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ON SOME WEAKER HESITANT FUZZY OPEN SETS

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Abstract. The purpose of this paper is to define and study some new types of hesitant fuzzy open sets namely, hesitant fuzzy α-open, hesitant fuzzy preopen, hesitant fuzzy semiopen, hesitant fuzzy b-open and hesitant fuzzy β-open in hesitant fuzzy topological space. Some properties and the relationships between these hesitant fuzzy sets are investigated. Furthermore, some relationships between them in hesitant fuzzy subspace are introduced.

1. Introduction

Hesitant fuzzy sets are very useful to deal with group decision making problems when experts have a hesitation among several possible memberships for an element to a set. During the evaluating process in practice, however, these possible memberships may be not only crisp values in $[0, 1]$, but also interval values. Then hesitant fuzzy set theory has many applications in various fields like decision making problems, decision support systems, clustering algorithms, algebras, etc. After that time, hesitant fuzzy set theory has been developed rapidly by some scholars in theory and practice. In 1965, Zadeh [16] introduced the concept of a fuzzy set as a generalization of a crisp set. Chang [3] defined initially the notion of fuzzy topological spaces. In 2010, Torra [14] introduced the notion of a hesitant fuzzy set as an extension of a fuzzy set. In 2011, Xia and Xu [15] applied a hesitant fuzzy set to decision making by defining "hesitant fuzzy information aggregation". Jun et al. [5] studied hesitant fuzzy bi-ideals in semigroups. Divakaran and John [4] introduced a basic version of hesitant fuzzy rough sets through hesitant fuzzy relations. On the other hand, Jun and Ahn [6] applied hesitant fuzzy sets to BCK/BCI-algebras. Kim et al. [7] gave characterizations of a hesitant fuzzy positive implicative ideal, a hesitant fuzzy implicative ideal, and a hesitant fuzzy commutative ideal, respectively

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in BCK-algebras. Recently, Lee and Hur [10] defined a hesitant fuzzy topology and introduced the concepts of a hesitant fuzzy neighborhood, closure, interior, hesitant fuzzy subspace and obtained some of their properties. Also, they defined a hesitant fuzzy continuous mapping and investigated some of its properties. In 1965, Njastad [13] defined the class of α-open sets in topological spaces. In 1982, Mashhour et al [12] introduced the concept of preopen sets. The study of semiopen sets and their properties was initiated by Levine [11]. In 1996, Andrijevic' [2] introduced and studied a class of generalized open sets in a topological space called b-open sets, this class of sets contained in the class of β-open sets [1] and contains all semiopen sets and all preopen sets.

2. HESITANT FUZZY OPEN SETS

Definition 1. *[14] Let X be a reference set, and $P[0,1]$ denote the power set of $[0,1]$. Then, a mapping $h : X \to P[0,1]$ is called a hesitant fuzzy set in X.*

The hesitant fuzzy empty (resp. whole) set, denoted by h^0 (resp. h^1), is a hesitant fuzzy set in X defined as $h^0(x) = \phi$ (resp. $h^1(x) = [0,1]$), for each $x \in X$. Especially, we will denote the set of all hesitant fuzzy sets in X as $HS(X)$ [8].

Definition 2. *Assume that X is a nonempty set and $h, h_i \in HS(X)$ for i belong to the set of natural numbers $\boldsymbol{N}$. Then,*

(1) *h_1 is a subset of h_2, denoted by $h_1 \subseteq h_2$, if $h_1(x) \subseteq h_2(x)$, for each $x \in X$ [4].*
(2) *h_1 is equal to h_2, denoted by $h_1 = h_2$, if $h_1(x) \subseteq h_2(x)$ and $h_2(x) \subseteq h_1(x)$ [4].*
(3) *the intersection of h_1 and h_2, denoted by $h_1 \widetilde{\cap} h_2$, is a hesitant fuzzy set in X defined as follows: for each $x \in X$,*
 $(h_1 \widetilde{\cap} h_2)(x) = h_1(x) \cap h_2(x)$ *[8].*
(4) *the union of h_1 and h_2, denoted by $h_1 \widetilde{\cup} h_2$, is a hesitant fuzzy set in X defined as follows: for each $x \in X$,*
 $(h_1 \widetilde{\cup} h_2)(x) = h_1(x) \cup h_2(x)$ *[8].*
(5) *the complement of h, denoted by h^c, is a hesitant fuzzy set in X defined as: for each $x \in X$,*
 $h^c(x) = h(x)^c = [0,1] \setminus h(x)$ *[8].*
(6) *the intersection of $\{h_i\}_{i \in \boldsymbol{N}}$, denoted by $\widetilde{\bigcap}_{i \in \boldsymbol{N}} h_i$, is a hesitant fuzzy set in X defined as follows: for each $x \in X$,*
 $(\widetilde{\bigcap}_{i \in \boldsymbol{N}} h_i)(x) = \bigcap_{i \in \boldsymbol{N}} h_i(x)$ *[8].*
(7) *the union of $\{h_i\}_{i \in \boldsymbol{N}}$, denoted by $\widetilde{\bigcup}_{i \in \boldsymbol{N}} h_i$, is a hesitant fuzzy set in X defined as follows: for each $x \in X$,*
 $(\widetilde{\bigcup}_{i \in \boldsymbol{N}} h_i)(x) = \bigcup_{i \in \boldsymbol{N}} h_i(x)$ *[8].*

Definition 3. *[9] Let $h \in HS(X)$. Then, h is called a hesitant fuzzy point with the support $x \in X$ and the value δ, denoted by x_δ, if $x_\delta : X \to P[0,1]$ is the mapping* *given by: for each $y \in X$,*

$$x_\delta(y) = \begin{cases} \delta \subseteq [0,1] & \text{if } y = x, \\ \phi & \text{otherwise.} \end{cases}$$

In particular, $H_P(X)$ is called the set of all hesitant fuzzy points in X. If $\delta \subseteq h(x)$, then x_δ is said to belong to h, denoted by $x_\delta \in h$. It is obvious that $h = \widetilde{\bigcup}_{x_\delta \in h} x_\delta$.

