On Weak Projection Invariant Semisimple Modules

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Abstract

We introduce and investigate the notion of weak projection invariant semisimple modules. We deal with the structural properties of this new class of modules. In this trend we have indecomposable decompositions of the special class of the former class of modules via some module theoretical properties. As a consequence, we obtain when the finite exchange property implies full exchange property for the latter class of modules.

1. Introduction

All rings are associative with unity and modules are unital right modules. Let R be a ring and M a right R-module. Recall that M is called CS (or, *extending*) if every submodule of M is essentially contained in a direct summand of M. This kind of modules are important generalizations of injective, semisimple and uniform modules. There have been several generalizations of CS modules as well as some classes of modules which are related to the direct summands of the module in literature (see [1]-[4]).

A submodule N of M is called *projection invariant*, if $f(N) \subseteq N$ for all $f^2 = f \in End(M_R)$ (see [3, 5, 6]). Note that torsion subgroup of a group, socle of a module and the radical of a ring are all projection invariant submodules of the corresponding modules, respectively. Recall from [6], a module M is called π-extending if every projection invariant submodule of M is essential in a direct summand of M. It is well-known that a CS-module is π-extending [3].

In this paper, we introduce and investigate the notion of weak projection invariant semisimple modules which is a generalization of semisimple and projection invariant semisimple modules [7]. We call a module M is weak projection invariant semisimple, denoted by $w\pi$-semisimple, provided that each semisimple projection invariant submodule of M is a direct summand of M. It is clear that the class of the $w\pi$-semisimple modules is contained in the class of π-extending modules. We deal with structural module properties of $w\pi$-semisimple modules. Moreover, we define special class of $w\pi$-semisimple modules and obtain indecomposable decomposition for the aforementioned modules via Abelian endomorphism rings over rings with ascending chain condition on the right annihilators. As a consequence, we obtain that the finite exchange property implies full exchange property.

Let $X \subseteq M$, then $X \leq M$, $SocM$ and $End(M_R)$ denote X is a submodule of M, the socle of M and the endomorphism ring of M_R, respectively. Recall that a module M over a ring R is said to have (finite) exchange property if for any (finite) index set I, whenever $M \oplus Y = \underset{i \in I}{\oplus} A_i$ for modules Y and A_i, then $M \oplus Y = M \oplus (\underset{i \in I}{\oplus} B_i)$ for submodules B_i of A_i [8]. A family $\{N_i : i \in I\}$ of independent submodules of a module M is said to be a local summand if for any finite subset F of I, $\underset{i \in F}{\oplus} N_i$ is a direct summand of M [3, 9]. Recall further that a ring R is called Abelian if every idempotent of R is central [3, 10].

Since $w\pi$-semisimple modules are based on semisimple projection invariant submodules, we start with the following basic result.

Lemma 1.1. *(i) If A is projection invariant in B and B is projection invariant in M then A is projection invariant in M.*
(ii) If $M = \bigoplus\limits_{i \in I} M_i$, and X is a semisimple projection invariant submodule of M, then $X = \bigoplus\limits_{i \in I}(X \cap M_i)$ and $X \cap M_i$ is semisimple projection invariant submodule of M_i for all $i \in I$.

Proof. Immediate by definitions (see [11, p.50]). $\qquad\square$

In [7, Lemma 1.2], the author attempts to obtain the following statement: Let M_R be a module and $N \leq K \leq M_R$. If N is projection invariant in M and K/N is projection invariant in M/N, then K is projection invariant in M. However, the proof therein is inconsistent. Since $f(N)$ would be nonzero, the function $\theta : M/N \to M$, $\theta(m+N) = f(m)$ for all $m \in M$ is not well-defined. Let us make it clear by the following example.

