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(n, d)-COCOHERENT RINGS, (n, d)-COSEMIHEREDITARY RINGS AND (n, d)-V-RINGS

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Abstract. Let R be a ring, n be an non-negative integer and d be a positive integer or ∞. A right R-module M is called *$(n, d)^*$-projective* if $\mathrm{Ext}^1_R(M, C) = 0$ for every n-copresented right R-module C of injective dimension $\leq d$; a ring R is called *right (n, d)-cocoherent* if every n-copresented right R-module C with $id(C) \leq d$ is $(n+1)$-copresented; a ring R is called *right (n, d)-cosemihereditary* if whenever $0 \to C \to E \to A \to 0$ is exact, where C is n-copresented with $id(C) \leq d$, E is finitely cogenerated injective, then A is injective; a ring R is called *right (n, d)-V-ring* if every n-copresented right R-module C with $id(C) \leq d$ is injective. Some characterizations of $(n, d)^*$-projective modules are given, right (n, d)-cocoherent rings, right (n, d)-cosemihereditary rings and right (n, d)-V-rings are characterized by $(n, d)^*$-projective right R-modules. $(n, d)^*$-projective dimensions of modules over right (n, d)-cocoherent rings are investigated.

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1. Introduction

Throughout this paper, R is an associative ring with identity and all modules considered are unitary, n is a non-negative integer, d is a positive integer or ∞ unless a special note.

In 1982, V. A. Hiremath [4] defined and studied *finitely corelated modules*. Following [4], a right R-module M is said to be finitely corelated if there is a short exact sequence $0 \to M \to N \to K \to 0$ of right R-modules with N finitely cogenerated, *cofree* and K is finitely cogenerated, where a right R-module N is said to be cofree if it is isomorphic to a direct product of the injective hulls of some simple right R-modules. Finitely corelated modules are also called *finitely copresented modules*

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in some literatures such as [7]. Following [12], a right R-module M is said to be *FCP-projective* if $\mathrm{Ext}_R^1(M, C) = 0$ for every finitely copresented right R-module C. In [12], right V-rings are characterized by FCP-projective right R-modules. We recall also that R is called *right co-semihereditary* [6,8,12] if every finitely cogenerated factor module of a finitely cogenerated injective right R-module is injective, R is called *right co-coherent* [12] if every finitely cogenerated factor module of a finitely cogenerated injective right R-module is finitely copresented. It is easy to see that right V-rings, right co-semihereditary rings and right co-coherent rings are the dual concepts of von Neumann regular rings, right semihereditary rings and right coherent rings. In this paper, right *cocoherent* rings will denote right co-coherent rings in order to facilitate. In [12], right V-rings, right co-semihereditary rings are characterized by FCP-projective right R-modules, FCP-projective dimensions of right R-modules over right cocoherent rings are investigated. For example, we show that a ring R is right co-semihereditary if and only if every submodule of an FCP-projective right R-module is FCP-projective if and only if every submodule of a projective right R-module is FCP-projective [12, Theorem 3], a ring R is a right V-ring if and only if every right R-module is FCP-projective [12, Theorem 4].

In 1999, Xue introduced n-copresented modules and n-cocoherent rings respectively in [9]. According to [9], M is said to be *n-copresented* if there is an exact sequence of right R-modules $0 \to M \to E_0 \to E_1 \to \cdots \to E_n$, where each E_i is a finitely cogenerated injective module. It is easy to see that a module M is finitely cogenerated if and only if it is 0-copresented, a module M is finitely copresented if and only if it is 1-copresented. We call any module (-1)-copresented. n-copresented modules have been studied in [2,9,11]; R is called *right n-cocoherent* [9] in case every n-copresented right R-module is $(n+1)$-copresented. It is easy to see that R is right cocoherent if and only if it is right 1-cocoherent. Following [5], a ring R is called right co-noetherian if every factor module of a finitely cogenerated right R-module is finitely cogenerated. By [4, Proposition 17], a ring R is right co-noetherian if and only if it is right 0-cocoherent. In [11], we extend the concepts of FCP-projective modules, cosemihereditary rings and V-rings to (n, d)-projective modules, n-cosemihereditary rings and n-V-rings respectively, right n-V-rings and right n-cosemihereditary rings are characterized by $(n, 0)$-projective right R-modules, $(n, 0)$-projective dimensions of right R-modules over right n-cocoherent rings are investigated. Following [11], a right R-module M is called (n, d)-projective if $\mathrm{Ext}_R^{d+1}(M, A) = 0$ for every n-copresented right R-module A; a ring R is called right n-cosemihereditary if every submodule of a projective right

