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On $\mathscr{I}_\theta$-convergence in Neutrosophic Normed Spaces

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Abstract

The purpose of this article is to investigate lacunary ideal convergence of sequences in neutrosophic normed space (NNS). Also, an original notion, named lacunary convergence of sequence in NNS, is defined. Also, lacunary $\mathscr{I}$-limit points and lacunary $\mathscr{I}$-cluster points of sequences in NNS have been examined. Furthermore, lacunary Cauchy and lacunary $\mathscr{I}$-Cauchy sequences in NNS are introduced and some properties of these notions are studied.

1. Introduction and background

Theory of fuzzy sets (FSs) was firstly given by Zadeh [1]. The publication of the paper affected deeply all the scientific fields. This notion is significant for real-life conditions, but has not adequate solution to some problems and so these problems lead to original quests.
Intuitionistic fuzzy sets (IFSs) for such cases were initiated by Atanassov [2]. Atanassov et al. [3] used this concept in decision-making problems. Kramosil and Michalek [4] investigated fuzzy metric space (FMS) utilizing the concepts fuzzy and probabilistic metric space. The FMS as a distance between two points to be a non-negative fuzzy number was examined by Kaleva and Seikkala [5]. George and Veeramani [6] gave some qualifications of FMS. Some basic features of FMS were given and significant theorems were proved in [7]. Moreover, FMS has used by practical researches as for example decision-making, fixed point theory, medical imaging. Park [8] generalized FMSs and defined IF metric space (IFMS). Park utilized George and Veeramani's [6] opinion of using t-norm and t-conorm to the FMS meantime describing IFMS and investigating its fundamental properties. Saadati and Park [9] initially examined properties of intuitionistic fuzzy normed space (IFNS).
The statistical convergence initially introduced by [10]. Statistical convergence in IFNS was given by Karakuş et al [11]. Notable results on this topic can be found in [12]-[17].
By a lacunary sequence we mean an increasing integer sequence $\theta = \{k_r\}$ such that $k_0 = 0$ and $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$. Throughout this paper the intervals determined by θ will be indicated by $I_r = (k_{r-1}, k_r]$. Using lacunary sequence, Fridy and Orhan [18] examined the concept of lacunary statistical convergence. The publication of the paper affected deeply all the scientific fields. Some works in lacunary statistical convergence can be found in [19]-[23].
The concept neutrosophy implies impartial knowledge of thought and then neutral describes the basic difference between neutral, fuzzy, intuitive fuzzy set and logic. The neutrosophic set (NS) was investigated by F. Smarandache [24] who defined the degree of indeterminacy (i) as indepedent component. In [25], neutrosophic logic was firstly examined. It is a logic where each proposition is determined to have a degree of truth (T), falsity (F), and indeterminacy (I). A Neutrosophic set (NS) is determined as a set where every component of the universe has a degree of T, F and I.
In IFSs the 'degree of non-belongingness' is not independent but it is dependent on the 'degree of belongingness'. FSs can be thought as a remarkable case of an IFS where the 'degree of non-belongingness' of an element is absolutely equal to '1- degree of belongingness'.
Uncertainty is based on the belongingness degree in IFSs, whereas the uncertainty in NS is considered independently from T and F values. Since no any limitations among the degree of T, F, I, NSs are actually more general than IFS.
Neutrosophic soft linear spaces (NSLSs) were considered by Bera and Mahapatra [26]. Subsequently, in [27], the concept neutrosophic soft normed linear (NSNLS) was defined and the features of (NSNLS) were examined. Significant results on this topic can be found in [28]-[32].

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Kirişçi and Şimşek [33] defined new concept known as neutrosophic metric space (NMS) with continuous t-norms and continuous t-conorms. Some notable features of NMS have been examined.
Neutrosophic normed space (NNS) and statistical convergence in NNS has been investigated by Kirişci and Şimşek [34]. Neutrosophic set and neutrosophic logic has used by applied sciences and theoretical science such as decision making, robotics, summability theory. Some noteworthy results on this topic can be examined in [35]-[39].
In [39], lacunary statistical convergence of sequences in NNS was examined. Also, lacunary statistically Cauchy sequence in NNS was given and lacunary statistically completeness in connection with a neutrosophic normed space was presented.
Firstly, we recall some definitions used throughout the paper.
For $K \subset \mathbb{N}$ and $j \in \mathbb{N}$, if

