



Buckling Analysis of Rectangular Beams Having Ceramic Liners at Its Top and Bottom Surfaces with the help of the Exact Transfer Matrix

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Abstract

In this study the elastic buckling behavior of beams with rectangular cross section is studied analytically. It is assumed that both the top and bottom surfaces of the beam are ceramic coated. The aluminum (Al) is chosen as a core material while the aluminum-oxide (Al_2O_3) is preferred as a liner (face) material. The transfer matrix method based on the Euler-Bernoulli beam theory is employed in the analysis. The exact transfer matrix in terms of equivalent bending stiffness is presented together with the exact buckling equations for hinged-hinged, clamped-hinged, clamped-free, and finally clamped-clamped boundary conditions. After verifying the results for beams without liners, dimensionless buckling loads of the beam with ceramic liners are numerically computed for each boundary condition. The effect of the thickness of the ceramic liner on the buckling loads is also investigated. It is found that a ceramic liner enhances noticeably the buckling loads. As an additional study those effects are also examined for the ratios of elasticity modulus of face material to core material in a wide range.

Keywords: Exact buckling, Euler-Bernoulli, transfer matrix, stability, sandwich beam, critical buckling loads

1. Introduction

Buckling of columns being a physical phenomenon is a matter of significance in the design of structural elements. Underestimation of this phenomenon may lead to disastrous results.

Buckling occurs in beams subjected to compressive loads. The longer and more slender the column is, the lower the safe compressive stress that it can stand. The maximum load at which the column tends to have lateral displacement or tends to buckle is known as critical buckling or crippling load. Therefore in the design of columns, determination of the critical buckling loads becomes an inevitable stage.

Research into buckling of columns dates back to late 1700s with Euler's study [1]. Greenhill's [2], Dinnik's [3], and Timoshenko and Gere's [4] studies are some subsequent fundamental works to Euler's [1] study in the related realm. Numerous analytical and numerical works on the stability of columns were conducted after those pioneers [5-42]. From those methods which can be used to determine the elastic critical buckling load may be summarized as the differential equation solution method [1-11], energy methods [12-16], the finite element method [17-22], the finite difference method [23], the modified slope deflection method [24], the effective-thickness concept [25], the multi-segment integration technique [23], the variational iteration



method [26-28], the homotopy perturbation method [28-33], Adomian decomposition method [28, 34], the transfer matrix method [35-39], the stiffness matrix method [39], the fictitious load method [40], the modified vibration modes [41] and much more [42-46]. In the solution of more complex problems, some of the solution methods mentioned above may also be used in a combined manner.

The governing buckling differential equation may be obtained based on the either beam or elasticity [9-11] theories. The beam theories allow to solve much more complex problems for a beam or a system of beams. The governing equation in differential form may then be solved by using exact or approximate solution techniques. As may be guessed it is possible to obtain exact solutions for relatively simple problems.

As is known, the gain of strain energy in the elements is less than the potential energy of the loads which are lower than the elastic critical load. If the change of these two energies is zero then the structure will not resist any disturbance. This stage at which the stiffness of the structure is zero is defined as a critical instability condition in the energy methods.

In the finite element method, in which the structure is subdivided into a series of fairly short elements, buckling is considered by adding a geometrical stiffness matrix to the element equations. The resulting eigenvalue problem is then solved by applying several techniques such as vector iteration methods (inverse iteration, forward iteration, and Rayleigh quotient iteration), transformation methods (Jacobi method, the subspace iteration method).

The transfer matrix method is one of the methods to the solution of initial value problem (IVP). Many problems from the simplest one to the complex ones may be solved with the help of this technique. The governing equations in canonical form, which is a relationship between the section quantities and their first derivatives, may be obtained from the either beam or elasticity theories. In the method, determination of the elements of the transfer matrix is crucial. The overall transfer matrix, which is obtained from the solution of a set of differential equation having either constant or variable coefficients, relates the section quantities at the initial point and at any point on the beam axis. The accuracy of the solution directly depends on the accuracy of the overall transfer matrix to be derived. It is possible to obtain some closed form solutions for the governing equation with constant coefficients. Otherwise, in case of existence of variable coefficients, the transfer matrix should be determined numerically. Contrary to the finite elements method, orders of the resulting matrices are independent from the number of elements to be considered. Therefore it is possible to construct easy-to-use algorithms with the transfer matrix method which are highly accurate and computationally efficient.

A sandwich structure usually consists of two relatively thin, stiff and strong faces separated by a relatively thick lightweight core. The main purpose of a sandwich structure is to achieve a stiff and simultaneously light component. That is higher stiffness and strength can be achieved by sandwich structures without increasing the weight dramatically. Sandwich constructions are also used for the aim of thermal insulation, corrosion insulation, vibration/noise damping, and water ingress prevention. Buckling phenomenon is a crucial task to be considered in the analysis of such structures [47-53]. This may be conducted by using any of the methods mentioned above. Recently, Sayyad and Ghugal [54] reviewed bending buckling, and free vibration of laminated composite and sandwich beams up to 2017s.

As is well known, Euler-Bernoulli theory is a simple beam theory by which one may get exact results which are reasonable for long and slender structural members. The theory offers overestimate buckling loads for relatively short columns. In other words, Euler buckling loads

are independent from the ratio of the total length of the beam to the width of the section. In the present study, the effect of the existence of the liners on the buckling loads of a rectangular beam is intended for an examination with the help of the transfer matrix method. As a basic work, Euler-Bernoulli beam theory is employed to achieve fast and reasonable buckling loads.