Example 1.2. *Let $M = (\mathbb{Z}/4\mathbb{Z}) \oplus (\mathbb{Z}/4\mathbb{Z})$ be the $\mathbb{Z}$-module, and $N = (2\mathbb{Z}/4\mathbb{Z}) \oplus 0$ be the submodule of $M_\mathbb{Z}$. Now, it is easy to see that $End(M_\mathbb{Z}) \cong \begin{bmatrix} \mathbb{Z}/4\mathbb{Z} & \mathbb{Z}/4\mathbb{Z} \\ \mathbb{Z}/4\mathbb{Z} & \mathbb{Z}/4\mathbb{Z} \end{bmatrix}$. Let $f^2 = f = \begin{bmatrix} \bar{1} & \bar{1} \\ \bar{0} & \bar{0} \end{bmatrix} \in End(M_\mathbb{Z})$. So, let us consider $\theta : M/N \to M$, $\theta(m+N) = f(m)$ for all $m \in M$. Since $(\bar{1}+\bar{3}) - (\bar{3}+\bar{3}) \in N$, $(\bar{1}+\bar{3}) + N = (\bar{3}+\bar{3}) + N$. But $\theta((\bar{1}+\bar{3})+N) = f(\bar{1}+\bar{3}) = \begin{bmatrix} \bar{1} & \bar{1} \\ \bar{0} & \bar{0} \end{bmatrix}\begin{bmatrix} \bar{1} \\ \bar{3} \end{bmatrix} = \begin{bmatrix} \bar{0} \\ \bar{0} \end{bmatrix}$, and $\theta((\bar{3}+\bar{3})+N) = f(\bar{3}+\bar{3}) = \begin{bmatrix} \bar{1} & \bar{1} \\ \bar{0} & \bar{0} \end{bmatrix}\begin{bmatrix} \bar{3} \\ \bar{3} \end{bmatrix} = \begin{bmatrix} \bar{2} \\ \bar{0} \end{bmatrix}$. Hence $\theta((\bar{1}+\bar{3})+N) \neq \theta((\bar{3}+\bar{3})+N)$.*

Notice that Proposition 2.3 (ii), Corollaries 2.4, 2.5 and one part of the proof of Theorem 2.6 in [7] use [7, Lemma 1.2]. By the previous example, the aforementioned results are also invalid.

2. Main results

In this section, we introduce and investigate the class of weak projection invariant semisimple modules. We focus on some structural properties of weak projection invariant semisimple modules as well as indecomposable decompositions for the special class of the weak projection invariant semisimple modules via some module theoretical conditions.

Definition 2.1. *We call an R-module M weak projection invariant semisimple, denoted by $w\pi$-semisimple, if each semisimple projection invariant submodule of M is a direct summand of M.*

Observe that any semisimple module and π-semisimple module is $w\pi$-semisimple. Moreover, any module which has zero socle (for example, a polynomial ring $R[x]$ over any ring R) is clearly a $w\pi$-semisimple module. Next, we provide $w\pi$-semisimple modules which are not π-semisimple.

Example 2.2. *(i) Let M be the $\mathbb{Z}$-module $\mathbb{Z}$. Obviously, $M_\mathbb{Z}$ is $w\pi$-semisimple. However, $M_\mathbb{Z}$ is not π-semisimple. For example, $N = 2\mathbb{Z}$ is a projection invariant in $M_\mathbb{Z}$ which is not a direct summand of $M_\mathbb{Z}$.*
(ii) Let M be the $\mathbb{Z}[x]$-module $\mathbb{Z}[x]$. Then $Soc M = 0$. Hence M is a $w\pi$-semisimple module. Since M is uniform, it is not π-semisimple.
(iii) [4, Example 2.4(ii)]. Let D be a simple domain which is not a division ring. Take $R = \begin{bmatrix} D & D \oplus D \\ 0 & D \end{bmatrix}$ then $I = \begin{bmatrix} 0 & 0 \oplus D \\ 0 & 0 \end{bmatrix}$ is an ideal of R. Thus, I is a projection invariant submodule of R_R which is not a direct summand of R_R. It follows that R_R is not π-semisimple. However, $Soc(R_R) = 0$, and hence R_R is $w\pi$-semisimple.