Definition 4. *[10] Let X be a nonempty set, and $\tau \subseteq HS(X)$. Then, τ is called a hesitant topology (HFT) on X, if it satisfies the following axioms:*

(1) *$h^0, h^1 \in \tau$.*
(2) *For any $h_1, h_2 \in \tau$, we have $h_1 \widetilde{\cap} h_2 \in \tau$.*
(3) *For each $h_i \in \tau$, we have $\widetilde{\cup}_{i \in \boldsymbol{N}} h_i \in \tau$.*

The pair (X, τ) is called a hesitant fuzzy topological space. Each member of τ is called a hesitant fuzzy open set (HFOS) in X. A hesitant fuzzy set h in X is called a hesitant fuzzy closed set (HFCS) in (X, τ), if $h^c \in \tau$. The set of all hesitant fuzzy closed sets is denoted by $HFC(X)$.

Definition 5. *[10] Let (X, τ) be a hesitant fuzzy topological space, and $h_A \in HS(X)$. Then:*

(1) *$int_H(h_A) = \widetilde{\bigcup}\{h_U \in \tau : h_U \subseteq h_A\}$.*
(2) *$cl_H(h_A) = \widetilde{\bigcap}\{h_F \in HFC(X) : h_A \subseteq h_F\}$.*

3. Weaker hesitant fuzzy open sets

Definition 6. *Let (X, τ) be a hesitant fuzzy topological space. A subset h of $HS(X)$ is called:*

(1) *hesitant fuzzy α-open if $h \subseteq int_H(cl_H(int_H(h)))$.*
(2) *hesitant fuzzy preopen if $h \subseteq int_H(cl_H(h))$.*
(3) *hesitant fuzzy semiopen if $h \subseteq cl_H(int_H(h))$.*
(4) *hesitant fuzzy b-open if $h \subseteq int_H(cl_H(h)) \widetilde{\cup} cl_H(int_H(h))$.*
(5) *hesitant fuzzy β-open if $h \subseteq cl_H(int_H(cl_H(h)))$.*

Theorem 1. Let (X, τ) be a hesitant fuzzy topological space, then the following statements are hold:

(1) Every hesitant fuzzy open set is hesitant fuzzy α-open.
(2) Every hesitant fuzzy α-open set is hesitant fuzzy preopen.
(3) Every hesitant fuzzy α-open set is hesitant fuzzy semiopen.
(4) Every hesitant fuzzy preopen set is hesitant fuzzy b-open.
(5) Every hesitant fuzzy semiopen set is hesitant fuzzy b-open.
(6) Every hesitant fuzzy b-open set is hesitant fuzzy β-open.

Proof. (1) If h_A is hesitant fuzzy open, then $h_A = int_H(h_A) \subseteq int_H(cl_H(h_A)) = int_H(cl_H(int_H(h_A)))$. Thus, h_A is hesitant fuzzy α-open.

(2) If h_A is hesitant fuzzy α-open, then $h_A \subseteq int_H(cl_H(int_H(h_A))) \subseteq int_H(cl_H(h_A))$. Thus, h_A is hesitant fuzzy preopen.
(3) If h_A is hesitant fuzzy α-open, then $h_A \subseteq int_H(cl_H(int_H(h_A))) \subseteq cl_H(in_H(h_A))$. Thus, h_A is hesitant fuzzy semiopen.
(4) If h_A is hesitant fuzzy preopen, then $h_A \subseteq int_H(cl_H(h_A)) \subseteq int_H(cl_H(h_A)) \widetilde{\cup} cl_H(int_H(h_A))$. Thus, h_A is hesitant fuzzy b-open.
(5) If h_A is hesitant fuzzy semiopen, then $h_A \subseteq cl_H(int_H(h_A)) \subseteq int_H(cl_H(h_A)) \widetilde{\cup} cl_H(int_H(h_A))$. Thus, h_A is hesitant fuzzy b-open.
(6) If h_A is hesitant fuzzy b-open, then $h_A \subseteq int_H(cl_H(h_A)) \widetilde{\cup} cl_H(int_H(h_A)) \subseteq cl_H(int_H(cl_H(h_A))) \widetilde{\cup} cl_H(int_H(h_A)) = cl_H[int_H(cl_H(h_A)) \widetilde{\cup} int_H(h_A)] \subseteq cl_H[int_H(cl_H(h_A)) \widetilde{\cup} int_H(cl_H(h_A))] = cl_H(int_H(cl_H(h_A)))$. Thus, h_A is hesitant fuzzy β-open.