R-module is $(n,0)$-projective, a ring R is called a right n-V-ring if every right R-module is $(n,0)$-projective. Clearly, a right R-module M is FCP-projective if and only if it is $(1,0)$-projective, a ring R is right cosemihereditary if and only if it is right 1-cosemihereditary, a ring R is a right V-ring if and only if it is a right 0-V-ring if and only if it is a right 1-V-ring. Characterizations of n-cosemihereditary rings and n-V-rings can be found in [11, Theorem 3.7] and [11, Theorem 3.9], respectively.

In this paper, we generalize the concepts of $(n,0)$-projective modules, n-cocoherent rings, n-cosemihereditary rings, n-V-rings to $(n,d)^*$-projective modules, (n,d)-cocoherent rings, (n,d)-cosemihereditary rings and (n,d)-V-rings respectively. (n,d)-cosemihereditary rings, (n,d)-V-rings will be characterized by $(n,d)^*$-projective modules, $(n,d)^*$-projective dimensions of modules over (n,d)-cocoherent rings will be investigated. As corollaries, some new characterizations of right V-rings will be given.

2. $(n,d)^*$-Projective modules and (n,d)-cocoherent rings

We start with the following definition.

Definition 2.1. A right R-module M is said to be $(n,d)^*$-projective if $\mathrm{Ext}^1_R(M,C)=0$ for every n-copresented right R-module C with $id(C)\leq d$. A right R-module C is said to be $(n,d)^*$-injective if $\mathrm{Ext}^1_R(M,C)=0$ for every $(n,d)^*$-projective right R-module M.

Remark 2.2. (1) It is easy to see that if a module M is $(n,d)^*$-projective, then it is $(n',d')^*$-projective for any $n'\geq n$ and $d'\leq d$.

(2) A module M is $(n,0)$-projective if and only if it is $(n,\infty)^*$-projective.

Recall that a short exact sequence of right R-modules $0\to A\to B\to C\to 0$ is said to be *n-copure* [11] if every n-copresented module is injective with respect to this sequence.

Definition 2.3. A short exact sequence of right R-modules $0\to A\to B\to C\to 0$ is said to be (n,d)-copure if every n-copresented module with injective dimension $\leq d$ is injective with respect to this sequence.

Remark 2.4. A short exact sequence of right R-modules $0\to A\to B\to C\to 0$ is n-copure if and only if it is (n,∞)-copure.

Theorem 2.5. *Let M be a right R-module. Then the following statements are equivalent:*

(1) *M is $(n,d)^*$-projective.*

(2) *M is projective with respect to the exact sequence $0 \to C \to B \to A \to 0$ of right R-modules, where C is n-copresented and $id(C) \leq d$.*
(3) *If E' is an $(n-1)$-copresented factor module of a finitely cogenerated injective right R-module E and $id(E') \leq d-1$, then every right R-homomorphism f from M to E' can be lifted to a homomorphism from M to E.*
(4) *Every exact sequence $0 \to M'' \to M' \to M \to 0$ is (n,d)-copure.*
(5) *There exists an (n,d)-copure exact sequence $0 \to K \to P \to M \to 0$ of right R-modules with P projective.*
(6) *There exists an (n,d)-copure exact sequence $0 \to K \to P \to M \to 0$ of right R-modules with P $(n,d)^*$-projective.*
(7) *M is projective with respect to every exact sequence $0 \to C \to B \to A \to 0$ of right R-modules with C $(n,d)^*$-injective.*
(8) *M is projective with respect to every exact sequence $0 \to C \to E \to A \to 0$ of right R-modules with C $(n,d)^*$-injective and E injective.*