Example 2.2 sheds light on the natural question, namely, when a $w\pi$-semisimple module is a π-semisimple. The second part of the following result provides an answer.

Proposition 2.3. *(i) Assume that M_R is an indecomposable module. Then M_R is semisimple if and only if M_R is $w\pi$-semisimple and SocM is essential in M.*
(ii) If M_R is a $w\pi$-semisimple module with essential socle then M_R is π-semisimple.

Proof. (i) ($\Rightarrow$) This implication is clear.
($\Leftarrow$) Let $X \leq M$. Since M is indecomposable, X is projection invariant in M. It follows that $SocX$ is projection invariant in M, by Lemma 1.1 (i). By hypothesis, $SocX$ is a direct summand of M. Hence $SocX = 0$ or $SocX = M$. Therefore $X = 0$ or M. Thus, X is a direct summand of M. So, M is semisimple.
(ii) Let X be any projection invariant submodule of M_R. Then $SocX$ is projection invariant in M, by Lemma 1.1 (i). It follows that $SocX$ is a direct summand of M. On the other hand,

$$SocX = X \cap SocM \leq X \cap M = X$$

gives that $SocX$ is essential in X. Thus $SocX = X$ i.e., X is a direct summand of M. So, M_R is π-semisimple. $\qquad\square$

Corollary 2.4. *If M_R is a $w\pi$-semisimple module with essential socle then M_R is π-extending.*

Proof. Let X be a projection invariant submodule of M. By Proposition 2.3, X is a direct summand of M. Since X is essential in itself, M_R is a π-extending module. $\qquad\square$

Lemma 2.5. *Let M_R be $w\pi$-semisimple and N a projection invariant submodule of M. Then N is $w\pi$-semisimple.*

Proof. Let X be any semisimple projection invariant submodule of N. By Lemma 1.1 (i), X is projection invariant in M. Therefore $M = X \oplus X'$ for some X' submodule of M. Now, by Lemma 1.1 (ii), $N = (N \cap X) \oplus (N \cap X') = X \oplus (N \cap X')$. Thus X is a direct summand of N which yields that N is $w\pi$-semisimple. $\qquad\square$

Lemma 2.6. *Let M_R be a $w\pi$-semisimple module. Then the following statements hold:*

(i) Every fully invariant submodule of M_R is $w\pi$-semisimple.
(ii) If $End(M_R)$ is Abelian then every direct summand of M_R is $w\pi$-semisimple.

Proof. (i) Since every fully invariant submodule is projection invariant, the proof follows from Lemma 2.5.
(ii) Let M_R be a $w\pi$-semisimple module with an Abelian endomorphism ring. Let $K = eM$ for some $e^2 = e \in End(M_R)$. Thus $g(eM) \subseteq eM$ for all $g^2 = g \in End(M_R)$. Hence K_R is a projection invariant submodule of M_R. By Lemma 2.5, K_R is a $w\pi$-semisimple module. $\qquad\square$

Proposition 2.7. *Let $M = M_1 \oplus M_2$ such that M_2 is semisimple fully invariant submodule of M. If M_R is $w\pi$-semisimple, then both M_1 and M_2 are $w\pi$-semisimple.*

Proof. It is clear that M_2 is $w\pi$-semisimple. Let X be a semisimple projection invariant submodule of M_1. Then $X \oplus M_2$ is a semisimple projection invariant submodule of M (see [6, Lemma 4.13]). By hypothesis, $X \oplus M_2$ is a direct summand of M. Hence $M = X \oplus M_2 \oplus L$ for some submodule L of M. Now, the modular law gives that

$$M_1 = M_1 \cap (X \oplus M_2 \oplus L) = X \oplus (M_1 \cap (M_2 \oplus L)).$$

Hence X is a direct summand of M_1 which yields that M_1 is $w\pi$-semisimple. $\qquad\square$