□

Remark 1. *The concepts of hesitant fuzzy preopen and hesitant fuzzy semiopen are independent.*

Remark 2. *The converse of the Theorem 1, need not be true as shown by the following examples.*

Example 1. *Consider the hesitant fuzzy sets in* $X = \{a, b, c\}$ *given by:*
$h_1(a) = [0.7, 1], h_1(b) = \{0.2, 0.5, 0.8\}, h_1(c) = [0.7, 1)$,
$h_2(a) = [0.5, 1), h_2(b) = \{0.2, 0.5, 0.7\}, h_2(c) = (0.7, 1]$,
$h_3(a) = [0.7, 1), h_3(b) = \{0.2, 0.5\}, h_3(c) = (0.7, 1)$, *and*
$h_4(a) = [0.5, 1], h_4(b) = \{0.2, 0.5, 0.7, 0.8\}, h_4(c) = [0.7, 1]$.
Then, $\tau = \{h^0, h^1, h_1, h_2, h_3, h_4\}$ *a hesitant topology on* X. *If* h_A *is the hesitant fuzzy set in* X *given by:*

(1) $h_A(a) = [0.6, 1], h_A(b) = \{0.2, 0.5, 0.6, 0.8, 0.9\}, h_A(c) = [0.3, 1)$,
then h_A *is hesitant fuzzy* α*-open but* h_A *is not hesitant fuzzy open.*
(2) $h_A(a) = [0, 1], h_A(b) = \phi, h_A(c) = \phi$,
then h_A *is both hesitant fuzzy preopen and hesitant fuzzy b-open but* h_A *is neither hesitant fuzzy* α*-open nor hesitant fuzzy semiopen.*

Example 2. *Consider the hesitant fuzzy sets in* $X = \{a, b, c\}$ *given by:*
$h_1(a) = \{0.4\}, h_1(b) = \{0.1\}, h_1(c) = \{0.8\}$,
$h_2(a) = \{0.3\}, h_2(b) = \{0.2\}, h_2(c) = \{0.7\}$, *and*
$h_3(a) = \{0.3, 0.4\}, h_3(b) = \{0.1, 0.2\}, h_3(c) = \{0.7, 0.8\}$.
Then, $\tau = \{h^0, h^1, h_1, h_2, h_3\}$ *a hesitant topology on* X. *If* h_A *is the hesitant fuzzy set in* X *given by:*
$h_A(a) = \{0.4, 0.6\}, h_A(b) = \{0.1, 0.6\}, h_A(c) = \{0.6, 0.8\}$,
then h_A *is both hesitant fuzzy semiopen and hesitant fuzzy b-open but* A *is neither hesitant fuzzy* α*-open nor hesitant fuzzy preopen.*

Example 3. *Consider the hesitant fuzzy sets in* $X = \{a\}$ *given by:*
$h_1(a) = \{0.1\}$,

$h_2(a) = \{0.2\}$,
$h_3(a) = \{0.1, 0.2\}$,
$h_4(a) = \{0.3, 0.4\}$,
$h_5(a) = \{0.1, 0.3, 0.4\}$,
$h_6(a) = \{0.2, 0.3, 0.4\}$,
$h_7(a) = \{0.1, 0.2, 0.3, 0.4\}$,
$h_8(a) = \{0.1, 0.3, 0.4, 0.5\}$, *and*
$h_9(a) = \{0.1, 0.2, 0.3, 0.4, 0.5\}$.
Then, $\tau = \{h^0, h^1, h_1, h_2, h_3, h_4, h_5, h_6, h_7, h_8, h_9\}$ *is a hesitant topology on* X. *If* h_A *is the hesitant fuzzy set in* X *given by:*
$h_A(a) = \{0.2, 0.3, 0.5\}$,
then h_A *is hesitant fuzzy* β*-open but* A *is not hesitant fuzzy b-open.*

Remark 3. From Theorem 1, we obtain the following diagram of implications:

$$\begin{tikzcd}
\text{hesitant fuzzy open} \arrow[d] & \text{hesitant fuzzy preopen} \arrow[d] & \\
\text{hesitant fuzzy } \alpha\text{-open} \arrow[ur] \arrow[dr] & \text{hesitant fuzzy b-open} \arrow[r] & \text{hesitant fuzzy } \beta\text{-open} \\
 & \text{hesitant fuzzy semiopen} \arrow[u] &
\end{tikzcd}$$

Theorem 2. Let (X, τ) be a hesitant fuzzy topological space and $h_A \in HS(X)$. Then:

(1) $cl_H(h_A)\widetilde{\cap}h_G \subseteq cl_H(h_A\widetilde{\cap}h_G)$, for every hesitant fuzzy open set h_G.
(2) $int_H(h_A\widetilde{\cup}h_F) \subseteq int_H(h_A)\widetilde{\cup}h_F$, for every hesitant fuzzy closed set h_F.

Proof. (1) Let $x_\delta \in cl_H(h_A)\widetilde{\cap}h_G$, then $x_\delta \in cl_H(h_A)$ and $x_\delta \in h_G$. If h_V is a hesitant fuzzy open set containing x_δ, then, $h_V\widetilde{\cap}h_G$ is also hesitant fuzzy open set containing x_δ. Since $x_\delta \in cl_H(h_A)$ implies $(h_V\widetilde{\cap}h_G)\widetilde{\cap}h_A \neq h^0$ and hence $h_V\widetilde{\cap}(h_G\widetilde{\cap}h_A) \neq h^0$. This is true for every h_V containing x_δ, so $x_\delta \in cl_H(h_G\widetilde{\cap}h_A)$. Therefore hesitant fuzzy $cl(h_A)\widetilde{\cap}h_G \subseteq cl_H(h_A\widetilde{\cap}h_G)$.
(2) Follows from (1) and so it is obvious.

□

Theorem 3. If $\{h_i : i \in \mathbf{N}\}$ is a collection of hesitant fuzzy b-open (resp. hesitant fuzzy α-open, hesitant fuzzy preopen, hesitant fuzzy semiopen and hesitant fuzzy β-open) sets of a hesitant fuzzy topological space (X, τ), then $\widetilde{\cup}_{i\in\mathbf{N}}h_i$ is a hesitant fuzzy b-open (resp. hesitant fuzzy α-open, hesitant fuzzy preopen, hesitant fuzzy semiopen and hesitant fuzzy β-open) set.