2. Theory

Consider a beam subjected to an axial compressive load N whose critical value called the critical buckling load satisfies the following fourth order Euler-Bernoulli differential equation in terms of transverse displacement, w [1-8].

$$\frac{d^4w}{dx^4} + \frac{N}{EI} \frac{d^2w}{dx^2} = 0 \quad (1)$$

Where, x is the coordinate along the beam axis, E is Young's modulus and I is the area moment of inertia about y axis (Fig. 1). Derivation of Eq. (1) may be found in References [1-8]. The general solution of the foregoing well-known ODE is

$$w(x) = A\cos\alpha x + B\sin\alpha x + Cx + D \quad (2)$$

where

$$\alpha = \sqrt{\frac{N}{EI}} \quad (3)$$

Solution to Eq. (2) is used with the following classical boundary conditions to determine the critical buckling loads of the beam. The boundary conditions for hinged ends are,

$$w = 0, \quad w'' = 0 \quad (4)$$

for clamped ends are,

$$w = 0, \quad w' = 0 \quad (5)$$

and for free ends are

$$w''' = 0, \quad w^{IV} = 0 \quad (6)$$

The number of the problems to be directly solved by Eq. (1) is limited. To consider a wider range applications of beams with initial axial force, the transfer matrix method is preferred in the present study. As stated in the introduction, one need to put the single fourth order differential equation given in Eq. (1) into a set of four differential equations of first order to be able to apply the transfer matrix method. The equations governing the elastic buckling behavior of an Euler-Bernoulli beam is given in canonical form as follows [5]

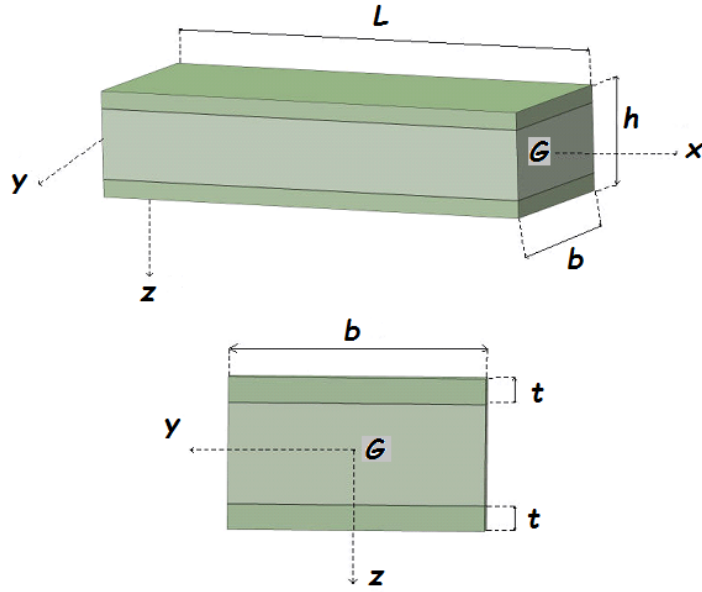


Fig. 1. The beam geometry and the coordinates

$$\frac{dw}{dx} = \theta \quad (7a)$$

$$\frac{d\theta}{dx} = -\frac{M}{EI} \quad (7b)$$

$$\frac{dM}{dx} = T + N\theta \quad (7c)$$

$$\frac{dT}{dx} = 0 \quad (7d)$$

where, w is still the transverse displacement, θ is the rotation, M is the bending moment, T is the shear force, N is the axial compressive constant initial force. Equation (7), which is identically equal to Eq. (1), may be written in a compact form as

$$\mathbf{S}'(x) = \mathbf{D} \mathbf{S}(x) \quad (8)$$

where the state vector which comprises the section quantities is defined by

$$\mathbf{S}(x) = \begin{bmatrix} w(x) \\ \theta(x) \\ M(x) \\ T(x) \end{bmatrix} \quad (9)$$

and the differential transfer matrix is

$$\mathbf{D} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{-1}{EI} & 0 \\ 0 & N & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (10)$$

There are a few ways for the determination of the elements of the transfer matrix, \mathbf{F} [5]. If the elements of the differential transfer matrix are constants as in beams having unchanged section and material properties along the beam axis, it is possible to get an exact solution for the element transfer matrix as in the present study.

Recalling that the element transfer matrix satisfy the similar differential equation for the state vector as in Eq. (8) the following may be written [5]

$$\mathbf{F}'(x) = \mathbf{D} \mathbf{F}(x) \quad (11)$$

Solution of Eq. (11) with the initial conditions

$$\mathbf{F}(x = 0) = \mathbf{I} \quad (12)$$

gives us the exact element transfer matrix in the form of a matrix exponential.

$$\mathbf{F}(x) = e^{x\mathbf{D}} = \mathbf{1} + x\mathbf{D} + \frac{x^2}{2!}\mathbf{D}^2 + \frac{x^3}{3!}\mathbf{D}^3 + \frac{x^4}{4!}\mathbf{D}^4 + \frac{x^5}{5!}\mathbf{D}^5 + \frac{x^6}{6!}\mathbf{D}^6 + \dots \quad (13)$$