Theorem 2.8. *Let $M = \underset{i \in I}{\oplus} M_i$ where M_i's are fully invariant submodules of M for $i \in I$. If M_i is $w\pi$-semisimple for all $i \in I$, then M is $w\pi$-semisimple.*

Proof. Assume each M_i is $w\pi$-semisimple for all $i \in I$ and $M = \underset{i \in I}{\oplus} M_i$. Let N be a semisimple projection invariant submodule of M. Then $N = \underset{i \in I}{\oplus}(N \cap M_i)$ where $N \cap M_i$ is a semisimple projection invariant submodule of M_i for all $i \in I$, from Lemma 1.1(ii). By assumption, M_i is $w\pi$-semisimple which gives that $N \cap M_i$ is a direct summand of M_i for all $i \in I$. It follows that N is a direct summand of M. Thus, M is $w\pi$-semisimple. $\qquad\square$

Observe that if M is a $w\pi$-semisimple module in the previous result then by Lemma 2.5, each M_i is also $w\pi$-semisimple for all $i \in I$. Our next aim is to obtain an indecomposable decomposition for special $w\pi$-semisimple modules. To do this, let us give the following definition.

Definition 2.9. *We call an R-module M $w\pi^*$-semisimple provided that whenever any semisimple projection invariant submodule is contained as projection invariant in a projection invariant submodule of M then the larger submodule is a direct summand of M.*

It can be seen easily that any $w\pi^*$-semisimple module is $w\pi$-semisimple (any projection invariant submodule has a semisimple projection invariant submodule, namely, its socle). However, there are several $w\pi$-semisimple modules which are not $w\pi^*$-semisimple. For example, let M be the $\mathbb{Z}$-module $\mathbb{Z}$ (see, Example 2.2 (i)).

Lemma 2.10. *Let R be a ring and M an R-module such that R satisfies ascending chain condition on right annihilators of the form $r(m)$ $(m \in M)$. If M is $w\pi^*$-semisimple with an Abelian endomorphism ring then M has an indecomposable decomposition.*

Proof. Let $\{X_\lambda : \lambda \in I\}$ be an independent family of submodules of M and $X = \underset{\lambda \in I}{\oplus} X_\lambda$ be a local summand of M. Now, let us define the canonical projection $\pi_k : X \to \underset{k \in I, k \neq \lambda}{\oplus} X_k$. Then $f(X) = f(\underset{\lambda \in I}{\oplus} X_\lambda) = \underset{\lambda \in I}{\oplus} f(X_\lambda) = \underset{\lambda \in I}{\oplus} f(ker\pi_\lambda)$ where $f^2 = f \in End(M_R)$. By the assumption that $End(M_R)$ is Abelian, $f(ker\pi_\lambda) \subseteq ker\pi_\lambda$. Thus $f(X) \subseteq X$. It follows that X is projection invariant in M_R. Since $SocX$ is projection invariant in X, by $w\pi^*$-semisimple, X is a direct summand of M. Hence [9, Theorem 2.17] yields that M has an indecomposable decomposition. $\qquad\square$

Next, we have the following result.

Theorem 2.11. *Let R be a ring and M an R-module such that R satisfies ascending chain condition on right annihilators of the form $r(m)$ $(m \in M)$. If M is $w\pi^*$-semisimple with an Abelian endomorphism ring then M is a direct sum of uniform submodules.*

Proof. Observe that being $w\pi^*$-semisimple implies π-extending, and an indecomposable module is uniform by [6, Proposition 3.8]. Now, we conclude the proof by Lemma 2.10 and Lemma 2.6 (ii). □

It would be worthy construct an example which shows that being Abelian endomorphism ring in the previous theorem is not superfluous. Incidentally, let us give the example.