Proof. We prove only the first case since the other cases are similarly shown. Since $h_i \subseteq int_H(cl_H(h_i))\widetilde{\cup}cl_H(int_H(h_i))$ for every $i \in \mathbf{N}$, we have

$$\widetilde{\cup}_{i\in\mathbf{N}}h_i \subseteq \widetilde{\cup}_{i\in\mathbf{N}}[int_H(cl_H(h_i))\widetilde{\cup}cl_H(int_H(h_i))]$$

$$\subseteq [\widetilde{\cup}_{i\in\mathbf{N}} int_H(cl_H(h_i))]\widetilde{\cup}[\widetilde{\cup}_{i\in\mathbf{N}} cl_H(int_H(h_i))]$$
$$\subseteq [int_H(\widetilde{\cup}_{i\in\mathbf{N}} cl_H(h_i))]\widetilde{\cup}[cl_H(\widetilde{\cup}_{i\in\mathbf{N}} int_H(h_i))]$$
$$\subseteq [int_H(cl_H(\widetilde{\cup}_{i\in\mathbf{N}} h_i))]\widetilde{\cup}[cl_H(int_H(\widetilde{\cup}_{i\in\mathbf{N}} h_i))].$$

Therefore, $\widetilde{\cup}_{i\in\mathbf{N}} h_i$ is hesitant fuzzy b-open. □

Theorem 4. Let (X,τ) be a hesitant fuzzy topological space, $h_U \in \tau$ and $h_A \in HS(X)$.

(1) If h_A is hesitant fuzzy preopen, then $h_U\widetilde{\cap}h_A$ is hesitant fuzzy preopen.
(2) If h_A is hesitant fuzzy semiopen, then $h_U\widetilde{\cap}h_A$ is hesitant fuzzy semiopen.

Proof. (1) Since h_A is hesitant fuzzy preopen and h_U is hesitant fuzzy open, then, $h_A \subseteq int_H(cl_H(h_A))$ and $int_H(h_U) = h_U$ and so by Theorem 2 (1), $h_U\widetilde{\cap}h_A \subseteq int_H(h_U)\widetilde{\cap}int_H(cl_H(h_A)) = int_H(h_U\widetilde{\cap}cl_H(h_A)) \subseteq int_H(cl_H(h_U\widetilde{\cap}h_A))$. Therefore, $h_U\widetilde{\cap}h_A$ is hesitant fuzzy preopen.

(2) Since h_A is hesitant fuzzy semiopen, then by Theorem 2 (1), $h_U\widetilde{\cap}h_A \subseteq h_U\widetilde{\cap}cl_H(in_H(h_A)) \subseteq cl_H(h_U\widetilde{\cap}in_H(h_A)) = cl_H(in_H(h_U)\widetilde{\cap}in_H(h_A)) = cl_H(in_H(h_U\widetilde{\cap}h_A))$. Therefore, $h_U\widetilde{\cap}h_A$ is hesitant fuzzy semiopen.

□

Theorem 5. Let (X,τ) be a hesitant fuzzy topological space, $h_U \in \tau$ and $h_A \in HS(X)$. If h_A is hesitant fuzzy β-open, then $h_U\widetilde{\cap}h_A$ is hesitant fuzzy β-open.

Proof. Since h_A is hesitant fuzzy β-open, then

$$h_U\widetilde{\cap}h_A \subseteq h_U\widetilde{\cap}cl_H(int_H(cl_H(h_A)))$$
$$\subseteq cl_H(h_U\widetilde{\cap}int_H(cl_H(h_A)))$$
$$= cl_H(int_H(h_U)\widetilde{\cap}int_H(cl_H(h_A)))$$
$$= cl_H(int_H(h_U\widetilde{\cap}cl_H(h_A)))$$
$$\subseteq cl_H(int_H(cl_H(h_U\widetilde{\cap}h_A))).$$

This shows that $h_U\widetilde{\cap}h_A$ is hesitant fuzzy β-open. □

Theorem 6. Let (X,τ) be a hesitant fuzzy topological space, $h_U \in \tau$ and $h_A \in HS(X)$. If h_A is hesitant fuzzy b-open, then $h_U\widetilde{\cap}h_A$ is hesitant fuzzy b-open.

Proof. Since h_A is hesitant fuzzy b-open, then

$$h_U\widetilde{\cap}h_A \subseteq h_U\widetilde{\cap}[int_H(cl_H(h_A))\widetilde{\cup}cl_H(int_H(h_A))]$$
$$= [h_U\widetilde{\cap}int_H(cl_H(h_A))]\widetilde{\cup}[h_U\widetilde{\cap}cl_H(int_H(h_A))]$$
$$= [int_H(h_U)\widetilde{\cap}int_H(cl_H(h_A))]\widetilde{\cup}[h_U\widetilde{\cap}cl_H(int_H(h_A))]$$
$$\subseteq [int_H(h_U\widetilde{\cap}cl_H(h_A))]\widetilde{\cup}[cl_H(h_U\widetilde{\cap}int_H(h_A))]$$

$$\subseteq [int_H(cl_H(h_U \widetilde{\cap} h_A))] \widetilde{\cup} [cl_H(int_H(h_U \widetilde{\cap} h_A))].$$

This shows that $h_U \widetilde{\cap} h_A$ is hesitant fuzzy b-open. □

Remark 4. *We note that the intersection of two hesitant fuzzy preopen (resp. hesitant fuzzy semiopen, hesitant fuzzy b-open and hesitant fuzzy β-open) sets need not be hesitant fuzzy preopen (resp. hesitant fuzzy semiopen, hesitant fuzzy b-open and hesitant fuzzy β-open) as can be seen from the following examples:*

