

A NOTE ON THE GENERATING SETS FOR THE MAPPING CLASS GROUPS

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ABSTRACT. In this short note, we obtain generating sets with two elements for the mapping class group of closed, oriented surfaces of genus three and four, containing elements of the lowest order known so far.

1. INTRODUCTION

Let Σ_g denote a closed, oriented surface of genus g . Let $MCG(\Sigma_g)$ denote the group of isotopy classes of orientation preserving self homeomorphisms, which is called the mapping class group of the surface Σ_g .

Dehn, [1] proved that the mapping class group of a compact, closed surface Σ_g , $MCG(\Sigma_g)$, is generated by twists, which are now called Dehn twists. Lickorish [6], unaware of Dehn’s work, proved that the group is generated by $3g-1$ Dehn twists about the non-separating curves that are given in Figure 1.

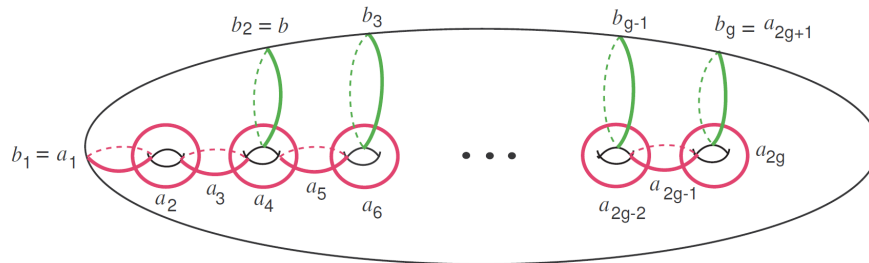


FIGURE 1. Lickorish generators

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Following these developments, Humphries [3] showed that, one can obtain the Dehn twists about the non-separating curves b_3, b_4, \dots, b_g , from the remaining collection of Lickorish generators, see Figure 1. Hence, Humphries proved that $2g+1$ of Lickorish generators are sufficient to generate the $MCG(\Sigma_g)$. These generators are the Dehn twists about the non-separating curves of Figure 2. In the same paper he showed that this number is in fact minimal.

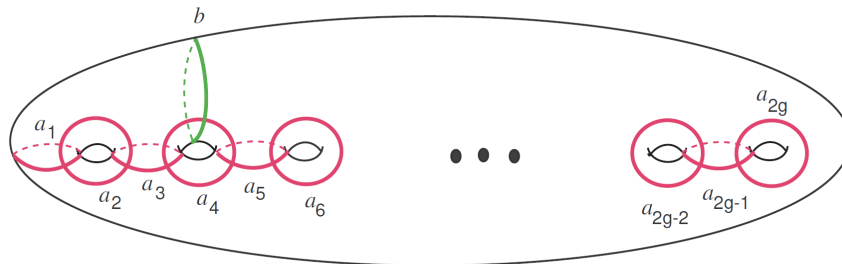


FIGURE 2. Humphries generators

These generators are of infinite order, hence the next question that comes to minds: Can we have generators of finite order? Let A_i and B_i denote Dehn twist about simple closed curves a_i and b_i given in Figure 3, respectively Wajnryb [7] proved that $MCG(\Sigma_g)$ is generated by $\langle S, B_{g-1}B_g^{-1} \rangle$, where $S = A_{2g}A_{2g-1} \cdots A_2A_1$, and is of order $4g + 2$, see Figure 3.