$$\delta_j(K) = \frac{|K \cap \{1,2,...,j\}|}{j},$$

then $\delta_j(K)$ is named jth partial density of K. If

$$\delta(K) = \lim_{n\to\infty} \frac{1}{n} |\{k \le n : k \in K\}|, \left(\text{i.e., } \delta(K) = \lim_{j\to\infty} \delta_j(K)\right)$$

exists, it is named the natural density of K. $\Psi = \{K \subset \mathbb{N} : \delta(K) = 0\}$ is denoted the zero density set.
A sequence (x_n) is said to be statistically convergent to ξ if for every $\varepsilon > 0$,

$$\delta(\{n \in \mathbb{N} : |x_k - \xi| \ge \varepsilon\}) = 0,$$

i.e., $\{n \in \mathbb{N} : |x_k - \xi| \ge \varepsilon\} \in \Psi$. We demonstrate $st - \lim x_n = \xi$ or $x_n \xrightarrow{st} \xi$, $(n \to \infty)$.
In the wake of the study of ideal convergence defined by Kostyrko et al. [40], there has been comprehensive research to discover applications and summability studies of the classical theories. Ideal convergence became a notable topic in summability theory after the researches of [41]-[52].
Let $\emptyset \ne S$ be a set, and then a non empty class $\mathscr{I} \subseteq P(S)$ is said to be an *ideal* on S iff (i) $\emptyset \in \mathscr{I}$, (ii) $\mathscr{I}$ is additive under union, (iii) for each $A \in \mathscr{I}$ and each $B \subseteq A$ we find $B \in \mathscr{I}$. An ideal $\mathscr{I}$ is called *non-trivial* if $\mathscr{I} \ne \emptyset$ and $S \notin \mathscr{I}$. A non-empty family of sets $\mathscr{F}$ is called *filter* on S iff (i) $\emptyset \notin \mathscr{F}$, (ii) for each $A, B \in \mathscr{F}$ we get $A \cap B \in \mathscr{F}$, (iii) for every $A \in \mathscr{F}$ and each $B \supseteq A$, we obtain $B \in \mathscr{F}$. Relationship between ideal and filter is given as follows:

$$\mathscr{F}(\mathscr{I}) = \{K \subset S : K^c \in \mathscr{I}\},$$

where $K^c = S - K$.
A non-trivial ideal $\mathscr{I}$ is (i) an *admissible ideal* on S iff it contains all singletons.
A sequence (x_n) is named to be ideal convergent to ξ if for every $\varepsilon > 0$, i.e.

$$A(\varepsilon) = \{n \in \mathbb{N} : |x_n - \xi| \ge \varepsilon\} \in \mathscr{I}.$$

We take $\mathscr{I}$ as admissible ideal throughout the paper.
Triangular norms (t-norms) (TN) were given by Menger [53]. TNs are used to generalize with the probability distribution of triangle inequality in metric space terms. Triangular conorms (t-conorms) (TC) known as dual operations of TNs. TNs and TCs are important for fuzzy operations (intersections and unions).

Definition 1.1. *([53]) Let $* : [0,1] \times [0,1] \to [0,1]$ be an operation. When $\circ$ satisfies following situations, it is called continuous TN. Take $p,q,r,s \in [0,1]$,*

(a) $p * 1 = p$,
(b) *If* $p \le r$ *and* $q \le s$, *then* $p * q \le r * s$,
(c) $*$ *is continuous,*
(d) $*$ *associative and commutative.*

Definition 1.2. *([53]) Let $\Diamond : [0,1] \times [0,1] \to [0,1]$ be an operation. When $\Diamond$ satisfies following situations, it is said to be continuous TC.*

(a) $p \Diamond 0 = p$,
(b) *If* $p \le r$ *and* $q \le s$, *then* $p \Diamond q \le r \Diamond s$,
(c) $\Diamond$ *is continuous,*
(d) $\Diamond$ *associative and commutative.*