Example 4. *Consider the hesitant fuzzy sets in $X = \{a\}$ given by $h(a) = \{0.3, 0.6\}$. Then, $\tau = \{h^0, h^1, h\}$ is a hesitant topology on X. If $h_A(a) = \{0.1, 0.3\}$ and $h_B(a) = \{0.1, 0.6\}$, then h_A and h_B are hesitant fuzzy preopen (resp. hesitant fuzzy b-open and hesitant fuzzy β-open), but $h_A \widetilde{\cap} h_B = \{0.1\} = h_C$ which is not hesitant fuzzy preopen (resp. hesitant fuzzy b-open and hesitant fuzzy β-open).*

Example 5. *From Example 2, if h_A is the hesitant fuzzy set in X given by:*
$h_A(a) = \{0.4, 0.6\}, h_A(b) = \{0.1, 0.6\}, h_A(c) = \{0.6, 0.8\}$,
and h_B is the hesitant fuzzy set in X given by:
$h_B(a) = \{0.3, 0.6\}, h_B(b) = \{0.2, 0.6\}, h_B(c) = \{0.7, 0.9\}$,
then h_A and h_B are hesitant fuzzy semiopen, but $h_A \widetilde{\cap} h_B = h_C$ which is not hesitant fuzzy semiopen, where h_C is the hesitant fuzzy set in X given by:
$h_C(a) = \{0.6\}, h_C(b) = \{0.6\}, h_C(c) = \phi$.

Remark 5. *From Remark 4, we notice that the family of all hesitant fuzzy preopen (resp. hesitant fuzzy semiopen, hesitant fuzzy b-open and hesitant fuzzy β-open) sets need not be a topology in general.*

Theorem 7. Let (X, τ) be a hesitant fuzzy topological space. If h_A and h_B are hesitant fuzzy α-open, then $h_B \widetilde{\cap} h_A$ is also hesitant fuzzy α-open.

Proof. Since h_A and h_B are hesitant fuzzy α-open, then

$$h_B \widetilde{\cap} h_A \subseteq int_H(cl_H(int_H(h_B))) \widetilde{\cap} int_H(cl_H(int_H(h_A)))$$

$$\subseteq int_H[cl_H(int_H(h_B)) \widetilde{\cap} int_H(cl_H(int_H(h_A)))]$$

$$\subseteq int_H cl_H[int_H(h_B) \widetilde{\cap} int_H(cl_H(int_H(h_A)))]$$

$$\subseteq int_H cl_H[int_H(h_B) \widetilde{\cap} cl_H(int_H(h_A))]$$

$$\subseteq int_H cl_H cl_H[int_H(h_B) \widetilde{\cap} int_H(h_A)]$$

$$\subseteq int_H cl_H int_H(h_B \widetilde{\cap} h_A).$$

Thus, $h_B \widetilde{\cap} h_A$ is hesitant fuzzy α-open. □

Remark 6. *From the Theorems 3 and 7, we notice that the family of all hesitant fuzzy α-open is a topology.*

Theorem 8. Let (X,τ) be a hesitant fuzzy topological space and $h_A \in HS(X)$. If h_A is both hesitant fuzzy semiopen and hesitant fuzzy preopen, then h_A is hesitant fuzzy α-open.

Proof. By assumption, $h_A \subseteq cl_H(int_H(h_A))$ and $h_A \subseteq int_H(cl_H(h_A))$. Then, $h_A \subseteq int_H(cl_H(h_A)) \subseteq int_H(cl_H(cl_H(int_H(h_A)))) = int_H(cl_H(int_H(h_A)))$. Therefore, h_A is hesitant fuzzy α-open. □

Theorem 9. Let (X,τ) be a hesitant fuzzy topological space and h_A be hesitant fuzzy α-open.

(1) If h_B is hesitant fuzzy semiopen, then $h_A \widetilde{\cap} h_B$ is hesitant fuzzy semiopen.
(2) If h_B is hesitant fuzzy preopen, then $h_A \widetilde{\cap} h_B$ is hesitant fuzzy preopen.

Proof. (1) By assumption, $h_A \subseteq int_H(cl_H(int_H(h_A)))$ and $h_B \subseteq cl_H(int_H(h_B))$, then by Theorem 2 (1), we have that

$$\begin{aligned} h_A \widetilde{\cap} h_B &\subseteq int_H(cl_H(int_H(h_A))) \widetilde{\cap} cl_H(int_H(h_B)) \\ &\subseteq cl_H[int_H(cl_H(int_H(h_A))) \widetilde{\cap} int_H(h_B)] \\ &\subseteq cl_H[cl_H(int_H(h_A)) \widetilde{\cap} int_H(h_B)] \\ &\subseteq cl_H[cl_H[int_H(h_A) \widetilde{\cap} int_H(h_B)]] \\ &= cl_H(int_H(h_A \widetilde{\cap} h_B)). \end{aligned}$$

Therefore, $h_A \widetilde{\cap} h_B$ is hesitant fuzzy semiopen.

(2) By assumption, $h_A \subseteq int_H(cl_H(int_H(h_A)))$ and $h_B \subseteq int_H(cl_H(h_B))$, then

$$\begin{aligned} h_A \widetilde{\cap} h_B &\subseteq int_H(cl_H(int_H(h_A))) \widetilde{\cap} int_H(cl_H(h_B)) \\ &= int_H[int_H(cl_H(int_H(h_A))) \widetilde{\cap} int_H(cl_H(h_B))] \\ &\subseteq int_H[cl_H(int_H(h_A)) \widetilde{\cap} int_H(cl_H(h_B))] \\ &\subseteq int_H[cl_H[int_H(h_A) \widetilde{\cap} int_H(cl_H(h_B))]] \\ &\subseteq int_H[cl_H[int_H(h_A) \widetilde{\cap} cl_H(h_B)]] \\ &\subseteq int_H[cl_H[cl_H[int_H(h_A) \widetilde{\cap} h_B]]] \\ &\subseteq int_H(cl_H(cl_H(h_A \widetilde{\cap} h_B))) \\ &= int_H(cl_H(h_A \widetilde{\cap} h_B)). \end{aligned}$$

Therefore, $h_A \widetilde{\cap} h_B$ is hesitant fuzzy preopen.