In the above, \mathbf{I} refers the unit matrix. In Eq. (13) the higher powers of the differential matrix which are equal or greater than four may be written in terms of the differential transfer matrices having smaller powers of up to three. To this end one may resort to Cayley-Hamilton theorem which states that every square matrix satisfies its own characteristic equation, $|\mathbf{D} - \mu\mathbf{I}| = 0$. Using Eq. (13) together with Cayley Hamilton theorem, Eq. (13) takes the following form in terms of up to the third powers of the differential transfer matrix [5].

$$\begin{aligned} \mathbf{F}(x) = \mathbf{1} + x\mathbf{D} + \left(\frac{x^2}{2!} - \frac{x^4}{4!}\alpha^2 + \frac{x^6}{6!}\alpha^4 - \frac{x^8}{8!}\alpha^6 + \frac{x^{10}}{10!}\alpha^8 - \dots \right) \mathbf{D}^2 \\ + \left(\frac{x^3}{3!} - \frac{x^5}{5!}\alpha^2 + \frac{x^7}{7!}\alpha^4 - \frac{x^9}{9!}\alpha^6 + \dots \right) \mathbf{D}^3 \end{aligned} \quad (14)$$

The coefficients of the differential transfer matrix, which are in series form, correspond explicitly to the following functions

$$\mathbf{F}(x) = \mathbf{1} + x\mathbf{D} + \frac{1 - \cos(\alpha x)}{\alpha^2} \mathbf{D}^2 + \frac{\alpha x - \sin(\alpha x)}{\alpha^3} \mathbf{D}^3 \quad (15)$$

The explicit forms of the elements of the exact element transfer matrix in Eq. (15) are given below in terms of the equivalent bending stiffness.

$$\begin{aligned}
 F_{1,1} &= F_{4,4} = 1 \\
 F_{2,1} &= F_{3,1} = F_{4,1} = F_{4,2} = F_{4,3} = 0 \\
 F_{1,2} &= F_{3,4} = \frac{\sin\left(x \sqrt{\frac{N}{E_{eq}I}}\right)}{\sqrt{\frac{N}{E_{eq}I}}} \\
 F_{1,3} &= F_{2,4} = \frac{\cos\left(x \sqrt{\frac{N}{E_{eq}I}}\right) - 1}{N} \\
 F_{1,4} &= \frac{\sqrt{E_{eq}I} \sin\left(x \sqrt{\frac{N}{E_{eq}I}}\right)}{N^{\frac{3}{2}}} - \frac{x}{N} \\
 F_{2,2} &= F_{3,3} = \cos\left(x \sqrt{\frac{N}{E_{eq}I}}\right) \\
 F_{2,3} &= -\frac{\sin\left(x \sqrt{\frac{N}{E_{eq}I}}\right)}{\sqrt{N E_{eq}I}} \\
 F_{3,2} &= \sqrt{N E_{eq}I} \sin\left(x \sqrt{\frac{N}{E_{eq}I}}\right)
 \end{aligned} \tag{16}$$

The overall transfer matrix relates the state vectors at both ends of the beam as follows

$$\mathbf{S}(L) = \mathbf{F}(L) \mathbf{S}(0) \tag{17}$$

This equation may be expanded as

$$\begin{bmatrix} W \\ \theta \\ M \\ T \end{bmatrix}_{x=L} = \begin{bmatrix} F_{1,1} & F_{1,2} & F_{1,3} & F_{1,4} \\ F_{2,1} & F_{2,2} & F_{2,3} & F_{2,4} \\ F_{3,1} & F_{3,2} & F_{3,3} & F_{3,4} \\ F_{4,1} & F_{4,2} & F_{4,3} & F_{4,4} \end{bmatrix}_{x=L} \begin{bmatrix} W \\ \theta \\ M \\ T \end{bmatrix}_{x=0} \tag{18}$$

In the present study the following boundary conditions are implemented (Fig. 2) for hinged (pinned) ends as

$$w = 0, \quad M = 0 \quad (19)$$

for clamped ends as

$$w = 0, \quad \theta = 0 \quad (20)$$

and for free ends as

$$T = 0, \quad M = 0 \quad (21)$$

After implementing those boundary conditions in Eq. (13), the buckling equations are obtained as follows

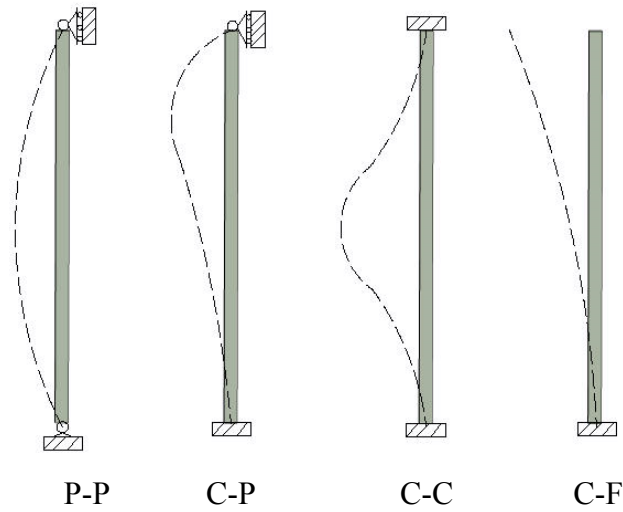


Fig. 2. Classical boundary conditions.