Definition 1.3. *([34]) Let F be a vector space, $\mathscr{N} = \{\langle u, \mathscr{G}(u), \mathscr{B}(u), \mathscr{Y}(u)\rangle : u \in F\}$ be a normed space (NS) such that $\mathscr{N} : F \times \mathbb{R}^+ \to [0,1]$. While following conditions hold, $V = (F, \mathscr{N}, *, \Diamond)$ is called to be NNS. For each $u, v \in F$ and $\lambda, \mu > 0$ and for all $\sigma \ne 0$,*

(a) $0 \le \mathscr{G}(u,\lambda) \le 1$, $0 \le \mathscr{B}(u,\lambda) \le 1$, $0 \le \mathscr{Y}(u,\lambda) \le 1$ $\forall \lambda \in \mathbb{R}^+$,
(b) $\mathscr{G}(u,\lambda) + \mathscr{B}(u,\lambda) + \mathscr{Y}(u,\lambda) \le 3$ *(for* $\lambda \in \mathbb{R}^+$*),*
(c) $\mathscr{G}(u,\lambda) = 1$ *(for* $\lambda > 0$*) iff* $u = 0$,
(d) $\mathscr{G}(\sigma u,\lambda) = \mathscr{G}\left(u, \frac{\lambda}{|\sigma|}\right)$,
(e) $\mathscr{G}(u,\mu) * \mathscr{G}(v,\lambda) \le \mathscr{G}(u+v, \mu+\lambda)$,
(f) $\mathscr{G}(u,.)$ *is non-decreasing continuous function,*
(g) $\lim_{\lambda\to\infty} \mathscr{G}(u,\lambda) = 1$,
(h) $\mathscr{B}(u,\lambda) = 0$ *(for* $\lambda > 0$*) iff* $u = 0$,
(i) $\mathscr{B}(\sigma u,\lambda) = \mathscr{B}\left(u, \frac{\lambda}{|\sigma|}\right)$,

(j) $\mathscr{B}(u,\mu)\lozenge\mathscr{B}(v,\lambda)\geq\mathscr{B}(u+v,\mu+\lambda)$,
(k) $\mathscr{B}(u,.)$ *is non-decreasing continuous function,*
(l) $\lim_{\lambda\to\infty}\mathscr{B}(u,\lambda)=0$,
(m) $\mathscr{Y}(u,\lambda)=0$ *(for* $\lambda>0$*) iff* $u=0$,
(n) $\mathscr{Y}(\sigma u,\lambda)=\mathscr{Y}\left(u,\frac{\lambda}{|\sigma|}\right)$,
(o) $\mathscr{Y}(u,\mu)\lozenge\mathscr{Y}(v,\lambda)\geq\mathscr{Y}(u+v,\mu+\lambda)$,
(p) $\mathscr{Y}(u,.)$ *is non-decreasing continuous function,*
(r) $\lim_{\lambda\to\infty}\mathscr{Y}(u,\lambda)=0$,
(s) If $\lambda\leq 0$*, then* $\mathscr{G}(u,\lambda)=0,\mathscr{B}(u,\lambda)=1$ *and* $\mathscr{Y}(u,\lambda)=1$.
Then $\mathscr{N}=(\mathscr{G},\mathscr{B},\mathscr{Y})$ *is called Neutrosophic norm (NN).*

Definition 1.4. *([34]) Let V be an NNS, the sequence* (x_k) *in* V, $\varepsilon\in(0,1)$ *and* $\lambda>0$*. Then, the sequence* (x_k) *is converges to* ξ *iff there is* $N\in\mathbb{N}$ *such that* $\mathscr{G}(x_k-\xi,\lambda)>1-\varepsilon$, $\mathscr{B}(x_k-\xi,\lambda)<\varepsilon$, $\mathscr{Y}(x_k-\xi,\lambda)<\varepsilon$*. That is,* $\lim_{n\to\infty}\mathscr{G}(x_k-\xi,\lambda)=1$, $\lim_{n\to\infty}\mathscr{B}(x_k-\xi,\lambda)=0$ *and* $\lim_{n\to\infty}\mathscr{Y}(x_k-\xi,\lambda)=0$ *as* $\lambda>0$*. In that case, the sequence* (x_k) *is named a convergent sequence in* V*. The convergent in NNS is indicated by* $\mathscr{N}-limx_k=\xi$.