□

Theorem 10. Let (X,τ) be a hesitant fuzzy topological space. If h_A is hesitant fuzzy preopen and h_B is hesitant fuzzy semiopen, then $h_A \widetilde{\cap} h_B$ is hesitant fuzzy β-open.

Proof. By assumption, $h_A \subseteq int_H(cl_H(h_A))$ and $h_B \subseteq cl_H(int_H(h_B))$, then by Theorem 2 (1), we have that

$$\begin{aligned} h_A \widetilde{\cap} h_B &\subseteq int_H(cl_H(h_A)) \widetilde{\cap} cl_H(int_H(h_B)) \\ &\subseteq cl_H[int_H(cl_H(h_A)) \widetilde{\cap} int_H(h_B)] \end{aligned}$$

$$
\begin{aligned}
&= cl_H[int_H[int_H(cl_H(h_A))\widetilde{\cap} int_H(h_B)]] \\
&\subseteq cl_H[int_H[cl_H(h_A)\widetilde{\cap} int_H(h_B)]] \\
&\subseteq cl_H[int_H[cl_H(h_A\widetilde{\cap} int_H(h_B))]] \\
&\subseteq cl_H(int_H(cl_H(h_A\widetilde{\cap} h_B))).
\end{aligned}
$$

Therefore, $h_A\widetilde{\cap} h_B$ is hesitant fuzzy β-open. □

Theorem 11. Let (X, τ) be a hesitant fuzzy topological space and $h_A, h_B \in HS(X)$. Then,

(1) h_A is hesitant fuzzy semiopen if and only if there exists a hesitant fuzzy open set h_U such that $h_U \subseteq h_A \subseteq cl_H(h_U)$.
(2) h_B is hesitant fuzzy semiopen if h_A is hesitant fuzzy semiopen and $h_A \subseteq h_B \subseteq cl_H(h_A)$.
(3) h_A is hesitant fuzzy semiopen if and only if $cl_H(h_A) = cl_H(int_H(h_A))$.

Proof. (1) Let h_A be hesitant fuzzy semiopen, then $h_A \subseteq cl_H(int_H(h_A))$. Take $h_U = int_H(h_A)$, then h_U is hesitant fuzzy open such that $h_U = int_H(h_A) \subseteq h_A \subseteq cl_H(int_H(h_A)) = cl_H(h_U)$.

Conversely, since $h_U \subseteq h_A$ implies that $h_U = int_H(h_U) \subseteq int_H(h_A)$ and so $h_A \subseteq cl_H(h_U) = cl_H(int_H(h_U)) \subseteq cl_H(int_H(h_A))$. Thus, h_A is hesitant fuzzy semiopen.

(2) Since h_A is hesitant fuzzy semiopen, then by (1) there exists a hesitant fuzzy open set h_U such that $h_U \subseteq h_A \subseteq cl_H(h_U)$. Since $h_A \subseteq h_B$, so $h_U \subseteq h_B$. But $cl_H(h_A) \subseteq cl_H(h_U)$, then $h_B \subseteq cl_H(h_U)$. Hence, $h_U \subseteq h_B \subseteq cl_H(h_U)$. Thus, h_B is hesitant fuzzy semiopen.

(3) Let h_A be hesitant fuzzy semiopen, then $h_A \subseteq cl_H(int_H(h_A))$ which implies that $cl_H(h_A) \subseteq cl_H(int_H(h_A)) \subseteq cl_H(h_A)$ and hence $cl_H(h_A) = cl_H(int_H(h_A))$.

Conversely, since by Theorem 1, $int_H(h_A)$ is hesitant fuzzy semiopen such that $int_H(h_A) \subseteq h_A \subseteq cl_H(h_A) = cl_H(int_H(h_A))$ and therefore h_A is hesitant fuzzy semiopen.

□

Definition 7. *[10] Let (X, τ) be a hesitant fuzzy topological space and $h \in HS(X)$. Then, the collection $\tau_h = \{U\widetilde{\cap} h : U \in \tau\}$ is called a hesitant fuzzy subspace topology or hesitant fuzzy relative topology on h. The pair (h, τ_h) is called a hesitant fuzzy subspace, and each member of τ_h is called a hesitant fuzzy open set in h.*

Proposition 1. *[10] Let (X, τ) be a hesitant fuzzy topological space, $h, h_A \in HS(X)$ and $h_A \subseteq h$. Then, $cl_{\tau_h}(h_A) = h\widetilde{\cap} cl_H(h_A)$, where $cl_{\tau_h}(h_A)$ denotes the closure of h_A in (h, τ_h).*

Definition 8. *Let (X, τ) be a hesitant fuzzy topological space, $h, h_A \in HS(X)$ and $h_A \subseteq h$. Then, $int_{\tau_h}(h_A) = \widetilde{\bigcup}\{h_U \in \tau_h : h_U \subseteq h_A\}$.*

Theorem 12. Let (X, τ) be a hesitant fuzzy topological space and $h_A, h_B \in HS(X)$. If h_A is hesitant fuzzy preopen in X and h_B is hesitant fuzzy semiopen in X, then

(1) $h_A \widetilde{\cap} h_B$ is hesitant fuzzy semiopen in h_A.
(2) $h_A \widetilde{\cap} h_B$ is hesitant fuzzy preopen in h_B.