$$\begin{aligned}
 |A|_{\text{hinged-hinged}} &= \begin{vmatrix} F(L)_{1,2} & F(L)_{1,4} \\ F(L)_{3,2} & F(L)_{3,4} \end{vmatrix} = 0 \\
 |A|_{\text{clamped-free}} &= \begin{vmatrix} F(L)_{3,3} & F(L)_{3,4} \\ F(L)_{4,3} & F(L)_{4,4} \end{vmatrix} = 0 \\
 |A|_{\text{clamped-hinged}} &= \begin{vmatrix} F(L)_{1,3} & F(L)_{1,4} \\ F(L)_{3,3} & F(L)_{3,4} \end{vmatrix} = 0 \\
 |A|_{\text{clamped-clamped}} &= \begin{vmatrix} F(L)_{1,3} & F(L)_{1,4} \\ F(L)_{2,3} & F(L)_{2,4} \end{vmatrix} = 0
 \end{aligned} \quad (22)$$

In the above the axial force making the corresponding determinants equal to zero is referred to as the critical buckling load, N_{cr} . These loads may be found by using the searching determinant method together with the bi-sectioned method, or other solution techniques.

The more compact forms of the determinants are given in Appendix.

3. Verifications of the results

For the sake of simplicity, the application of this method may be shown on the simple model of a column with uniform cross-section that is subjected to the axial compressive force N . The column is assumed to be made of a single isotropic and homogeneous material. The following dimensionless buckling load is defined to verify the results with the open literature

$$\beta = \frac{L^2}{EI}N \quad (23)$$

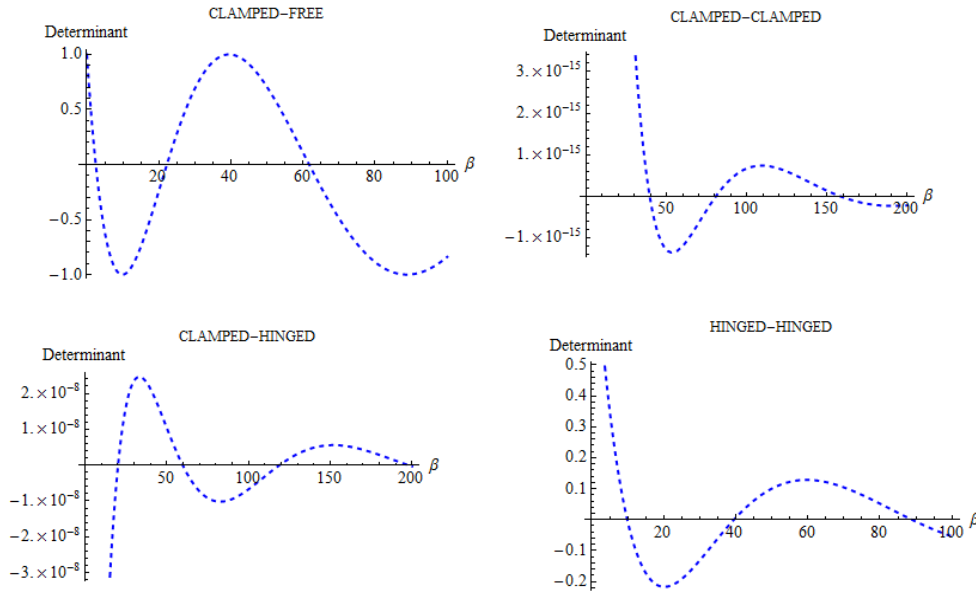


Fig. 3. Determinant-dimensionless frequency curves under classical boundary conditions.

Dimensionless buckling loads are listed in Table 1 in a comparative manner with the literature. A perfect harmony is observed among the results. The corresponding determinant curves are illustrated in Fig. 3. Further analytical verifications are given in Appendix A.

Table 1. Comparison of dimensionless critical Euler buckling loads of uniform columns without liners

	C-F	P-P	C-P	C-C
Present (Transfer matrix method)	2.4674	9.8696	20.1907	39.4784
Wang et al. [6] (Exact)	2.4674	9.8696	20.1907	39.4784
Saha and Banu [23] (Finite difference method)	--	9.8892	20.2044	39.786
Saha and Banu [23] (Multi-segment integration)	--	9.8728	20.1876	39.6408
Coşkun and Atay [27] (Variational iteration)	2.4674	9.8696	20.1908	39.4916
Eryılmaz et al. [33] (Homotopy analysis)	2.4674	9.8696	20.1907	39.4784

4. Effect of the liner thickness on the buckling loads

As stated before without increasing the weight dramatically, higher stiffness and strength can be achieved by sandwich structures with soft cores. Chakrabartia et al. [48] verified this for the

buckling of laminated sandwich beams with soft cores. Although the proposed method may be applied to the laminated structures having anisotropic characteristics after a certain effort, both the face and core material are assumed to be isotropic and homogeneous in the present study for simplicity.