Proof. (1) ⇒ (2) By the exact sequence

$$\mathrm{Hom}(M,B) \to \mathrm{Hom}(M,A) \to \mathrm{Ext}_R^1(M,C) = 0.$$

(2) ⇒ (3) Since E is finitely cogenerated injective and E' is $(n-1)$-copresented with $id(E') \leq d-1$, the kernel K of the natural epimorphism $E \to E'$ is n-copresented and $id(K) \leq d$. So (3) follows immediately from (2).

(3) ⇒ (1) For any n-copresented module C with $id(C) \leq d$, there exists an exact sequence $0 \to C \to E \to E' \to 0$, where E is finitely cogenerated injective, E' is $(n-1)$-copresented , and $id(E') \leq d-1$. Hence we get an exact sequence $\mathrm{Hom}(M,E) \to \mathrm{Hom}(M,E') \to \mathrm{Ext}_R^1(M,C) \to \mathrm{Ext}_R^1(M,E) = 0$, and thus $\mathrm{Ext}_R^1(M,C) = 0$ by (3).

(1) ⇒ (4) Assume (1). Then we have an exact sequence

$$\mathrm{Hom}(M',C) \to \mathrm{Hom}(M'',C) \to \mathrm{Ext}_R^1(M,C) = 0$$

for every n-copresented module C with $id(C) \leq d$, and so (4) follows.

(4) ⇒ (5) ⇒ (6) are obvious.

(6) ⇒ (1) By (6), we have an (n,d)-copure exact sequence $0 \to K \xrightarrow{f} P \to M \to 0$ of right R-modules with P $(n,d)^*$-projective, and so, for each n-copresented module C with $id(C) \leq d$, we have an exact sequence $\mathrm{Hom}(P,C) \xrightarrow{f^*} \mathrm{Hom}(K,C) \to \mathrm{Ext}_R^1(M,C) \to \mathrm{Ext}_R^1(P,C) = 0$ with f^* epic. This implies that $\mathrm{Ext}_R^1(M,C) = 0$, and therefore (1) follows.

(1) ⇒ (7) ⇒ (8) ⇒ (1) are similar to the proofs of (1) ⇒ (2) ⇒ (3) ⇒ (1). □

Definition 2.6. (1) The $(n,d)^*$-projective dimension of a module M_R is defined by

$$(n,d)^* - pd(M_R) = inf\{m : \text{Ext}_R^{m+1}(M,C) = 0 \text{ for every n-copresented module } C \text{ with } id(C) \leq d\}$$

(2) r. $(n,d)^*$-PD(R) is defined by

$$\text{r. } (n,d)^*\text{-}PD(R) = sup\{(n,d)^*\text{-}pd(M) : M \text{ is a right } R\text{-module}\}.$$

Definition 2.7. A ring R is called right (n,d)-cocoherent, if every n-copresented right R-module with injective dimension $\leq d$ is $(n+1)$-copresented.

Remark 2.8. (1) It is easy to see that if a ring R is right (n,d)-cocoherent, then it is right (n',d')-cocoherent for any $n' \geq n$ and $d' \leq d$.

(2) Every ring R is right $(n,1)$-cocoherent.

(3) A ring R is right n-cocoherent if and only if it is right (n,∞)-cocoherent.

Lemma 2.9. *Let R be a right (n,d)-cocoherent ring and M a right R-module. Then the following statements are equivalent:*

(1) $(n,d)^*$-*pd*$(M) \leq k$.