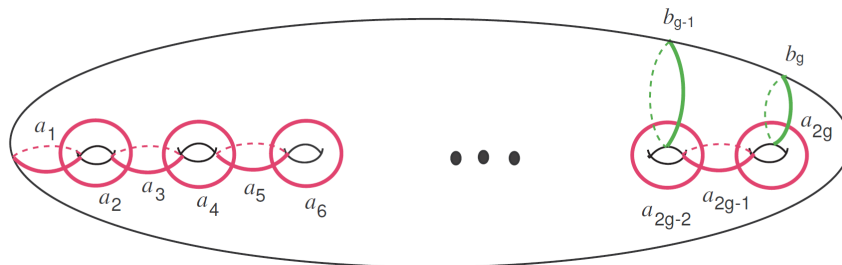


FIGURE 3. Wajnryb generators

Then Korkmaz improved this result, by the following theorem:

Theorem 1. [4] *Suppose that $g \geq 2$ and Σ_g is a closed oriented surface of genus g . The mapping class group $MCG(\Sigma_g)$ of Σ_g is generated by S and B .*

In Theorem 1, Korkmaz proved that $\text{MCG}(\Sigma_g)$ of a closed surface Σ_g is generated by two elements. One of the generators is B , Dehn twist about the curve b , hence it is of infinite order. The other generator is $S = A_{2g}A_{2g-1} \cdots A_2A_1$ and is of order $4g + 2$, see Figure 2 for the curves $b, a_1, a_2, \dots, a_{2g}$. In this note we lower this order using the homeomorphism $Q = A_{2g+1}A_{2g} \cdots A_2A_1$ for a genus 4 surface and $Q' = A_{2g+1}A_{2g} \cdots A_2A_1A_1$ for a genus 3 surface. In the following theorem we show that $\text{MCG}(\Sigma_4)$ is generated by $\langle Q, B \rangle$ where $Q = A_9A_8 \cdots A_2A_1$, and is of order 10 (that is $2g + 2$ for $g = 4$).

Theorem 2. *Mapping class group of a closed, oriented genus 4 surface is generated by Q , an element of order 10, and the Dehn twist B .*

Our second result gives even a lower order generator $Q' = A_7A_6 \cdots A_2A_1A_1$, which is of order 7 (that is $2g + 1$ for $g = 3$), for the group of mapping classes of the closed genus 3 surface:

Theorem 3. *Mapping class group of a closed, oriented genus 3 surface is generated by Q' , an element of order 7, and the Dehn twist B .*

Remark 4. *Even though we tried to generalize Theorem 2 and Theorem 3 for higher genus surfaces, it was not possible with the current technique. One needs other approaches to prove such generalizations.*

2. PRELIMINARIES

In the next section, we will prove Theorem 2 and Theorem 3, using some basic properties of the group of mapping classes. In this section we will review the basic properties that we need, without giving their proofs. For the proofs see [2], [5].

Convention: In this note we consider simple closed curves and homeomorphisms up to isotopy and the usual composition of functions, meaning that, if there are several number of functions (Dehn twists) to be composed, we first apply the function on the right then continue from right to left. Throughout the paper, when we use an equality sign between the curves or homeomorphisms, we mean the equivalence up to isotopy.

Notation: We use lower case letters (a_i, b, b_j, \dots) for the simple closed curves, and capital letters (A_i, B, B_j, \dots) to denote the Dehn twists about these curves (a_i, b, b_j, \dots) .

Relations:

- (1) Let c, d be two simple closed curves on an oriented surface Σ_g and let H be an orientation preserving self homeomorphism of the surface such that $H(c) = d$. Then

$$HCH^{-1} = D.$$

- (2) **Commutativity relation:** Let c, d be two disjoint simple closed curves on an oriented surface Σ_g , then the Dehn twists around the curves c and d commute:

$$CD = DC$$

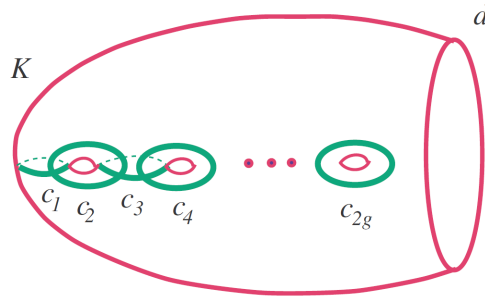


FIGURE 4. K , the neighborhood of union of even number of curves in the chain

- (3) **Braid relation:** Let c, d be two simple closed curves, intersecting transversely at one point, on an oriented surface Σ_g , then the Dehn twists around the curves c and d satisfy the relation below:

$$CDC = DCD$$

- (4) **Chain relation:**

First we define the chain, then we will give the chain relation.