Definition 1.5. *([34]) A sequence* (x_k) *in* V, $\varepsilon\in(0,1)$ *and* $\lambda>0$*. Then, the sequence* (x_k) *is Cauchy in NNS V if there is a* $N\in N$ *such that* $\mathscr{G}(x_k-x_m,\lambda)>1-\varepsilon$, $\mathscr{B}(x_k-x_m,\lambda)<\varepsilon$, $\mathscr{Y}(x_k-x_m,\lambda)<\varepsilon$ *for* $k,m\geq N$.

Definition 1.6. *([34]) A sequence* (x_m) *is said to be statistically convergent to* $\xi\in F$ *with regards to NN (SC-NN), if, for each* $\lambda>0$ *and* $\varepsilon>0$ *the set*

$$P_\varepsilon:=\{m\leq n:\mathscr{G}(x_m-\xi,\lambda)\leq 1-\varepsilon \text{ or } \mathscr{B}(x_m-\xi,\lambda)\geq\varepsilon,\ \mathscr{Y}(x_m-\xi,\lambda)\geq\varepsilon\}$$

or equivalently

$$P_\varepsilon:=\{m\leq n:\mathscr{G}(x_m-\xi,\lambda)>1-\varepsilon \text{ or } \mathscr{B}(x_m-\xi,\lambda)<\varepsilon,\ \mathscr{Y}(x_m-\xi,\lambda)<\varepsilon\}.$$

has ND zero. That is $d(P_\varepsilon)=0$ *or*

$$\lim_{n\to\infty}\frac{1}{n}\left|\{m\leq n:\mathscr{G}(x_m-\xi,\lambda)\leq 1-\varepsilon \text{ or } \mathscr{B}(x_m-\xi,\lambda)\geq\varepsilon,\ \mathscr{Y}(x_m-\xi,\lambda)\geq\varepsilon\}\right|=0.$$

It is denoted by $S_{\mathscr{N}}$-$\lim x_m=\xi$ *or* $x_k\to\xi\ (S_{\mathscr{N}})$*. The set of SC-NN will be denoted by* $S_{\mathscr{N}}$.

Definition 1.7. *([34]) The sequence* (x_k) *is called statistical Cauchy with regards to NN* $\mathscr{N}$ *(SCa-NN) in NNS V, if there exists* $N=N(\varepsilon)$*, for every* $\varepsilon>0$ *and* $\lambda>0$ *such that*

$$C_\varepsilon:=\{m\leq n:\mathscr{G}(x_m-x_N,\lambda)\leq 1-\varepsilon \text{ or } \mathscr{B}(x_m-x_N,\lambda)\geq\varepsilon,\ \mathscr{Y}(x_m-x_N,\lambda)\geq\varepsilon\}$$

has ND zero. That is, $d(C_\varepsilon)=0$.

Definition 1.8. *([34]) Let V be an NNS. For* $\lambda>0$, $w\in F$ *and* $\varepsilon\in(0,1)$,

$$B(w,\varepsilon,\lambda)=\{u\in F:\mathscr{G}(w-u,\lambda)>1-\varepsilon,\ \mathscr{B}(w-u,\lambda)<\varepsilon,\ \mathscr{Y}(w-u,\lambda)<\varepsilon\}$$

is called open ball with center w, radius ε.

2. Main results

Definition 2.1. *Take an NNS V. For a lacunary sequence* θ*, a sequence* $x=(x_k)$ *is named to be lacunary convergent to* $\xi\in F$ *with regards to NN (LC-NN), if for every* $\lambda>0$ *and* $\varepsilon\in(0,1)$*, there is* $r_0\in\mathbb{N}$ *such that*

$$\frac{1}{h_r}\sum_{k\in I_r}\mathscr{G}(x_k-\xi,\lambda)>1-\varepsilon \text{ and } \frac{1}{h_r}\sum_{k\in I_r}\mathscr{B}(x_k-\xi,\lambda)<\varepsilon,\ \frac{1}{h_r}\sum_{k\in I_r}\mathscr{Y}(x_k-\xi,\lambda)<\varepsilon$$

for all $r\geq r_0$*. We indicate* $(\mathscr{G},\mathscr{B},\mathscr{Y})^\theta-\lim x=\xi$.