(2) $\text{Ext}_R^{k+1}(M,C) = 0$ *for all n-copresented modules C with* $id(C) \leq d$.

Proof. (1) $\Rightarrow$ (2) Use induction on k. Clear if $(n,d)^*$-$pd(M) = k$. If $(n,d)^*$-$pd(M) \leq k-1$. Since C is n-copresented, there exists an exact sequence $0 \to C \to E \to E' \to 0$, where E is finitely cogenerated injective, and E' is $(n-1)$-copresented. Since $id(C) \leq d$, we have $id(E') \leq d$. But R is right (n,d)-cocoherent, C is $(n+1)$-copresented, so E' is n-copresented, and thus $\text{Ext}_R^{k+1}(M,A) \cong \text{Ext}_R^{k}(M,E') = 0$ by induction hypothesis.

(2) $\Rightarrow$ (1) is clear. □

Corollary 2.10. *Let R be a right (n,d)-cocoherent ring and let M_R be $(n,d)^*$-projective. Then* $\text{Ext}_R^k(M,C) = 0$ *for all n-copresented modules C with $id(C) \leq d$ and all positive integers k.*

Corollary 2.11. *Let R be a right (n,d)-cocoherent ring and let M be a right R-module. If the sequence* $0 \to P_k \overset{d_k}{\to} P_{k-1} \overset{d_{k-1}}{\to} \cdots \to P_0 \overset{d_0}{\to} M \to 0$ *is exact with* $P_0, \ldots, P_{k-1}$ $(n,d)^*$*-projective, then* $\text{Ext}_R^{k+1}(M,C) \cong \text{Ext}_R^1(P_k,C)$ *for any n-copresented modules C with $id(C) \leq d$.*

Proof. Since R is right (n,d)-cocoherent and $P_0, P_1, \ldots, P_{k-1}$ are (n,d)-projective, by Corollary 2.10, we have

$$\text{Ext}_R^{k+1}(M,C) \cong \text{Ext}_R^{k}(\text{Ker}(d_0),C) \cong \text{Ext}_R^{k-1}(\text{Ker}(d_1),C) \cong \cdots \cong$$

$$\mathrm{Ext}^1_R(\mathrm{Ker}(d_{k-1}), C) \cong \mathrm{Ext}^1_R(P_k, C).$$ □

Theorem 2.12. *Let R be a right (n, d)-cocoherent ring and M be a right R-module. Then the following statements are equivalent:*

(1) $(n, d)^*$-$pd(M_R) \le k$.

(2) $\mathrm{Ext}^{k+l}_R(M, C) = 0$ *for all n-copresented modules C with $id(C) \le d$ and all positive integers l.*

(3) $\mathrm{Ext}^{k+1}_R(M, C) = 0$ *for all n-copresented modules C with $id(C) \le d$.*

(4) *If the sequence $0 \to P_k \to P_{k-1} \to \cdots \to P_0 \to M \to 0$ is exact with $P_0, \ldots, P_{k-1}$ $(n, d)^*$-projective, then P_k is also $(n, d)^*$-projective.*

(5) *There exists an exact sequence $0 \to P_k \to P_{k-1} \to \cdots \to P_0 \to M \to 0$ of right R-modules with $P_0, \ldots, P_{k-1}, P_k$ $(n, d)^*$-projective.*

Proof. (1) ⇒ (2) Assume (1). Then $(n, d) - pd(M_R) \le k + l - 1$, and so (2) follows from Lemma 2.9.

(2) ⇒ (3) and (4) ⇒ (5) are obvious. (3) ⇒ (4) and (5) ⇒ (1) by Corollary 2.11. □

3. (n, d)-Cosemihereditary rings and (n, d)-V-rings

As the beginning of this section, we extend the concept of n-cosemihereditary rings as follows.