Definition 5. Chain: Let c_1, c_2, \dots, c_n be a sequence of simple closed curves on an orientable surface. If only the consecutive ones intersect transversely at one point, and the others are disjoint, then this sequence of simple closed curves c_1, c_2, \dots, c_n is called a chain.

Let K be a tubular neighborhood of union of curves in the chain. There are two cases according to the parity of number of curves (n is even or odd) in the chain:

- If $n = 2g$, then K is an orientable genus g surface of one boundary component, call that boundary component d , see Figure 4.
- If $n = 2g + 1$, then K is an orientable genus g surface of two boundary components, call them d_1 and d_2 .

Then we have the following relations, which are called the chain relations in $MCG(K)$:

- If $n = 2g$, then we have $(C_1 C_2 \cdots C_{2g})^{4g+2} = D$
- If $n = 2g + 1$, then we have $(C_1 C_2 \cdots C_{2g+1})^{2g+2} = D_1 D_2$.

3. PROOFS

We will start this section with the proof of Theorem 2. In Theorem 2, we show that $MCG(\Sigma_4) = \langle Q, B \rangle$, where $Q = A_9 A_8 \cdots A_2 A_1$, see Figure 1 for the curves a_1, a_2, \dots, a_9 . Then we show that the order of the element Q is 10, in Lemma 8.

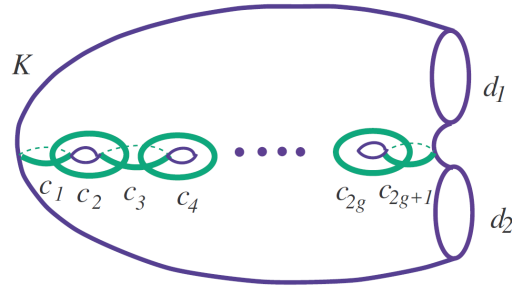


FIGURE 5. K , the neighborhood of union of odd number of curves in the chain

Hence, it is a lower order element than the one in Theorem 1, in which Korkmaz proved that $\text{MCG}(\Sigma_4) = \langle S, B \rangle$ and $S = A_8 A_7 \cdots A_2 A_1$. The element S is of order 14.

In order to prove these results we need some preparatory results. Let $G = \langle Q, B \rangle$ be the subgroup of $\text{MCG}(\Sigma_4)$ generated by Q and B . Let $C_1, C_2, C_3, D_1, D_2, E_1, E_2, F_1$ and B_3 be the Dehn twists around the curves $c_1, c_2, c_3, d_1, d_2, e_1, e_2, f_1$ and b_3 , respectively, which are shown in Figure 6.

Lemma 6. *The Dehn twists $C_1, C_2, C_3, D_1, D_2, E_1, E_2, F_1$ and B_3 are contained in $G = \langle Q, B \rangle$.*

Proof of Lemma 6. We will show that we can get the Dehn twists $C_1, C_2, C_3, D_1, D_2, E_1, E_2, F_1, B_3$ from the homeomorphisms Q and B by applying the Relation (1) several times. First, applying the homeomorphism Q^{-1} to the simple closed curve b , we get the simple closed curve c_1 . Hence, we have

- $Q^{-1}(b) = c_1$ and by Relation (1), we can write $C_1 = Q^{-1}BQ$. Since Q and B are already in the group G , we conclude that $C_1 \in G$.

Similarly, we repeat this process:

- $Q^{-1}(c_1) = d_1$, using Relation (1) and since $C_1, Q \in G$ we have $D_1 \in G$.
- $Q^{-1}(d_1) = c_2$ which implies $C_2 \in G$.
- $Q^{-1}(c_2) = d_2$ which implies $D_2 \in G$.
- $Q^{-1}(d_2) = c_3$ which implies $C_3 \in G$.
- $Q^{-1}(c_3) = b_3$ which implies $B_3 \in G$.
- $Q^{-1}(b_3) = e_1$ which implies $E_1 \in G$.
- $Q^{-1}(e_1) = f_1$ which implies $F_1 \in G$.
- $Q^{-1}(f_1) = e_2$ which implies $E_2 \in G$.