To study the effect of the total thickness of the bottom and top liners on the buckling loads, the following dimensionless quantity is defined.

$$\lambda = \frac{2t}{h} \quad (24)$$

where t is the thickness of a layer, h is the width of the rectangular section having length b (Fig. 1). The equivalent bending stiffness of the uniform section is derived as

$$E_{eq}I = ((3\lambda^2 - \lambda^3 - 3\lambda + 1)E_1 + (-3\lambda^2 + \lambda^3 + 3\lambda)E_2) \frac{bh^3}{12} \quad (25)$$

In the above E_1 is Young's modulus of the core material while E_2 stands for the elasticity modulus of the liner material (face material). The dimensionless buckling load may now be defined in terms of Young's modulus of the core material.

$$\beta = \frac{L^2}{E_1 I} N \quad (26)$$

The material and geometrical properties used in the parametric study are: $E_1 = E_{core} = 70.0 \cdot 10^9 \text{ GPa (Al)}$, $E_2 = E_{liner} = 393.0 \cdot 10^9 \text{ GPa (Al}_2\text{O}_3)$, $b = 2h$; $L = 1.0 \text{ m}$; $L/h = 10$. Effect of the total thickness of the liners with respect to the height of the section is seen in Table 2 and Fig. 4 under all classical boundary conditions.

Table 2. Dimensionless critical buckling loads of uniform columns with liners

λ	C-F	P-P	C-P	C-C
0.0	2.4674	9.8696	20.1907	39.4784
0.01	2.80556	11.2222	22.9578	44.8889
0.02	3.13695	12.5478	25.6696	50.1912
0.05	4.09123	16.3649	33.4785	65.4597
0.1	5.55282	22.2113	45.4387	88.8451
0.2	8.02342	32.0937	65.6556	128.375
0.3	9.94754	39.7902	81.4007	159.161
0.4	11.3935	45.5739	93.2327	182.296
0.5	12.4295	49.7181	101.711	198.873
0.6	13.124	52.4961	107.394	209.985
0.7	13.5453	54.1812	110.841	216.725
0.8	13.7616	55.0464	112.611	220.186
0.9	13.8413	55.3652	113.263	221.461
1.0	13.8527	55.4108	113.357	221.643

