootstrap (Boot-p) suggested by [16] is a popular bootstrap method. By using Boot-p method, the estimation of bootstrap confidence interval can be summarized as follows:

Step 1. Generate upper record sample $\left(x_{U(1)}, x_{U(2)}, ..., x_{U(m)}\right)$ from $EP\,(\alpha_1, \beta_1)$ and $\left(y_{U(1)}, y_{U(2)}, ..., y_{U(k)}\right)$ from $EP\,(\alpha_2, \beta_2)$.

Step 2. Compute $\hat{R}_{MLE}$

Step 3. Generate a bootstrap sample $\left(x^*_{U(1)}, x^*_{U(2)}, ..., x^*_{U(m)}\right)$ and $\left(y^*_{U(1)}, y^*_{U(2)}, ..., y^*_{U(k)}\right)$ by using $\hat{R}_{MLE}$ and upper records. Compute the bootstrap estimate of R, $\hat{R}^*_{MLE}$

Step 4. Repeat Step 3 NBOOT times

Step 5. Let $F^*\,(x) = P\left(\hat{R}^*_{MLE} \le x\right)$ be the cdf of $\hat{R}^*_{MLE}$. The approximate $100\,(1-\eta)\,\%$ confidence interval for R is given as follows:

$$\left(\hat{R}^*_{MLEBoot-p}\left(\frac{\eta}{2}\right), \hat{R}^*_{MLEBoot-p}\left(1-\frac{\eta}{2}\right)\right)$$

where $\hat{R}^*_{MLEBoot-p} = F^{*-1}\,(x)$ [7].

2.4. Approximate Bayesian estimation of R

Let $X_{U(1)}, X_{U(2)}, ..., X_{U(m)}$ and $Y_{U(1)}, Y_{U(2)}, ..., Y_{U(k)}$ are upper record values taken from $EP\,(\alpha_1, \beta_1)$ and $EP\,(\alpha_2, \beta_2)$, respectively. For Bayesian estimation of the parameters, it is needed to prior distributions for these parameters. We consider independent gamma priors for α_1, β_1, α_2 and β_2 as follows:

$$\pi\,(\alpha_1) = \frac{e_1^{d_1}}{\Gamma\,(d_1)}\alpha_1^{d_1-1}e^{-\alpha_1 e_1} \qquad \alpha_1, e_1, d_1 > 0$$

$$\pi\,(\beta_1) = \frac{e_2^{d_2}}{\Gamma\,(d_2)}\beta_1^{d_2-1}e^{-\beta_1 e_2} \qquad \beta_1, e_2, d_2 > 0$$

$$\pi\,(\alpha_2) = \frac{e_3^{d_3}}{\Gamma\,(d_3)}\alpha_2^{d_3-1}e^{-\alpha_2 e_3} \qquad \alpha_2, e_3, d_3 > 0$$

$$\pi\,(\beta_2) = \frac{e_4^{d_4}}{\Gamma\,(d_4)}\beta_2^{d_4-1}e^{-\beta_2 e_4} \qquad \beta_2, e_4, d_4 > 0.$$

The joint prior and posterior distributions of $\Theta = (\alpha_1, \beta_1, \alpha_2, \beta_2)$ are given as in (2.9) and (2.10) respectively.

$$\begin{aligned}\pi(\Theta) &= \pi(\alpha_1)\,\pi(\beta_1)\,\pi(\alpha_2)\,\pi(\beta_2)\\ &= \frac{e_1^{d_1}e_2^{d_2}e_3^{d_3}e_4^{d_4}}{\Gamma(d_1)\,\Gamma(d_2)\,\Gamma(d_3)\,\Gamma(d_4)}\alpha_1^{d_1-1}\beta_1^{d_2-1}\alpha_2^{d_3-1}\beta_2^{d_4-1}\\ &\quad\times e^{-(\alpha_1e_1+\beta_1e_2+\alpha_2e_3+\beta_2e_4)}\end{aligned} \tag{2.9}$$

$$\begin{aligned}\pi(\Theta|\mathbf{x},\mathbf{y}) &= \frac{f(x,y|\Theta)\,\pi(\Theta)}{f_x(x)\,f_y(y)}\\ &= \frac{w(x,y;\Theta)\,t(\Theta)}{\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty w(x,y;\Theta)\,t(\Theta)\,d\Theta},\end{aligned} \tag{2.10}$$