□

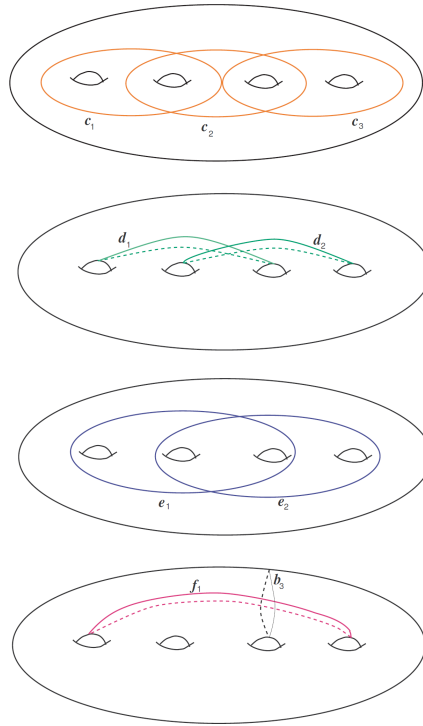


FIGURE 6. $c_1, c_2, c_3, d_1, d_2, e_1, e_2, f_1, b_3$, curves on a genus 4 surface

The main idea of the proof of Theorem 2 is to show that the G -orbit of the curve b contains the simple closed curves a_1, a_2, \dots, a_8 . As a result of this, all the Humphries generators A_1, A_2, \dots, A_8 and B of $\text{MCG}(\Sigma_4)$ are contained in G , hence we conclude that $\text{MCG}(\Sigma_4) = G$.

Proof of Theorem 2. Let h be the self homeomorphism of the surface Σ_4 which is given by the product of the Dehn twists: $C_2D_1BB_3^{-1}$. We will apply the homeomorphism h to the curve c_2 , and obtain the non-separating simple closed curve a_3 , which is shown in detail in Figure 7. Using Relation (1) and Lemma 6, we can say that the Dehn twist A_3 is in G . Then applying the homeomorphism Q^{-1} to the curve a_3 , we get:

- $Q^{-1}(a_3) = a_4$, using Relation (1) we say that $Q^{-1}A_3Q = A_4$ and since $Q, A_3 \in G$ we have $A_4 \in G$. Using repeatedly this process, we get the following:
- $Q^{-1}(a_4) = a_5$ which implies that $A_5 \in G$.
- $Q^{-1}(a_5) = a_6$ which implies that $A_6 \in G$.

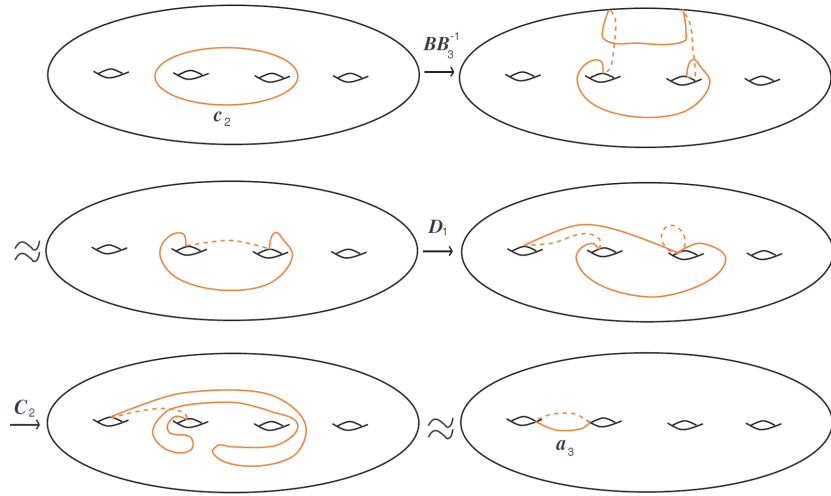


FIGURE 7. $h(c_2) = a_3$

- $Q^{-1}(a_6) = a_7$ which implies that $A_7 \in G$.
- $Q^{-1}(a_7) = a_8$ which implies that $A_8 \in G$.