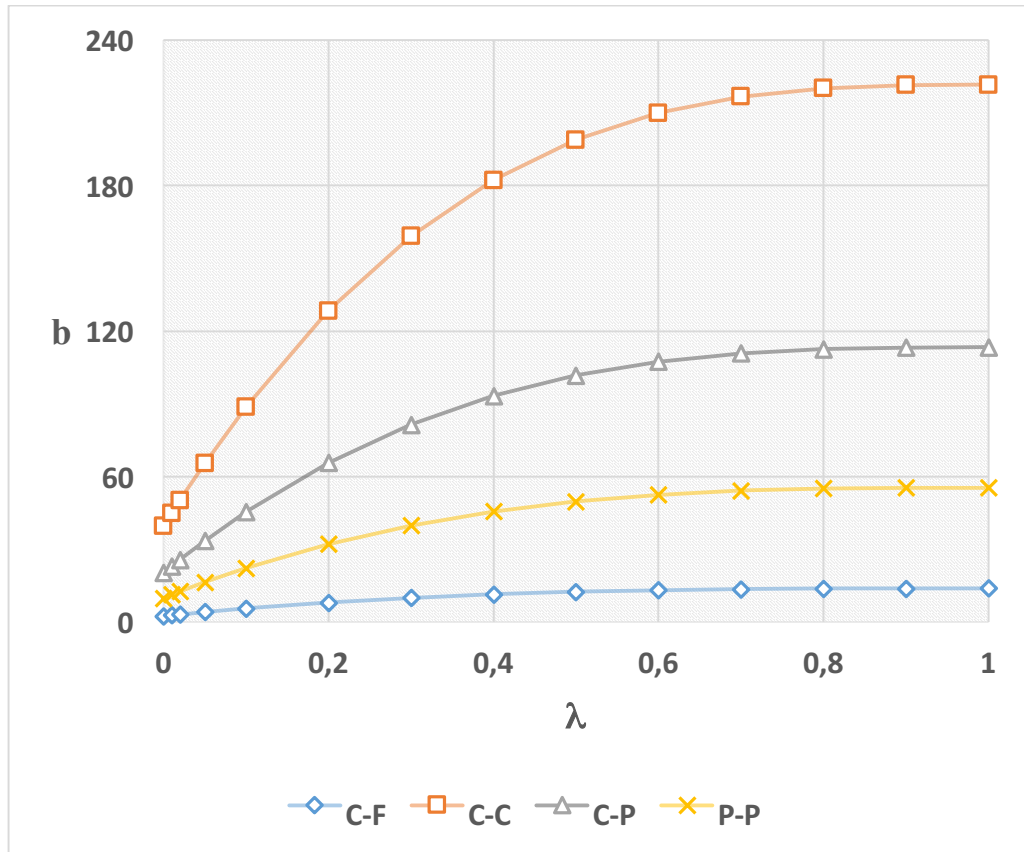


Fig. 4. Variation of the buckling loads with the total thickness of the liners

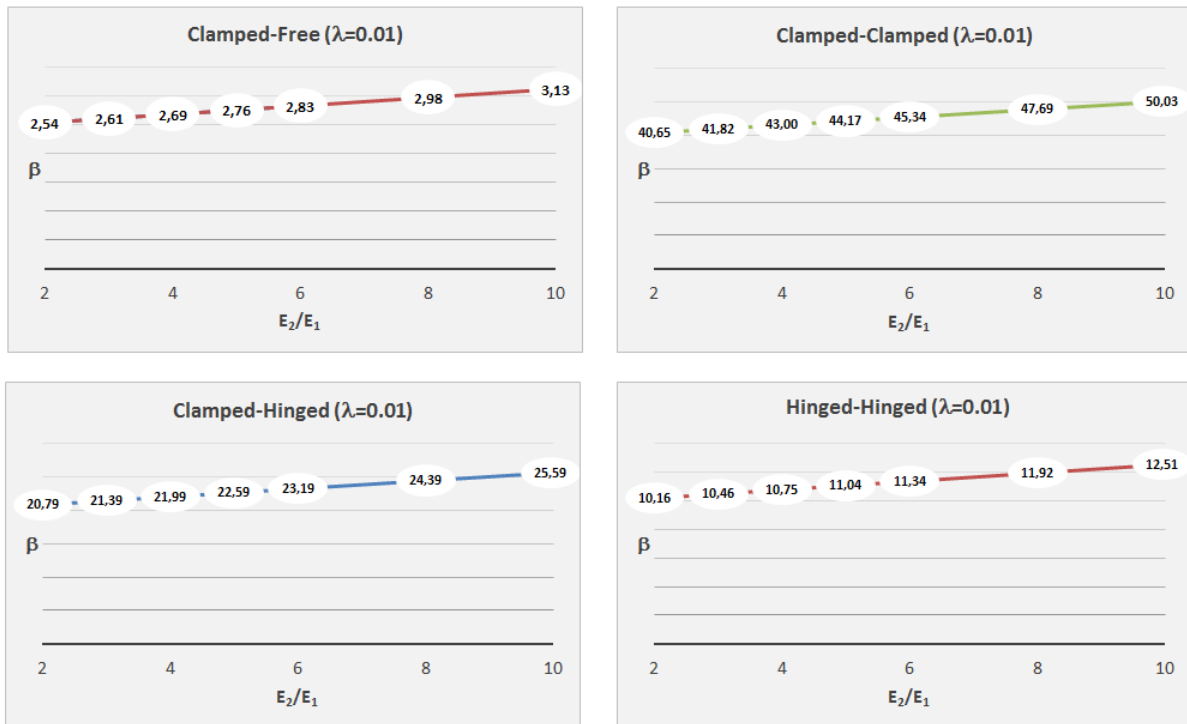


Fig. 5. Variation of the buckling loads with E_2/E_1 ratios and boundary conditions under all boundary conditions for $\lambda = 0.01$

From Table 2, it is revealed that for even very thin liners, $\lambda = 0.01$, the buckling load increases rapidly by over 10%. For $\lambda = 0.1$, the beam can withstand greater forces of around 22% more than the beam without liners. From Fig. 4, it may be concluded that the total thickness of the liners more than 50% is not feasible to enhance the buckling loads for the example considered.

To more generalize the problem for practical applications, let's consider a sandwich beam having a soft core and investigate the variation of the dimensionless buckling loads with E_2/E_1 ratios, liner thickness and boundary conditions ($b = 2h$; $L = 1.0\text{ m}$; $L/h = 10$). The results are presented in Fig. 5, and Table 3.

Table 3. Variation of the buckling loads with the total thickness of the liners and E_2/E_1 ratios under all boundary conditions

λ	E_2/E_1									
	2	3	5	8	10	2	3	5	8	10
	Clamped-Free					Clamped-Clamped				
0.0	2.4674	2.4674	2.4674	2.4674	2.4674	39.4784	39.4784	39.4784	39.4784	39.4784
0.01	2.54069	2.61397	2.76054	2.98039	3.12696	40.651	41.8235	44.1686	47.6863	50.0314
0.02	2.6125	2.75761	3.04781	3.48312	3.77333	41.8001	44.1217	48.765	55.7299	60.3732
0.05	2.81931	3.17123	3.87505	4.93079	5.63462	45.109	50.7396	62.0009	78.8927	90.1539
0.1	3.13607	3.80473	5.14206	7.14806	8.48539	50.1771	60.8757	82.273	114.369	135.766
0.2	3.67149	4.87558	7.28377	10.896	13.3042	58.7439	78.0094	116.54	174.337	212.868
0.3	4.08848	5.70957	8.95173	13.815	17.0571	65.4157	91.3531	143.228	221.04	272.914
0.4	4.40184	6.33629	10.2052	16.0085	19.8774	70.4295	101.381	163.283	256.136	318.038
0.5	4.62638	6.78535	11.1033	17.5802	21.8982	74.022	108.566	177.653	281.284	350.371
0.6	4.77689	7.08638	11.7054	18.6338	23.2528	76.4302	113.382	187.286	298.141	372.045
0.7	4.86818	7.26896	12.0705	19.2729	24.0744	77.8909	116.303	193.128	308.366	385.191
0.8	4.91506	7.36272	12.258	19.601	24.4964	78.641	117.804	196.129	313.617	391.942
0.9	4.93233	7.39727	12.3271	19.7219	24.6518	78.9174	118.356	197.234	315.551	394.429
1.0	4.9348	7.4022	12.337	19.7392	24.674	78.9568	118.435	197.392	315.827	394.784
	Clamped-Hinged					Hinged-Hinged				
0.0	20.1907	20.1907	20.1907	20.1907	20.1907	9.8696	9.8696	9.8696	9.8696	9.8696
0.01	20.7904	21.3901	22.5895	24.3885	25.5879	10.1627	10.4559	11.0422	11.9216	12.5078
0.02	21.3781	22.5655	24.9402	28.5024	30.8771	10.45	11.0304	12.1913	13.9325	15.0933
0.05	23.0704	25.9501	31.7095	40.3486	46.1081	11.2773	12.6849	15.5002	19.7232	22.5385
0.1	25.6624	31.1341	42.0775	58.4925	69.4359	12.5443	15.2189	20.5683	28.5922	33.9416
0.2	30.0438	39.8969	59.603	89.1623	108.868	14.686	19.5023	29.1351	43.5842	53.2169
0.3	33.456	46.7213	73.252	113.048	139.579	16.3539	22.8383	35.8069	55.2599	68.2286
0.4	36.0203	51.8498	83.5089	130.997	162.657	17.6074	25.3451	40.8207	64.034	79.5095
0.5	37.8576	55.5245	90.8583	143.859	179.193	18.5055	27.1414	44.4132	70.3209	87.5927
0.6	39.0893	57.9878	95.7848	152.48	190.277	19.1076	28.3455	46.8214	74.5353	93.0112
0.7	39.8363	59.4819	98.773	157.71	197.001	19.4727	29.0759	48.2821	77.0915	96.2977
0.8	40.2199	60.2491	100.308	160.395	200.454	19.6603	29.4509	49.0322	78.4041	97.9854
0.9	40.3613	60.5318	100.873	161.384	201.726	19.7293	29.5891	49.3085	78.8877	98.6072
1.0	40.3815	60.5722	100.954	161.526	201.907	19.7392	29.6088	49.348	78.9568	98.696