Definition 3.1. A ring R is called right (n, d)-cosemihereditary, if for every finitely cogenerated injective right R-module E, each $(n-1)$-copresented factor module E' of E with $id(E') \le d - 1$ is injective. A ring R is called right cohereditary if it is right $(0, \infty)$-cosemihereditary.

Remark 3.2. (1) It is easy to see that if a ring R is right (n, d)-cosemihereditary, then it is right (n', d')-cosemihereditary for any $n' \ge n$ and $d' \le d$.

(2) Every ring R is right $(n, 1)$-cosemihereditary.

(3) A ring R is right n-cosemihereditary if and only if it is right (n, ∞)-cosemihereditary.

(4) A ring R is right cohereditary if and only if every factor module of a finitely cogenerated injective right R-module is injective.

(5) A ring R is right cosemihereditary if and only if it is right $(1, \infty)$-cosemihereditary.

Theorem 3.3. *The following statements are equivalent for a ring R:*

(1) *R is a right (n,d)-cosemihereditary ring.*

(2) *R is right (n,d)-cocoherent and* $r.(n,d)^*$-$PD(R) \leq 1$.

(3) $\mathrm{Ext}^2_R(M,C) = 0$ *for any right R-module M and any n-copresented right R-module C with* $id(C) \leq d$.

(4) *Every submodule of an* $(n,d)^*$*-projective right R-module is* $(n,d)^*$*-projective.*

(5) *Every submodule of a projective right R-module is* $(n,d)^*$*-projective.*

Proof. (1) ⇒ (2) Let C be an n-copresented right R-module with injective dimension $\leq d$. Then there exists an exact sequence $0 \to C \to E \to E' \to 0$, where E is finitely cogenerated injective, E' is $(n-1)$-copresented and $id(E') \leq d-1$. Since R is right (n,d)-cosemihereditary, E' is finitely cogenerated injective, and so C is $(n+1)$-copresented, it shows that R is right (n,d)-cocoherent. Now let M be a right R-module. Then for any n-copresented right R-module C with $id(C) \leq d$, we have an exact sequence $0 \to C \to E \to E' \to 0$ of right R-modules, where E is finitely cogenerated injective, E' is $(n-1)$-copresented and $id(E') \leq d-1$. Since R is right (n,d)-cosemihereditary, by the above proof, E' is injective. Thus the exact sequence $0 = \mathrm{Ext}^1_R(M,E') \to \mathrm{Ext}^2_R(M,C) \to \mathrm{Ext}^2_R(M,E) = 0$ implies that $\mathrm{Ext}^2_R(M,C) = 0$. This follows that $r.(n,d)^*$-$PD(R) \leq 1$.

(2) ⇒ (3) It follows from Theorem 2.12.

(3) ⇒ (4) Let M be an $(n,d)^*$-projective right R-module and K be its submodule. Then for any n-copresented module C with $id(C) \leq d$, we have an exact sequence $0 = \mathrm{Ext}^1_R(M,C) \to \mathrm{Ext}^1_R(K,C) \to \mathrm{Ext}^2_R(M/K,C) = 0$ by (3), it follows that $\mathrm{Ext}^1_R(K,C) = 0$, as required.

(4) ⇒ (5) It is obvious.