Theorem 2.2. *Let V be an NNS. If x is lacunary convergent with regards to NN, then* $(\mathscr{G},\mathscr{B},\mathscr{Y})^\theta-\lim x$ *is unique.*

Proof. Presume that $(\mathscr{G},\mathscr{B},\mathscr{Y})^\theta-\lim x=\xi_1$ and $(\mathscr{G},\mathscr{B},\mathscr{Y})^\theta-\lim x=\xi_2$. Given $\varepsilon>0$, select $\rho\in(0,1)$ such that $(1-\rho)*(1-\rho)>1-\varepsilon$ and $\rho\lozenge\rho<\varepsilon$. For each $\lambda>0$, there is $r_1\in\mathbb{N}$ such that

$$\frac{1}{h_r}\sum_{k\in I_r}\mathscr{G}(x_k-\xi_1,\lambda)>1-\varepsilon \text{ and } \frac{1}{h_r}\sum_{k\in I_r}\mathscr{B}(x_k-\xi_1,\lambda)<\varepsilon,\ \frac{1}{h_r}\sum_{k\in I_r}\mathscr{Y}(x_k-\xi_1,\lambda)<\varepsilon$$

for all $r\geq r_1$. Also, there is $r_2\in\mathbb{N}$ such that

$$\frac{1}{h_r}\sum_{k\in I_r}\mathscr{G}(x_k-\xi_2,\lambda)>1-\varepsilon \text{ and } \frac{1}{h_r}\sum_{k\in I_r}\mathscr{B}(x_k-\xi_2,\lambda)<\varepsilon,\ \frac{1}{h_r}\sum_{k\in I_r}\mathscr{Y}(x_k-\xi_2,\lambda)<\varepsilon$$

for all $r \geq r_2$. Think $r_0 = \max\{r_1, r_2\}$. Then, for $r \geq r_0$, we take a $m \in \mathbb{N}$ such that

$$\mathscr{G}\left(x_m - \xi_1, \frac{\lambda}{2}\right) > \frac{1}{h_r} \sum_{k \in I_r} \mathscr{G}\left(x_k - \xi_1, \frac{\lambda}{2}\right) > 1 - \rho,$$

$$\mathscr{G}\left(x_m - \xi_2, \frac{\lambda}{2}\right) > \frac{1}{h_r} \sum_{k \in I_r} \mathscr{G}\left(x_k - \xi_2, \frac{\lambda}{2}\right) > 1 - \rho.$$

Then, we obtain

$$\begin{aligned} \mathscr{G}(\xi_1 - \xi_2, \lambda) &\geq \mathscr{G}\left(x_m - \xi_1, \frac{\lambda}{2}\right) * \mathscr{G}\left(x_m - \xi_2, \frac{\lambda}{2}\right) \\ &> (1-\rho) * (1-\rho) > 1 - \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is abritrary, we get $\mathscr{G}(\xi_1 - \xi_2, \lambda) = 1$ for all $\lambda > 0$, which gives that $\xi_1 = \xi_2$. □

Definition 2.3. *Let $\theta = (k_r)$ be a lacunary sequence, $\mathscr{I} \subset 2^{\mathbb{N}}$ and let V be an NNS. A sequence $x = (x_k)$ is said to be lacunary $\mathscr{I}$-convergent to $\xi \in F$ with regards to NN ($\mathscr{I}_\theta$C-NN), if, for every $\varepsilon \in (0,1)$ and $\lambda > 0$, the set*

$$\left\{ \begin{array}{c} r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} \mathscr{G}(x_k - \xi, \lambda) \leq 1 - \varepsilon \\ \text{or } \frac{1}{h_r} \sum_{k \in I_r} \mathscr{B}(x_k - \xi, \lambda) \geq \varepsilon, \ \frac{1}{h_r} \sum_{k \in I_r} \mathscr{Y}(x_k - \xi, \lambda) \geq \varepsilon \end{array} \right\} \in \mathscr{I}.$$

ξ is called the lacunary $\mathscr{I}$-limit of the sequence of (x_k), and we demonstrate $\mathscr{I}_\theta^{(\mathscr{G},\mathscr{B},\mathscr{Y})} - \lim x = \xi$.

Now, we prepare an example to denote the sequence $\mathscr{I}_\theta$-convergent in an NNS.