As stated above, if the aluminum is a core material and the aluminum-oxide is a face material, the buckling load increases rapidly over 10% for $\lambda = 0.01$. This contribution is linearly changed with the E_2/E_1 ratios for the same ratio of λ (Fig. 5). For instance, under all boundary conditions and for $\lambda=0.01$, the buckling loads increase by 3% if $E_2/E_1 = 2$, by 12% if $E_2/E_1 = 5$, and by 27% if $E_2/E_1 = 10$.

For $\lambda=0.1$, the improved buckling load reaches about 1.3 times the buckling load of the beam made from only the core material if $E_2/E_1 = 2$. It is around 2 times of the buckling load without liners if $E_2/E_1 = 5$, and is approximately 3.5 times that load if $E_2/E_1 = 10$.

7. Conclusions

In the present study the effect of the thickness of liners on the critical buckling loads of a beam having uniform rectangular cross-section is investigated based on the Euler-Bernoulli beam theory under several boundary conditions. Real-life materials together with hypothetical ones are used in the examples.

The transfer matrix method is chosen for the solution procedure due to its effective, economical, and accurate results together with its wider applications in the engineering realm. The element transfer matrix is obtained analytically by solving a set of four differential equations of first order. The effective bending rigidity is used in the determination of the elements of the exact element transfer matrix. This approach is reasonably suitable for especially industrial applications

As a first stage of the present work, the critical buckling loads are obtained for a uniform beam without liners and compared with the literature. Perfect agreement is observed among the buckling loads.

In the next stage, a rectangular sectioned beam is handled to observe the variation of the effect of the liner thickness on the buckling loads. The aluminum (*Al*) is used for a core material and the aluminum-oxide (*Al₂O₃*) for a liner (face) material. It is discovered that for even very thin liners, $\lambda = 0.01$, the buckling load increases rapidly by over 10%. For $\lambda = 0.1$, the beam can tolerate greater buckling loads of around 22% more than the buckling loads of the beam without liners.

In the last stage, a generalized parametric study is conducted for various ratios of Young's modulus of the core material to the face material from 2 to 10. It is observed that under all boundary conditions and for $\lambda=0.01$, the buckling loads increase by 3% if $E_2/E_1 = 2$, by 27% if $E_2/E_1 = 10$. For $\lambda=0.1$, the improved buckling load reaches about 1.3 times the buckling load of the beam made from only the core material if $E_2/E_1 = 2$, and is around 3.5 times that load if $E_2/E_1 = 10$.

It is chiefly concluded that the thickness of the liner strongly affects the buckling loads. However the ratio of the total thickness of the liners to the total width of the section is not feasible if it reaches over 50%.

It is also revealed that the transfer matrix method leading to exact solutions may be used effectively in the analysis of elastic stability problems of such structures. The method offered here may also be applied to the multi-spanned beams, beam systems having different bending rigidities under classical/non-classical boundary conditions.

Notations

α	Dimensionless buckling parameter
β	Dimensionless buckling load
θ	Rotation about y-axis
A	Characteristic buckling coefficient matrix
b	Base length of rectangular cross section of the beam
D	Differential transfer matrix
E	Elasticity modulus of the beam material
$E_{eq}I$	Equivalent bending stiffness
F	Transfer matrix
h	Width of rectangular section
I	Area moment of inertia about y-axis
I	Unit matrix
L	Length of the beam
M	Bending moment about y-axis
N	Axial compressive load
N_{cr}	Critical buckling load
S	State vector
t	Thickness of one of liners
T	Shearing force
w	Transverse displacement along z-axis
x	Position coordinate along the beam axis

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Appendix: Analytical Verification of the Results

Consider Eq. (16) at section $x = L$ for a beam without liners.