Example 2.12. *Let R be the real field and n be any odd integer with $n \geq 1$. Let S be the polynomial ring $R[x_1, \ldots, x_n]$ indeterminates $x_1, x_2, \ldots, x_n$ over R. Let R be the ring S/Ss, where $s = x_1^2 + x_2^2 + \cdots + x_n^2 - 1$. Then the free R-module $M = \bigoplus_{i=1}^{n} R$ contains a submodule K_R which is indecomposable and has uniform dimension $n-1$ (see [12, Corollary 16]).*
Note that K_R is not uniform. Since $SocM = 0$, then $Soc(K_R) = 0$. Now, let $Y = \begin{bmatrix} S & K \\ 0 & R \end{bmatrix}$ be the split null extension ring where $S = End(K_R)$. Observe that $_SK$ is faithful. Therefore $Soc(Y_Y) = 0$. Hence Y is a $w\pi^*$-semisimple module. Moreover, $Y = \begin{bmatrix} S & K \\ 0 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & 0 \\ 0 & R \end{bmatrix}$ and $\begin{bmatrix} S & K \\ 0 & 0 \end{bmatrix}$ is indecomposable with uniform dimension $n-1$. It follows that Y has no decomposition into uniform submodules. It can be seen that Y_Y is Noetherian. However, $End(Y_Y) \cong Y$ is not Abelian. For, let $a = \begin{bmatrix} f & x \\ 0 & 0 \end{bmatrix}$, and $b = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ be two elements of $End(Y_Y)$ where $f \in S$ and $0 \neq x \in K_R$. Then $ab = \begin{bmatrix} f & x \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}$ and $ba = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} f & x \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

Now, we have the following consequences of the Theorem 2.11. The first one is the result on exchange property of modules which was pointed out in the introduction and the last is based on locally Noetherian modules. Recall that a module is called locally Noetherian provided that every finitely generated submodule is Noetherian (see [3]).

Corollary 2.13. *Let R be a right Noetherian ring and M an R-module with an Abelian endomorphism ring. If M is $w\pi^*$-semisimple then the finite exchange property implies full exchange property.*

Proof. By Theorem 2.11 and [8, Corollary 6]. □

Corollary 2.14. *Let M be a locally Noetherian module with an Abelian endomorphism ring. If M is $w\pi^*$-semisimple then the finite exchange property implies full exchange property.*

Proof. Let $m \in M$. Then $R/r(m) \cong mR$ is right Noetherian module. It follows that R satisfies ascending chain condition on right annihilators of the form $r(m)$ $(m \in M)$. Thus Theorem 2.11 gives the result. □

Finally, we have the next result on endomorphism ring of a $w\pi^*$-semisimple module. First, recall that a ring R is π-Baer if the right annihilator of a projection invariant left ideal of R is of the form eR for some $e^2 = e \in R$ (see [5, 13]).

Theorem 2.15. *Assume that M is a $w\pi^*$-semisimple module. Then the endomorphism ring of M is a π-Baer ring.*

Proof. Let $S = End(M_R)$ and I be a projection invariant left ideal of S. We want to show that $r_S(I) = eS$ for some $e^2 = e \in S$. It can be seen that $r_M(I)$ is a projection invariant submodule of M_R. Hence $Soc(r_M(I))$ is a projection invariant submodule of $r_M(I)$. By hypothesis, $r_M(I) = eM$ for some $e^2 = e \in S$. Thus $IeM = 0$, so $Ie = 0$, as $_SM$ faithful. Therefore $eS \subseteq r_S(I)$. Now, let $a \in r_S(I)$. Hence $Ia = 0$ which gives that $l(aM) = 0$. It follows that $aM \subseteq r_M(I) = eM$. Thus $a \in eS$, so $r_S(I) \subset eS$. Therefore, S is a π-Baer ring. □

Corollary 2.16. *If M is a π-semisimple module then the endomorphism ring of M is a π-Baer ring.*

Proof. Since π-semisimple implies $w\pi^*$-semisimple the result follows from Theorem 2.15. □

Acknowledgements

Funding

Availability of data and materials

Not applicable.

Competing interests

The authors declare that they have no competing interests.

Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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