(5) ⇒ (1) Let E' be an $(n-1)$-copresented factor module of a finitely cogenerated injective right R-module E and $id(E') \leq d-1$. Let f be an epimorphism from E to E'. Then for any projective right R-module P and any submodule K of P, K is $(n,d)^*$-projective by (4). So for any n-copresented right R-module C with $id(C) \leq d$, we have an exact sequence $0 = \mathrm{Ext}^1_R(K,C) \to \mathrm{Ext}^2_R(P/K,C) \to \mathrm{Ext}^2_R(P,C) = 0$, which implies that $\mathrm{Ext}^2_R(P/K,C) = 0$. Note that $\mathrm{Ker}(f)$ is n-copresented and $id(\mathrm{Ker}(f)) \leq d$, we get an exact sequence $0 = \mathrm{Ext}^1_R(P/K,E) \to \mathrm{Ext}^1_R(P/K,E') \to \mathrm{Ext}^2_R(P/K,\mathrm{Ker}(f)) = 0$, and then $\mathrm{Ext}^1_R(P/K,E') = 0$, which shows that E'_R is P_R-injective from the exact sequence $\mathrm{Hom}(P,E') \to \mathrm{Hom}(K,E') \to \mathrm{Ext}^1_R(P/K,E')$. Therefore, E' is injective. □

Our following Corollary 3.4 improves [11, Theorem 3.7] partly.

Corollary 3.4. *The following statements are equivalent for a ring R:*

(1) *R is a right n-cosemihereditary ring.*

(2) *R is right n-cocoherent and* $r.(n,0)$-$PD(R) \leq 1$.
(3) $\mathrm{Ext}^2_R(M,C) = 0$ *for any right R-module M and any n-copresented right R-module C.*
(4) *Every submodule of an* $(n,0)$*-projective right R-module is* $(n,0)$*-projective.*
(5) *Every submodule of a projective right R-module is* $(n,0)$*-projective.*

Corollary 3.5. *The following statements are equivalent for a ring R:*

(1) *R is a right cosemihereditary ring.*
(2) *R is right cocoherent and* $r.FCP$-$PD(R) \leq 1$.
(3) $\mathrm{Ext}^2_R(M,C) = 0$ *for any right R-module M and any finitely copresented right R-module C.*
(4) *Every submodule of an FCP-projective right R-module is FCP-projective.*
(5) *Every submodule of a projective right R-module is FCP-projective.*

Corollary 3.6. *The following statements are equivalent for a ring R:*

(1) *R is a right cohereditary ring.*
(2) *R is right co-noetherian and* $r.FCG$-$PD(R) \leq 1$.
(3) $\mathrm{Ext}^2_R(M,C) = 0$ *for any right R-module M and any finitely cogenerated right R-module C.*
(4) *Every submodule of an FCG-projective right R-module is FCG-projective.*
(5) *Every submodule of a projective right R-module is FCG-projective.*

Next we extend the concept of right n-V-rings as follows.

Definition 3.7. A ring R is called right (n,d)-V-ring if every right R-module is $(n,d)^*$-projective.

Remark 3.8. (1) It is easy to see that if $n' \geq n$ and $d' \leq d$, then a right (n,d)-V-ring is a right (n',d')-V-ring.
(2) A ring R is a right n-V-ring if and only if it is a right (n,∞)-V-ring.

Now we give some characterizations of right (n,d)-V-rings.

Theorem 3.9. *The following conditions are equivalent for a ring R:*

(1) *R is a right* (n,d)*-V-ring.*
(2) *Every* $(n-1)$*-copresented right R-module with injective dimension* $\leq d-1$ *is* $(n,d)^*$*-projective.*
(3) *R is right* (n,d)*-cosemihereditary and* $E(S)$ *is* $(n,d)^*$*-projective for every simple right R-module S.*

(4) *R is right (n,d)-cocoherent and for every finitely cogenerated injective right R-module E, every n-copresented factor module E' of E with $id(E') \le d-1$ is $(n,d)^*$-projective.*

(5) *For every finitely cogenerated injective right R-module E, every $(n-1)$-copresented factor module E' of E with $id(E') \le d-1$ is $(n,d)^*$-projective.*

(6) *Every n-copresented right R-module with injective dimension $\le d$ is injective.*

Proof. (1) $\Rightarrow$ (2) and (6) $\Rightarrow$ (1) are obvious.