Example 2.4. *Let $(F, \|.\|)$ be a NNS, $\mathscr{I}$ be a non-trivial admissible ideal. For all $u, v \in [0,1]$, take the TN $u * v = uv$ and the TC $u \Diamond v = \min\{u + v, 1\}$. For all $x \in F$ and every $\lambda > 0$, we contemplate $\mathscr{G}(x, \lambda) = \frac{\lambda}{\lambda + \|x\|}$, $\mathscr{B}(x, \lambda) = \frac{\|x\|}{\lambda + \|x\|}$ and $\mathscr{Y}(x, \lambda) = \frac{\|x\|}{\lambda}$. Then, V is an NNS. We define a sequence (x_k) by*

$$x_k = \begin{cases} 1, & \text{if } k = t^2 \ (t \in \mathbb{N}) \\ 0, & \text{otherwise.} \end{cases}$$

Then, for any $\lambda > 0$ and for all $\varepsilon \in (0,1)$, the following set

$$\begin{aligned} A(\varepsilon, \lambda) &= \left\{ k \in \mathbb{N} : \frac{\lambda}{\lambda + \|x_k\|} \leq 1 - \varepsilon \text{ or } \frac{\|x_k\|}{\lambda + \|x_k\|} \geq \varepsilon, \ \frac{\|x_k\|}{\lambda} \geq \varepsilon \right\} \\ &= \left\{ k \in \mathbb{N} : \|x_k\| \geq \frac{\lambda\varepsilon}{1-\varepsilon}, \text{ or } \|x_k\| \geq \lambda\varepsilon \right\} \\ &= \{k \in \mathbb{N} : \|x_k\| = 1\} = \left\{ k \in \mathbb{N} : k = t^2 \ (t \in \mathbb{N}) \right\} \end{aligned}$$

i.e.,

$$A(\varepsilon, \lambda) = \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} \mathscr{G}(x_k, \lambda) \leq 1 - \varepsilon \text{ or } \frac{1}{h_r} \sum_{k \in I_r} \mathscr{B}(x_k, \lambda) \geq \varepsilon, \ \frac{1}{h_r} \sum_{k \in I_r} \mathscr{Y}(x_k, \lambda) \geq \varepsilon \right\}$$

will be a finite set. So, $\delta(A(\varepsilon, \lambda)) = 0$, and as a result $A(\varepsilon, \lambda) \in \mathscr{I}$. We show that $\mathscr{I}_\theta^{(\mathscr{G},\mathscr{B},\mathscr{Y})} - \lim x = 0$.

Lemma 2.5. *For every $\varepsilon > 0$ and $\lambda > 0$, the following situations are equivalent.*

(a) $\mathscr{I}_\theta^{(\mathscr{G},\mathscr{B},\mathscr{Y})} - \lim x = \xi$,

(b) $\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} \mathscr{G}(x_k - \xi, \lambda) \leq 1 - \varepsilon \right\} \in \mathscr{I}$ *and*

$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} \mathscr{B}(x_k - \xi, \lambda) \geq \varepsilon \right\} \in \mathscr{I}$,

$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} \mathscr{Y}(x_k - \xi, \lambda) \geq \varepsilon \right\} \in \mathscr{I}$,

(c) $\left\{ \begin{array}{c} r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} \mathscr{G}(x_k - \xi, \lambda) > 1 - \varepsilon \\ \text{and } \frac{1}{h_r} \sum_{k \in I_r} \mathscr{B}(x_k - \xi, \lambda) < \varepsilon \\ \frac{1}{h_r} \sum_{k \in I_r} \mathscr{Y}(x_k - \xi, \lambda) < \varepsilon \end{array} \right\} \in \mathscr{F}(\mathscr{I})$,

(d) $\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} \mathscr{G}(x_k - \xi, \lambda) > 1 - \varepsilon \right\} \in \mathscr{F}(\mathscr{I})$ *and*

$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} \mathscr{B}(x_k - \xi, \lambda) < \varepsilon \right\} \in \mathscr{F}(\mathscr{I})$,

$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} \mathscr{Y}(x_k - \xi, \lambda) < \varepsilon \right\} \in \mathscr{F}(\mathscr{I})$ *and*

(e) $\mathscr{I}_\theta^{(\mathscr{G},\mathscr{B},\mathscr{Y})} - \lim \mathscr{G}(x_k - \xi, \lambda) = 1$ *and* $\mathscr{I}_\theta^{(\mathscr{G},\mathscr{B},\mathscr{Y})} - \lim \mathscr{B}(x_k - \xi, \lambda) = 0$, $\mathscr{I}_\theta^{(\mathscr{G},\mathscr{B},\mathscr{Y})} - \lim \mathscr{Y}(x_k - \xi, \lambda) = 0$.