Proof. By assumption, $h_A \subseteq int_H(cl_H(h_A))$ and $h_B \subseteq cl_H(int_H(h_B))$.

(1) Then,

$$\begin{aligned} h_A \widetilde{\cap} h_B &\subseteq int_H(cl_H(h_A)) \widetilde{\cap} cl_H(int_H(h_B)) \\ &\subseteq cl_H[int_H(cl_H(h_A)) \widetilde{\cap} int_H(h_B)] \\ &\subseteq cl_H[cl_H(h_A) \widetilde{\cap} int_H(h_B)] \\ &\subseteq cl_H[cl_H[h_A \widetilde{\cap} int_H(h_B)]] \\ &= cl_H[h_A \widetilde{\cap} int_H(h_B)]. \end{aligned}$$

Hence, $h_A \widetilde{\cap} h_B \subseteq cl_H(h_A \widetilde{\cap} int_H(h_B))$ and so $h_A \widetilde{\cap} h_B \subseteq cl_H(h_A \widetilde{\cap} int_H(h_B)) \widetilde{\cap} h_A = cl_{\tau_{h_A}}(h_A \widetilde{\cap} int_H(h_B))$. Since $h_A \widetilde{\cap} int_H(h_B)$ is a hesitant fuzzy open set in h_A, so $h_A \widetilde{\cap} h_B \subseteq cl_{\tau_{h_A}}(h_A \widetilde{\cap} int_H(h_B)) = cl_{\tau_{h_A}}(int_{\tau_{h_A}}(h_A \widetilde{\cap} int_H(h_B))) \subseteq cl_{\tau_{h_A}}(int_{\tau_{h_A}}(h_A \widetilde{\cap} h_B))$. Therefore, $h_A \widetilde{\cap} h_B$ is hesitant fuzzy semiopen in h_A.

(2) Now,

$$\begin{aligned} h_A \widetilde{\cap} h_B &\subseteq int_H(cl_H(h_A)) \widetilde{\cap} h_B \\ &= int_{\tau_{h_B}}[int_H(cl_H(h_A)) \widetilde{\cap} h_B] \\ &\subseteq int_{\tau_{h_B}}[int_H(cl_H(h_A)) \widetilde{\cap} cl_H(int_H(h_B))] \\ &\subseteq int_{\tau_{h_B}}[cl_H[int_H(cl_H(h_A)) \widetilde{\cap} int_H(h_B)]] \\ &\subseteq int_{\tau_{h_B}}[cl_H[cl_H(h_A) \widetilde{\cap} int_H(h_B)]] \\ &\subseteq int_{\tau_{h_B}}[cl_H[cl_H[h_A \widetilde{\cap} int_H(h_B)]]] \\ &\subseteq int_{\tau_{h_B}}[cl_H[cl_H[h_A \widetilde{\cap} h_B]]] \\ &= int_{\tau_{h_B}}(cl_H(h_A \widetilde{\cap} h_B)). \end{aligned}$$

Since $int_{\tau_{h_B}}(cl_H(h_A \widetilde{\cap} h_B))$ is hesitant fuzzy open in h_B, then $int_{\tau_{h_B}}(cl_H(h_A \widetilde{\cap} h_B)) \widetilde{\cap} h_B = int_{\tau_{h_B}}(cl_H(h_A \widetilde{\cap} h_B) \widetilde{\cap} h_B)$, and hence $h_A \widetilde{\cap} h_B \subseteq int_{\tau_{h_B}}(cl_H(h_A \widetilde{\cap} h_B) \widetilde{\cap} h_B) = int_{\tau_{h_B}}(cl_{\tau_{h_B}}(h_A \widetilde{\cap} h_B))$.

Therefore, $h_A \widetilde{\cap} h_B$ is hesitant fuzzy preopen in h_B. □

Theorem 13. Let (X, τ) be a hesitant fuzzy topological space, $h_A, h_B \in HS(X)$, $h_A \subseteq h_B$ and h_B be hesitant fuzzy semiopen in X. Then, h_A is hesitant fuzzy semiopen in X if and only if h_A is hesitant fuzzy semiopen in h_B.

Proof. Let h_A be hesitant fuzzy semiopen in X, then there is a hesitant fuzzy open set h_U such that $h_U \subseteq h_A \subseteq cl_H(h_U)$ implies that $h_U \subseteq h_A \subseteq h_B$. Hence,

$h_U \subseteq h_A \subseteq cl_H(h_U)\widetilde{\cap}h_B = cl_{\tau_{h_B}}(h_U)$. Since $h_U\widetilde{\cap}h_B = h_U$ is also hesitant fuzzy open in h_B, then h_A is hesitant fuzzy semiopen in h_B.

Conversely, let h_A be hesitant fuzzy semiopen in h_B. Then there is a hesitant fuzzy open set h_U in h_B such that $h_U \subseteq h_A \subseteq cl_{\tau_{h_B}}(h_U)$. Since h_U is hesitant fuzzy open in h_B, there exists a hesitant fuzzy open set h_V such that $h_U = h_V\widetilde{\cap}h_B$. Then, $h_V\widetilde{\cap}h_B = h_U \subseteq h_A \subseteq cl_{\tau_{h_B}}(h_U) = cl_{\tau_{h_B}}(h_V\widetilde{\cap}h_B) \subseteq cl_H(h_V\widetilde{\cap}h_B)$. By Theorem 4 (2), $h_V\widetilde{\cap}h_B$ is hesitant fuzzy semiopen, then by Theorem 11 (2), h_A is hesitant fuzzy semiopen in X. □

Theorem 14. Let (X,τ) be a hesitant fuzzy topological space, $h_A, h_B \in HS(X)$, $h_A \subseteq h_B$ and h_B be hesitant fuzzy preopen in X. Then, h_A is hesitant fuzzy preopen in X if and only if h_A is hesitant fuzzy preopen in h_B.