where $d\Theta = d\alpha_1 d\beta_1 d\alpha_2 d\beta_2$,

$$\begin{aligned}w(x,y;\Theta) &= \left(\frac{\beta_1}{\alpha_1}\right)^m e^{\left[1-e^{\left(\frac{x_{U(m)}}{\alpha_1}\right)^{\beta_1}}\right]}\prod_{i=1}^{m}\left(\frac{x_{U(i)}}{\alpha_1}\right)^{\beta_1-1}e^{\left(\frac{x_{U(i)}}{\alpha_1}\right)^{\beta_1}}\\ &\quad\times\left(\frac{\beta_2}{\alpha_2}\right)^k e^{\left[1-e^{\left(\frac{y_{U(k)}}{\alpha_2}\right)^{\beta_2}}\right]}\prod_{j=1}^{k}\left(\frac{y_{U(j)}}{\alpha_2}\right)^{\beta_2-1}e^{\left(\frac{y_{U(j)}}{\alpha_2}\right)^{\beta_2}}\end{aligned}$$

$$t(\Theta) = \alpha_1^{d_1-1}\beta_1^{d_2-1}\alpha_2^{d_3-1}\beta_2^{d_4-1}e^{-(\alpha_1e_1+\beta_1e_2+\alpha_2e_3+\beta_2e_4)}.$$

In this case, Bayes estimator for $u(\Theta|\mathbf{x},\mathbf{y})$ under squared loss function is as follows;

$$\begin{aligned}\hat{u}_b(\Theta) &= E\left[u(\Theta|\mathbf{x},\mathbf{y})\right]\\ &= \int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty u(\Theta\,|\mathbf{x},\mathbf{y})\,\pi(\Theta|\mathbf{x},\mathbf{y})\,d\Theta\\ &= \frac{\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty u(\Theta|\mathbf{x},\mathbf{y})\,v(\alpha_1,\beta_1,\alpha_2,\beta_2|\mathbf{x},\mathbf{y})\,d\theta}{\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty v(\Theta|\mathbf{x},\mathbf{y})\,d\Theta},\end{aligned}$$

where $v(\Theta|\mathbf{x},\mathbf{y}) = e^{[\ell(\Theta|\mathbf{x},\mathbf{y})+\rho(\Theta|textbfx,\mathbf{y})]}$, $\ell(\Theta|\mathbf{x},\mathbf{y})$ is log-likelihood function and $\rho(\Theta|\mathbf{x},\mathbf{y})$ is logarithm of joint prior distribution. It is difficult to solve the above equation in closed-form. For solution of this equation, some approximate methods are used. One of these methods is Tierney Kadane's approximation. In the following, we consider the Tierney Kadane's approximation for the solution of aboved equation.

2.5. Bayes estimation with Tierney Kadane's method

Tierney and Kadane's approximation is also known as Laplace approach. This approximation is an important method which is used in asymptotic expansion of integrals. This approximation method suggested by [38] has been studied by many authors such as [17], [18], [21], [27], [33] and [34] for finding integral ratios in Bayes analysis. Tierney and Kadane approximation for case with two parameters can be summarized as follows:

$$I(\Theta) = \frac{1}{m+k}\left\{\rho(\Theta|\mathbf{x},\mathbf{y}) + \ell(\Theta|\mathbf{x},\mathbf{y})\right\},$$

$$I^*(\Theta) = \frac{1}{m+k}\left\{\log u(\Theta|\mathbf{x},\mathbf{y})\right\} + I(\Theta|\mathbf{x},\mathbf{y}),$$

where $u(\Theta|\mathbf{x},\mathbf{y})$ is any functions of $\Theta=(\alpha_1,\beta_1,\alpha_2,\beta_2)$, $\ell(\Theta|\mathbf{x},\mathbf{y})$ is log-likelihood function and $\rho(\Theta|\mathbf{x},\mathbf{y})$ is logarithm of joint prior distribution. For $\Theta=(\alpha_1,\beta_1,\alpha_2,\beta_2)$, Tierney Kadane Bayes estimator of $u(\Theta)$ under squared error loss function is defined as follows:

$$\begin{aligned}\hat{u}_b(\Theta) &= E\left[u(\Theta)|\mathbf{x},\mathbf{y}\right]=\frac{\int_0^\infty e^{nI^*(\Theta)}d(\Theta)}{\int_0^\infty e^{nI(\Theta)}d(\Theta)}\\ &= \left(\frac{\det\Sigma^*}{\det\Sigma}\right)^{\frac{1}{2}}\exp\left[n\left(I^*\left(\hat{\Theta}_{I^*}\right)-I\left(\hat{\Theta}_I\right)\right)\right]\end{aligned}$$

where $\left(\hat{\Theta}_{I^*}\right)$ and $\left(\hat{\Theta}_I\right)$ maximize $I^*\left(\hat{\Theta}_{I^*}\right)$ and $I\left(\hat{\Theta}_I\right)$, respectively. Σ^* and Σ are minus the inverse Hessians of $I^*(\Theta)$ and $I(\Theta)$ at $\left(\hat{\theta}_{I^*}\right)$ and $\left(\hat{\Theta}_I\right)$, respectively. $u(\Theta)$ is replaced by $\hat{R}$, It is obtained Bayes estimation for R.