(2) $\Rightarrow$ (3) Assume (2). Then it is clear that $E(S)$ is $(n,d)^*$-projective for every simple right R-module S. Let E be a finitely cogenerated injective module and E' an $(n-1)$-copresented factor module of E with $id(E') \le d-1$. By (2), E' is $(n,d)^*$-projective, so by Theorem 2.5(3), we have that E' is isomorphic to a direct summand of E and hence E' is injective. Therefore, R is right (n,d)-cosemihereditary.

(3) $\Rightarrow$ (4) Assume (3). Since R is right (n,d)-cosemihereditary, it is right (n,d)-cocoherent by Theorem 3.3. Now let E be a finitely cogenerated injective right R-module and E' an $(n-1)$-copresented factor module of E with $id(E') \le d-1$. Since R is right (n,d)-cocoherent, E' is n-copresented and hence finitely cogenerated. Thus, the injective envelope $E(E')$ of E' is a finitely cogenerated injective module, and so $E(E') \cong \oplus_{i=1}^{k} E(S_i)$ for some simple modules $E_i, i = 1, 2, \dots, k$. Since each E_i is $(n,d)^*$-projective by (3), $E(E')$ is also $(n,d)^*$-projective. Observing that R is right (n,d)-cosemihereditary, by Theorem 3.3, E' is also $(n,d)^*$-projective.

(4) $\Rightarrow$ (5) Let E be a finitely cogenerated injective module and E' an $(n-1)$-copresented factor module of E with $id(E') \le d-1$. Since R is right (n,d)-cocoherent, E' is n-copresented. By (4), E' is $(n,d)^*$-projective.

(5) $\Rightarrow$ (6) Let C be an n-copresented right R-module with $id(C) \le d$. Then there exists an exact sequence $0 \to C \to E \to E' \to 0$ of right R-modules, where E is finitely cogenerated injective, E' is $(n-1)$-copresented and $id(E') \le d-1$. By (5), E' is $(n,d)^*$-projective, so E' is projective respect to this exact sequence by Theorem 2.5(3). This follows that C is isomorphic to a direct summand of E, and therefore C is injective. □

Recall that a right R-module M is called *FCG-projective* [11] if $\mathrm{Ext}_R^1(M, A) = 0$ for every finitely cogenerated right R-module A. By Remark 2.2, a right R-module is FCG-projective if and only if it is $(0,\infty)^*$-projective, a right R-module is FCP-projective if and only if it is $(1,\infty)^*$-projective, every FCG-projective module is FCP-projective.

Corollary 3.10. *The following conditions are equivalent for a ring R:*

(1) *R is a right V-ring.*
(2) *R is a right $(0,\infty)$-V-ring.*
(3) *R is a right $(1,\infty)$-V-ring.*
(4) *Every right R-module is FCG-projective.*
(5) *R is right cohereditary and $E(S)$ is FCG-projective for every simple right R-module S.*
(6) *R is right co-noetherian and for every finitely cogenerated injective right R-module E, every finitely cogenerated factor module E' of E is FCG-projective.*
(7) *For every finitely cogenerated injective right R-module E , every factor module E' of E is FCG-projective.*
(8) *Every finitely cogenerated right R-module is injective.*
(9) *Every finitely cogenerated right R-module is FCP-projective.*
(10) *R is right cosemihereditary and $E(S)$ is FCP-projective for every simple right R-module S.*
(11) *R is right cocoherent and for every finitely cogenerated injective right R-module E, every finitely copresented factor module E' of E is FCP-projective.*
(12) *For every finitely cogenerated injective right R-module E , every finitely cogenerated factor module E' of E is FCP-projective.*
(13) *Every finitely copresented right R-module is injective.*

Proof. (2) $\Rightarrow$ (3) is obvious. By Theorem 3.9, we have

(2) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) $\Leftrightarrow$ (6) $\Leftrightarrow$ (7) $\Leftrightarrow$ (8); and (3) $\Leftrightarrow$ (9) $\Leftrightarrow$ (10) $\Leftrightarrow$ (11) $\Leftrightarrow$ (12) $\Leftrightarrow$ (13).