Similarly applying the homeomorphism Q and Q^2 to the curve a_3 , we get a_2 and a_1 :

- $Q(a_3) = a_2$ which implies that $A_2 \in G$.
- $Q^2(a_3) = a_1$ which implies that $A_1 \in G$.

Therefore, we get the result that all Humphries generators of $MCG(\Sigma_4)$ are contained in G , and hence, conclude that $G = MCG(\Sigma_4)$. \square

Now we are going to prove Theorem 3, which implies that $MCG(\Sigma_3) = \langle Q', B \rangle$, where $Q' = A_7A_6A_5A_4A_3A_2A_1A_1$. The idea in the proof of Theorem 3 is similar to the one in the proof of Theorem 2. In order to prove Theorem 3, we need the following lemma. For the simple closed curves that we use in Lemma 7, see Figure 8.

Lemma 7. *The Dehn twists C_1, C_2, D_1 are in the group $G' = \langle Q', B \rangle$.*

Proof of Lemma 7. We start with the simple closed curve b , and apply the homeomorphism Q' . We get $Q'(b) = c_2$ and by Relation (1) we have, $C_2 = Q'BQ'^{-1}$. Since Q' and B are in the group G' , we deduce that $C_2 \in G'$.

Similarly,

- $Q'(c_2) = d_1$ which implies that $D_1 \in G'$.
- $Q'(d_1) = c_1$ which implies that $C_1 \in G'$.

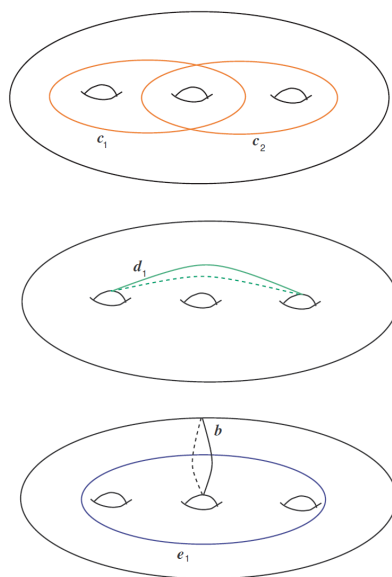


FIGURE 8. c_1, c_2, d_1, e_1, b curves on a genus 3 surface

□

Proof of Theorem 3. Let h be the self-homeomorphism of the genus 3 surface to itself, given by $h = C_1^{-1}(Q')^{-1}$. We apply the homeomorphism h to the simple closed curve b , and see that $h(b) = a_1$, as in Figure 9. Using Relation (1), we get that $A_1 \in \langle Q', B \rangle = G'$. Note that for a genus 3 surface, the homeomorphism Q is given by $Q = A_7A_6A_5A_4A_3A_2A_1$ and $Q = Q'A_1^{-1}$, hence, $Q \in G'$.

- $Q^{-1}(a_1) = a_2$, using Relation (1) we say that $Q^{-1}A_1Q = A_2$ and since $Q, A_1 \in G'$ we have $A_2 \in G'$. Upon repeating this process we get the following results:
- $Q^{-1}(a_2) = a_3$ which implies that $A_3 \in G'$.
- $Q^{-1}(a_3) = a_4$ which implies that $A_4 \in G'$.
- $Q^{-1}(a_4) = a_5$ which implies that $A_5 \in G'$.
- $Q^{-1}(a_5) = a_6$ which implies that $A_6 \in G'$.

Therefore all the Humphries generators of $MCG(\Sigma_3)$, are contained in G' , hence, we conclude that $MCG(\Sigma_3) = G'$.