3. Simulation study

In this section, a Monte-Carlo simulation study is taken place to investigate the performances of MLE and Bayes estimators of R. These estimators are compared according to the bias and MSE. We describe the following algorithm to generate upper record values from EP distribution.

1. Generate the random sample of n sizes $T_1,T_2,\ldots,T_n$ from the $U(0,1)$ distribution.
2. Generate the random sample of n sizes $Z_1,Z_2,\ldots,Z_n$ from the from standard exponential distribution by using the transformation $Z_i=-\ln(1-T_i)$.
3. The i^{th} upper record value from the standard exponential distribution is obtained by using the transformation $Y_i=Z_1+Z_2+\cdots+Z_i$.
4. The i^{th} upper record value taken from $U(0,1)$ distribution is obtained with $U_i=1-e^{-Y_i}$
5. The i^{th} upper record value from EP distribution is obtained by using the inverse transformation $X_{U(i)}=F_{EP}^{-1}(U_i)\,, i=1,2,...n$. Thus, we generate n upper records from EP distribution via this algorithm.

By using the above method, we have generated the records $X_{U(1)},X_{U(2)},...,X_{U(m)}$ and $Y_{U(1)},Y_{U(2)},...,Y_{U(k)}$ from the $EP(\alpha_1,\beta_1)$ and $EP(\alpha_2,\beta_2)$ distributions, respectively. Based on the generated record samples, we computed the MLE and Bayes estimators of R as described in Section 2. Then, we computed the average biases and MSEs of the these estimators of R based on 1000 repetitions. Table 1 provides the MLE and Bayesian estimates of R. The average bias and MSE of the estimators are given in Table 2. The interval estimation results of the simulation study are presented in Table 3. In this simulation study, we considered $NBOOT=300$ for bootstrap estimation. For Bayes analysis, we used the Gamma distribution as the prior distribution with three priors as follows:

Prior 1: $d_1=1,e_1=2;d_2=2,e_2=3;d_3=1,e_3=3;d_4=1,e_4=4$.
Prior 2: $d_1=1.5,e_1=1;d_2=2.5,e_2=1.5;d_3=2,e_3=1;d_4=4,e_4=2$.
Prior 3: $d_1=e_1=d_2=e_2=d_3=e_3=d_4=e_4=0.0001$.

The parameter settings are given by

Case 1: $\alpha_1=0.2,\beta_1=0.4;\alpha_2=0.2,\beta_2=0.3$.
Case 2: $\alpha_1=0.2,\beta_1=0.3;\alpha_2=0.2,\beta_2=0.4$.
Case 3: $\alpha_1=0.1,\beta_1=0.4;\alpha_2=0.2,\beta_2=0.4$.

Table 1. MLE and Bayesian Estimates of R

		R	$\hat{R}_{MLE}$	$\hat{R}_{BAYES}$		
Parameters	(n,k)			prior 1	prior 2	prior 3
	(10,10)		0.530	0.267	0.293	0.229
Case 1	(15,10)	0.400	0.530	0.3181	0.368	0.303
	(15,15)		0.456	0.335	0.32	0.322
	(10,10)		0.556	0.249	0.27	0.172
Case 2	(15,10)	0.485	0.570	0.313	0.365	0.252
	(15,15)		0.534	0.336	0.316	0.266
	(10,10)		0.639	0.3686	0.537	0.368
Case 3	(15,10)	0.536	0.645	0.4187	0.591	0.454
	(15,15)		0.609	0.4514	0.525	0.479