(1) $\Rightarrow$ (8) Let R be a right V-ring. Then every simple right R-module is injective. For any finitely cogenerated right R-module M, we have $E(M) \cong E(S_1) + \cdots + E(S_n)$ for some finite set $S_1, \ldots, S_n$ of simple modules by [1, Proposition 18.18], so $E(M) \cong S_1 + \cdots + S_n$ is semisimple. Thus M is a direct summand of $E(M)$, and therefore M is injective.

(13) $\Rightarrow$ (1) Let S be any simple right R-module. Suppose S is not injective. Let $x \in E(S)\backslash S$ and let A be a submodule of $E(S)$ maximal with respect to $S \subseteq A$ and $x \notin A$, then $0 \neq x + A \in \cap\{K \leq E(S)/A \mid K \neq 0\}$, which implies that $E(S)/A$ is finitely cogenerated and whence A is finitely copresented. By (13), A is injective. It follows that $A = E(S)$, which contradicts the fact that $x \notin A$. Hence S is injective and so R is a right V-ring. $\square$

Recall that a right R-module M is called *n-presented* [3] if there is an exact sequence of right R-modules $F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to M \to 0$ where each F_i is a finitely generated free, equivalently projective right R-module; a left R-module M is called $(n,0)$-flat [10] if $\mathrm{Tor}_1^R(A,M)=0$ for every n-presented right R-module A. A ring R is called right n-regular [10] if every n-presented right R-module is projective. By [10, Theorem 3.9], a ring R is right n-regular if and only if every left R-module M is $(n,0)$-flat.

Theorem 3.11. *Let R be a commutative ring. Then every $(n,0)$-projective module is $(n,0)$-flat.*

Proof. Let M be an $(n,0)$-projective module. To prove M is $(n,0)$-flat, we need prove $\mathrm{Tor}_1^R(A,M)=0$ for every n-presented R-module A. Since A is n-presented, $\mathrm{Hom}_R(A,E(S))$ is n-copresented for any simple module S. Let $0\to K\to P\to M\to 0$ be an exact sequence of R-modules with P projective. Then by Theorem 2.5, this exact sequence is n-copure. And so we get an exact sequence of R-modules

$$0\to \mathrm{Hom}_R(M,\mathrm{Hom}_R(A,E(S)))\to \mathrm{Hom}_R(P,\mathrm{Hom}_R(A,E(S)))\to$$

$$\mathrm{Hom}_R(K,\mathrm{Hom}_R(A,E(S)))\to 0.$$

By [1, Proposition 20.6, Proposition 20.7], this induces an exact sequence

$$0\to \mathrm{Hom}_R(M\otimes_R A,E(S))\to \mathrm{Hom}_R(P\otimes_R A,E(S))\to \mathrm{Hom}_R(K\otimes_R A,E(S))\to 0.$$

Let $\mathscr{S}_0$ denote an irredundant set of representatives of the simple R-modules and let $C=\prod_{S\in\mathscr{S}_0}E(S)$. Then by [1, Corollary 18.16], C is a cogenerator. And we have an exact sequence of R-modules

$$0\to \mathrm{Hom}_R(M\otimes_R A,C)\to \mathrm{Hom}_R(P\otimes_R A,C)\to \mathrm{Hom}_R(K\otimes_R A,C)\to 0.$$

So, by [1, Proposition 18.14], the sequence

$$0\to K\otimes_R A\to P\otimes_R A\to M\otimes_R A\to 0$$

of R-modules is exact. This shows that $\mathrm{Tor}_1^R(A,M)=0$, as required. □

Corollary 3.12. *Let R be a commutative n-V-ring. Then it is an n-regular ring.*

The following result is well-known.

Corollary 3.13. *Let R be a commutative V-ring. Then it is a regular ring.*

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