$$F(L) = \begin{bmatrix} 1 & \frac{\sin(\alpha L)}{\alpha} & \frac{\cos(\alpha L) - 1}{EI\alpha^2} & \frac{\sin(\alpha L) - \alpha L}{EI\alpha^3} \\ 0 & \cos(\alpha L) & -\frac{\sin(\alpha L)}{EI\alpha} & \frac{\cos(\alpha L) - 1}{EI\alpha^2} \\ 0 & EI\alpha \sin(\alpha L) & \cos(\alpha L) & \frac{\sin(\alpha L)}{\alpha} \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (A.1)$$

The elements of the transfer matrix given above is used for the expansion of the determinants given by Eq. (22) as follows

Beam with hinged ends

The expansion of the determinant leads to

$$|A|_{P-P} = \frac{\sin^2(\alpha L)}{\alpha^2} - \frac{\sin^2(\alpha L)}{\alpha^2} + \frac{\sin(\alpha L) L}{\alpha} \quad (A.2)$$

After simplification we are left with

$$|A|_{hinged-hinged} = \sin(\alpha L) = 0 \quad (A.3)$$

For $L \neq 0$ and $n = 0,1,2,3 \dots$ solution is found as

$$\alpha = \sqrt{\frac{N}{EI}} = \frac{\pi n}{L} \quad (A.4)$$

This gives

$$N = \frac{\pi^2 EI}{L^2} n^2 \quad (\text{A.5})$$

Where $n = 0$ corresponds to the trivial solution. So, for a nontrivial solution, $n = 1$ should be taken to determine the critical buckling load of the beam with hinged ends.

$$(N_{cr})_{\text{hinged-hinged}} = \frac{\pi^2 EI}{L^2} \quad (\text{A.6})$$

Beam with clamped-free ends

The following is used for the characteristic equation of the beam.

$$|A|_{\text{clamped-free}} = \begin{vmatrix} \cos(\alpha L) & \frac{\sin(\alpha L)}{\alpha} \\ 0 & 1 \end{vmatrix} = \cos(\alpha L) = 0 \quad (\text{A.7})$$

For $L \neq 0$ and $n = 0, 1, 2, 3 \dots$ solution is

$$\alpha = \sqrt{\frac{N}{EI}} = (2n + 1) \frac{\pi}{2L} \quad (n = 0, 1, 2, \dots) \quad (\text{A.8})$$

From this we get

$$N = \frac{\pi^2 EI}{4L^2} (2n + 1)^2 \quad (\text{A.9})$$

The critical buckling load occurs when $n = 0$.

$$(N_{cr})_{\text{clamped-free}} = \frac{\pi^2 EI}{4L^2} \quad (\text{A.10})$$

Beam with clamped-hinged ends

The expansion of the characteristic determinant gives

$$|A|_{\text{clamped-hinged}} = -\frac{\sin(\alpha L) - \alpha L \cos(\alpha L)}{EI \alpha^3} = 0 \quad (\text{A.11})$$

Simplification leads to

$$\tan(\alpha L) = \alpha L \quad (\text{A.12})$$

There is no symbolic solution to this transcendental equation. It is satisfied to the four digits after period if the smallest root is taken as $\alpha L \cong 4.4934$.

$$\tan(4.49341001) = 4.4934211571 \quad (\text{A.13})$$

Therefore

$$\alpha L = \sqrt{\frac{N}{EI}} L \cong 4.4934 \quad (\text{A.14})$$

or

$$(N_{cr})_{clamped-hinged} = \frac{20.19064356 EI}{L^2} = 2.046 \frac{\pi^2 EI}{L^2} \quad (\text{A.15})$$

is obtained.

Beam with clamped ends

For C-C ends we have

$$|A|_{C-C} = 2 - L \sin(L\alpha) - 2 \cos(L\alpha) = 0 \quad (\text{A.16})$$

or

$$|A|_{C-C} = 4 \sin^2\left(\frac{L\alpha}{2}\right) - \alpha L \sin(L\alpha) = 0 \quad (\text{A.17})$$

By using the following trigonometric identity

$$\sin(L\alpha) = 2 \sin\left(\frac{L\alpha}{2}\right) \cos\left(\frac{L\alpha}{2}\right) \quad (\text{A.18})$$

The expansion of the determinant reduces to

$$|A|_{C-C} = \sin\left(\frac{L\alpha}{2}\right) \left\{ 4 \sin\left(\frac{L\alpha}{2}\right) - 2\alpha L \cos\left(\frac{L\alpha}{2}\right) \right\} = 0 \quad (\text{A.19})$$

From this we get the solution as follows

$$\sin\left(\frac{L\alpha}{2}\right) = 0 \quad (\text{A.20})$$

or

$$\frac{L\alpha}{2} = \frac{L}{2} \sqrt{\frac{N}{EI}} = n\pi \quad (n = 1, 2, \dots) \quad (\text{A.21})$$

If the axial load is isolated from the above

$$N = \frac{4\pi^2 EI}{L^2} n^2 \quad (n = 1, 2, \dots) \quad (\text{A.22})$$

The corresponding critical load is obtained for $n = 1$ as in the following.

$$(N_{cr})_{c-c} = \frac{4\pi^2 EI}{L^2} \quad (\text{A.23})$$

It is revealed that this critical load is exactly quadruple of the pinned-pinned Euler column. Thus fixing two ends has increased the critical load to a large extent.