Table 2. The biases and MSEs of MLE and Bayesian Estimators of R

		$\hat{R}_{MLE}$		$\hat{R}_{BAYES}$					
				prior 1		prior 2		prior 3	
Case	(n, k)	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
	$(10, 10)$	0.129	0.033	-0.133	0.020	-0.107	0.020	-0.171	0.033
1	$(15, 10)$	0.130	0.032	-0.082	0.010	-0.032	0.011	-0.097	0.015
	$(15, 15)$	0.056	0.019	-0.065	0.007	-0.080	0.011	-0.078	0.011
	$(10, 10)$	0.071	0.034	-0.235	0.058	-0.215	0.064	-0.313	0.104
2	$(15, 10)$	0.085	0.029	-0.171	0.033	-0.120	0.031	-0.233	0.062
	$(15, 15)$	0.049	0.029	-0.148	0.025	-0.169	0.039	-0.219	0.054
	$(10, 10)$	0.103	0.027	-0.167	0.034	0.001	0.026	-0.168	0.041
3	$(15, 10)$	0.109	0.024	-0.117	0.021	0.055	0.026	-0.082	0.022
	$(15, 15)$	0.073	0.023	-0.085	0.013	-0.011	0.018	-0.057	0.014

Table 3. The lengths and CPs of MLEs of R

		RMLE		RMLE*	
Parameters	(n,k)	Length	CP	Length	CP
	(10,10)	0.676	0.913	0.682	0.997
Case 1	(15,10)	0.650	0.911	0.648	0.994
	(15,15)	0.527	0.902	0.501	0.982
	(10,10)	0.782	0.917	0.881	0.998
Case 2	(15,10)	0.749	0.934	0.845	0.998
	(15,15)	0.667	0.877	0.717	0.987
	(10,10)	0.720	0.947	0.822	0.999
Case 3	(15,10)	0.692	0.955	0.786	0.996
	(15,15)	0.592	0.888	0.646	0.981

CP: Coverage probability.

According to Table 1, as the number of records increases MSEs and biases of the MLEs of R decreases as expected. From Table 2, it can be concluded that the performances of Bayes estimators are better than MLEs in terms of MSE criterion. In this Monte Carlo simulation study, it is seen that the best choice is prior 1 among used three priors for Bayesian estimation. It provides the smallest bias and MSE in most cases considered. Prior 2 is the second best prior and Prior 3 is the worst among three priors. Table 3 shows that the estimation of approximate bootstrap interval is closer to 0.95 than the asymptotic confidence interval estimation.

4. Real data analysis

In this section, we provide a numerical example to illustrate the use of examined methods of estimation. Crowder [14] considered data sets representing the times to failure of steel specimens subjected to cyclic stress loading of various amplitudes. The data are for 20 specimens at each of the 14 stress amplitudes: $32.0, 32.5, 33.0, \cdots, 38.0, 38.5$. These data are taken from [24] (Page 574). Here, we analyze the data for 33.0 (Data Set 1) and

32.0 (Data Set 2) stress amplitudes, which are divided by 1000. The data set 1 and data set 2 are independent. The data are presented as follows:

Data Set 1 (The level of stress: 33): 0.184, 0.241, 0.273, 1.842, 0.371, 0.830, 0.683, 1.306, 0.562, 0.166, 0.981, 1.867, 0.493, 0.418, 2.978, 1.463, 2.220, 0.312, 0.251, 0.076,

Data Set 2 (The level of stress: 32): 1.144, 0.231, 0.523, 0.474, 4.510, 3.107, 0.815, 6.297, 1.580, 0.605, 1.786, 0.206, 1.943, 0.935, 0.283, 1.336, 0.727, 0.370, 1.056, 0.413, 0.619, 2.214, 1.826, 0.597.

We interested in estimating the stress-strength parameter $R = Pr(X < Y)$ where X and Y denote the amount of the times to failure of steel specimens in data set 1 and 2, respectively. First, we check to see whether the EP distribution is adequate to fit these data sets or not.

For two data sets, the MLEs and their standard errors (in parentheses) of EP distribution parameters, Kolmogorov-Smirnov (K-S) statistics and relevant p values are given in Table 4.

Table 4. The MLEs(standard errors), K-S distances and their p-values

	$\widehat{\alpha}$	$\widehat{\beta}$	K-S	p-value
Data Set 1	1.6113(0.2724)	0.8220(0.1540)	0.1427	0.7585
Data Set 2	2.7265(0.4698)	0.7435(0.1183)	0.1460	0.6334

From Table 4, it is clear that the EP distribution fits quite well to both data sets. Moreover, Figures 1-2 illustrate the fitted density functions and fitted cdfs. Thus, it can be easily assess fits of EP distribution to both data sets. It can be concluded that the EP distribution fits the current data sets according to the results of K-S tests.