□

Lemma 8. *The order of the element $Q = A_9A_8 \cdots A_2A_1$ is 10.*

Proof of Lemma 8. The curves defining Q form a chain on a closed genus 4-surface Σ_4 . Then using the chain relation we get, $Q^{10} = id$. Hence the order of Q is at

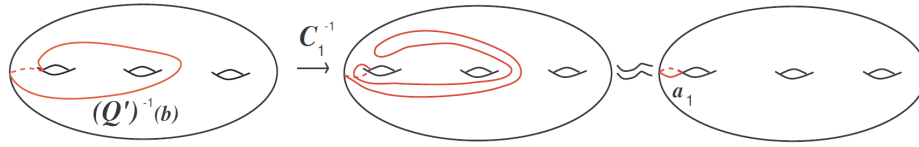


FIGURE 9. $h(b) = a_1$

most 10. Now, take the simple closed curve e_2 on Σ_4 , see Figure 6. From the Proof of Lemma 6, we observe that $Q^i(e_2) \neq e_2$ for $1 \leq i \leq 9$. Therefore the order of Q is 10. \square

Lemma 9. *The order of the element $Q' = A_7A_6 \cdots A_2A_1A_1$ is 7.*

Proof of Lemma 9. Using commutativity and braid relation, we have $(Q')^7 = Q^8$, where $Q = A_7A_6A_5A_4A_3A_2A_1$ on a closed genus 3-surface Σ_3 :

$$\begin{aligned}
 (Q')^7 &= A_7A_6A_5A_4A_3A_2A_1A_1(Q')^6 \\
 &= (Q)A_1(Q')^6 \\
 &= (Q)A_1(A_7A_6A_5A_4A_3A_2A_1A_1)(Q')^5 \\
 &= (Q)^2A_2A_1(Q')^5 \\
 &= (Q)^3A_3A_2A_1(Q')^4 \\
 &= (Q)^4A_4A_3A_2A_1(Q')^3 \\
 &= (Q)^5A_5A_4A_3A_2A_1(Q')^2 \\
 &= (Q)^6A_6A_5A_4A_3A_2A_1(Q') \\
 &= (Q)^7A_7A_6A_5A_4A_3A_2A_1 \\
 &= (Q)^8
 \end{aligned}$$

Moreover since Q is a chain on a closed Σ_3 surface, from the chain relation we have, $Q^8 = id$, which implies that, $(Q')^7 = Q^8 = id$, hence, the order of Q' is at most 7.

On the other hand, one can easily check that $(Q')^i(a_7) \neq a_7$ for $1 \leq i \leq 6$ which is shown in the Figure 10. Therefore the order of Q' is 7. \square

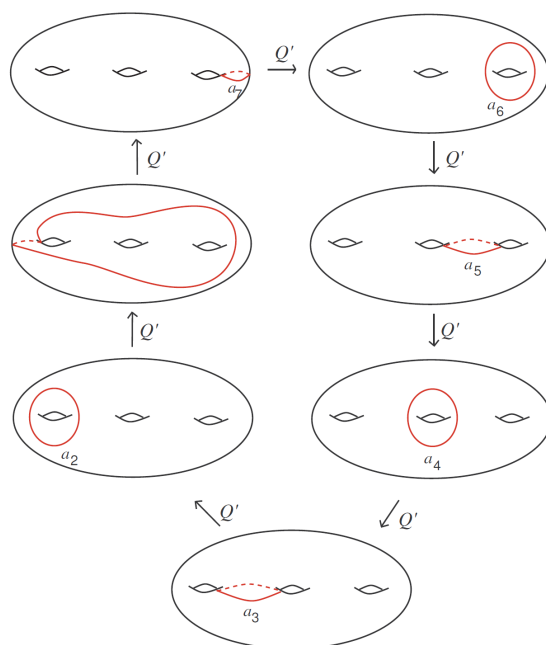


FIGURE 10. $(Q')^i(a_7) \neq a_7$ for $1 \leq i \leq 6$.

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