Proof. Suppose that h_A is hesitant fuzzy preopen in X, then $h_A = h_A\widetilde{\cap}h_B \subseteq int_H(cl_H(h_A))\widetilde{\cap}h_B$. Since $int_H(cl_H(h_A))\widetilde{\cap}h_B$ is hesitant fuzzy open in h_B, then $h_A \subseteq int_H(cl_H(h_A))\widetilde{\cap}h_B \subseteq int_{\tau_{h_B}}[int_H(cl_H(h_A))\widetilde{\cap}h_B] \subseteq int_{\tau_{h_B}}[cl_H(h_A)\widetilde{\cap}h_B] = int_{\tau_{h_B}}(cl_{\tau_{h_B}}(h_A))$. Hence, h_A is hesitant fuzzy preopen in h_B.

Conversely, assume that h_A is hesitant fuzzy preopen in h_B. Then, $h_A \subseteq int_{\tau_{h_B}}(cl_{\tau_{h_B}}(h_A))$. Since $int_{\tau_{h_B}}(cl_{\tau_{h_B}}(h_A))$ is hesitant fuzzy open in h_B, so there a hesitant fuzzy open set h_U in X such that $int_{\tau_{h_B}}(cl_{\tau_{h_B}}(h_A)) = h_U\widetilde{\cap}h_B$. By Theorem 9 (2), $int_{\tau_{h_B}}(cl_{\tau_{h_B}}(h_A))$ is hesitant fuzzy preopen in X. Therefore,

$$\begin{aligned} h_A &\subseteq int_{\tau_{h_B}}(cl_{\tau_{h_B}}(h_A)) \\ &\subseteq int_H(cl_H(int_{\tau_{h_B}}(cl_{\tau_{h_B}}(h_A)))) \\ &= int_H(cl_H(int_{\tau_{h_B}}[cl_H(h_A)\widetilde{\cap}h_B])) \\ &\subseteq int_H(cl_H[cl_H(h_A)\widetilde{\cap}h_B]) \\ &\subseteq int_H(cl_H(cl_H(h_A))) \\ &= int_H(cl_H(h_A)). \end{aligned}$$

This shows that h_A is hesitant fuzzy preopen in X. □

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References

[1] Abd El-Monsef, M. E., El-Deeb, S. N., Mahmoud, R. A., β-open sets and β-continuous mappings, *Bull. Fac. Sci. Assuit Univ.*, 12 (1983), 77–90.

[2] Andrijevic, D., On b-open sets, *Mat. Vesnik*, 48 (1996), 59–64.

[3] Chang, C. L., Fuzzy topological spaces, *J. Math. Anal. Appl.*, 24(1) (1968), 182–190.

[4] Divakaran, D., John, S. J., Hesitant fuzzy rough sets through hesitant fuzzy relations, *Ann. Fuzzy Math. Inform.*, 8 (2014), 33–46.

[5] Jun, Y. B., Lee, K. J., Song, S. Z., Hesitant fuzzy bi-ideals in semigroups, *Commun. Korean Math. Soc.*, 30(3) (2015), 143–154. http://dx.doi.org/10.4134/CKMS.2015.30.3.143

[6] Jun, Y. B., Ahn, S. S., Hesitant fuzzy sets theory applied to BCK/BCI-algebras, *J. Comput. Anal. Appl.*, 20 (2016), 635–646.

[7] Kim, J., Lim, P. K., Lee, J. G., Hur, K., Hesitant fuzzy sets applied to BCK/BCI-algebras, *Ann. Fuzzy Math. Inform.*, 18(3) (2019), 209–231. https://doi.org/10.30948/afmi.2019.18.3.209

[8] Kim, J., Jun, Y. B., Lim, P. K., Lee, J. G., Hur, K., The category of hesitant H-fuzzy sets, *Ann. Fuzzy Math. Inform.*, 18(1) (2019), 57–74. https://doi.org/10.30948/afmi.2019.18.1.57

[9] Kim, J., Lim, P. K., Lee, J. G., Hur, K., Hesitant fuzzy subgroups and subrings, *Ann. Fuzzy Math. Inform.*, 18(2) (2019), 105–122. https://doi.org/10.30948/afmi.2019.18.2.105

[10] Lee, J.-G., Hur, K., Hesitant fuzzy topological spaces, *Mathematics*, 8(2) (2020), 1–21. https://doi.org/10.3390/math8020188

[11] Levine, N., Semi-open sets and semi-continuity in topological spaces, *Amer. Math. Monthly,* 70 (1) (1963), 34–41.

[12] Mashhour, A. S., Abd El-Monsef, M. E., El-Deeb, S. N., On precontinuous and weak precontinuous functions, *Proc. Math. Phys. Soc. Egypt*, 51 (1982), 47–53.

[13] Njstad, O., On some classes of nearly open sets, *Pacific J. Math.,* 15 (1965), 961–970.

[14] Torra, V., Hesitant fuzzy sets, *Int. J. Intell. Syst.*, 25(6) (2010), 529–539. https://doi.org/10.1002/int.20418

[15] Xia, M., Xu, Z., Hesitant fuzzy information aggregation in decision making, *Int. J. Approx. Reason.*, 52(3) (2011), 395–407. https://doi.org/10.1016/j.ijar.2010.09.002

[16] Zadeh, L. A., Fuzzy sets, *Inf. Control.*, 8(3) (1965), 338–353.