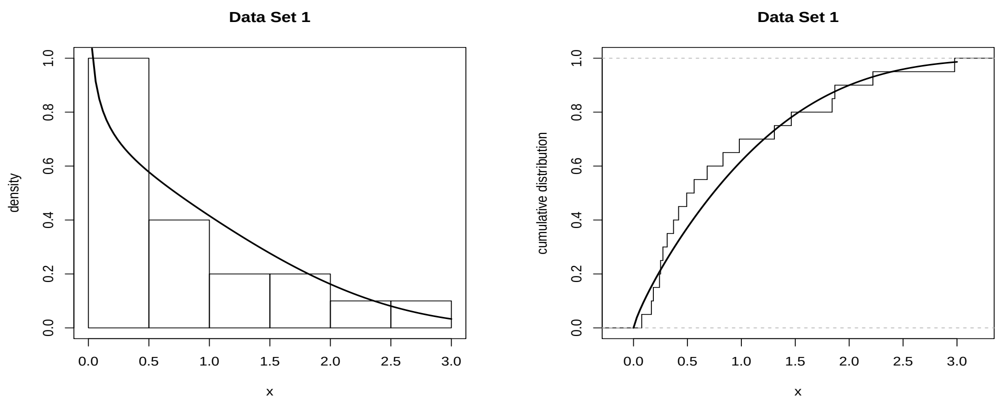

Figure 1. Estimated densities and empirical and estimated cdf in data set 1

From the whole sequence of data, the observed upper record values are obtained as follows:

$x_{U(i)}$: 0.184, 0.241, 0.273, 1.842, 1.867, 2.978

$y_{U(i)}$: 1.144, 4.510, 6.297

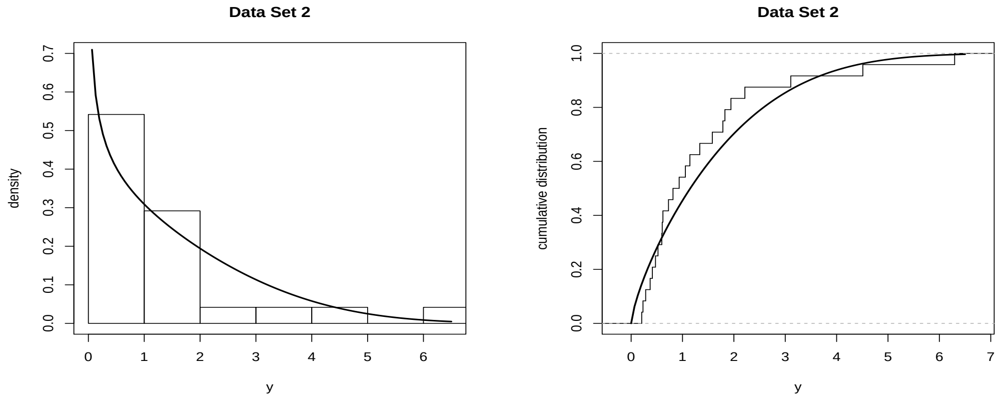

Figure 2. Estimated densities and empirical and estimated cdf in data set 2

Thus, based on these upper record data, we obtain the MLE of R as $\hat{R}_{MLE} = 0.855$ and the asymptotic 95 % confidence interval of R as $(0.7636, 0.9464)$. Also, we compute the MLE of R based on boostrap method as $\hat{R}^*_{MLE} = 0.596$ and the boostrap 95 % confidence interval of R as $(0.4919, 0.6596)$. The Bayes estimate of R is $\hat{R}_{BAYES} = 0.8568$. Note that for computing Bayes estimate, since we dont have any prior information, we used very small (close to zero) values of the hyper-parameters, i.e., $d_1 = e_1 = d_2 = e_2 = d_3 = e_3 = d_4 = e_4 = 0.0001$. Therefore, in this case, the priors are proper priors but they are almost improper. We obtain the MLE of R as $\hat{R}_{MLE} = 0.6169$ based on complete sample. It can be concluded that the boostrap estimate based on upper records is very close estimate based on complete sample.

5. Conclusion

In this paper, we considered MLE and Bayes estimatior of $R = Pr(X < Y)$ based on upper record values when X and Y are two independent EP random variables with (α_1, β_1) and (α_2, β_2) parameters. We obtained asymptotic confidence interval of R. Since the Bayes estimates cannot be obtained in closed form, we used Tierney kadane approximation to compute the Bayes estimate of R under squared loss function based on the independent gamma priors. We provided a real data analysis in the problem of estimation based on upper record values for EP distribution. Monte Carlo simulation study is used to evaluate the performance of MLE and Bayes estimators. Based on the simulation results, it is observed that the Bayes estimator under Prior 1 works better that the MLE and Bayes estimators under Priors 2 and 3. Also, the results of the simulation study shows that the approximate bootstrap interval is closer to 0.95 than the asymptotic confidence interval estimation.

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