Theorem 2.6. *If a sequence $x=(x_k)$ is lacunary $\mathscr{I}$-convergent with regards to the NN, then $\mathscr{I}_\theta^{(\mathscr{G},\mathscr{B},\mathscr{Y})}-\lim x$ is unique.*

Proof. Presume that $\mathscr{I}_\theta^{(\mathscr{G},\mathscr{B},\mathscr{Y})}-\lim x=\xi_1$ and $\mathscr{I}_\theta^{(\mathscr{G},\mathscr{B},\mathscr{Y})}-\lim x=\xi_2$. Select $\varepsilon\in(0,1)$. Then, for a given $\rho\in(0,1)$, $(1-\rho)*(1-\rho)>1-\varepsilon$ and $\rho\diamond\rho<\varepsilon$. For any $\lambda>0$, let's denote the following sets:

$$K_{\mathscr{G}1}(\rho,\lambda)=\left\{r\in\mathbb{N}:\frac{1}{h_r}\sum_{k\in I_r}\mathscr{G}\left(x_k-\xi_1,\frac{\lambda}{2}\right)\leq 1-\rho\right\},$$
$$K_{\mathscr{G}2}(\rho,\lambda)=\left\{r\in\mathbb{N}:\frac{1}{h_r}\sum_{k\in I_r}\mathscr{G}\left(x_k-\xi_2,\frac{\lambda}{2}\right)\leq 1-\rho\right\},$$
$$K_{\mathscr{B}1}(\rho,\lambda)=\left\{r\in\mathbb{N}:\frac{1}{h_r}\sum_{k\in I_r}\mathscr{B}\left(x_k-\xi_1,\frac{\lambda}{2}\right)\geq\rho\right\},$$
$$K_{\mathscr{B}2}(\rho,\lambda)=\left\{r\in\mathbb{N}:\frac{1}{h_r}\sum_{k\in I_r}\mathscr{B}\left(x_k-\xi_2,\frac{\lambda}{2}\right)\geq\rho\right\},$$
$$K_{\mathscr{Y}1}(\rho,\lambda)=\left\{r\in\mathbb{N}:\frac{1}{h_r}\sum_{k\in I_r}\mathscr{Y}\left(x_k-\xi_1,\frac{\lambda}{2}\right)\geq\rho\right\},$$
$$K_{\mathscr{Y}2}(\rho,\lambda)=\left\{r\in\mathbb{N}:\frac{1}{h_r}\sum_{k\in I_r}\mathscr{Y}\left(x_k-\xi_2,\frac{\lambda}{2}\right)\geq\rho\right\}.$$

Since $\mathscr{I}_\theta^{(\mathscr{G},\mathscr{B},\mathscr{Y})}-\lim x=\xi_1$, using Lemma 2.5, we obtain $K_{\mathscr{G}1}(\rho,\lambda)$, $K_{\mathscr{B}1}(\rho,\lambda)$, $K_{\mathscr{Y}1}(\rho,\lambda)\in\mathscr{I}$. Utilizing $\mathscr{I}_\theta^{(\mathscr{G},\mathscr{B},\mathscr{Y})}-\lim x=\xi_2$, we get $K_{\mathscr{G}2}(\rho,\lambda)$, $K_{\mathscr{B}2}(\rho,\lambda)$, $K_{\mathscr{Y}2}(\rho,\lambda)\in\mathscr{I}$.
Let

$$\begin{aligned}K_{\mathscr{G},\mathscr{B},\mathscr{Y}}(\rho,\lambda) \quad &:=(K_{\mathscr{G}1}(\rho,\lambda)\cup K_{\mathscr{G}2}(\rho,\lambda))\cap(K_{\mathscr{B}1}(\rho,\lambda)\cup K_{\mathscr{B}2}(\rho,\lambda))\\ &\cap(K_{\mathscr{Y}1}(\rho,\lambda)\cup K_{\mathscr{Y}2}(\rho,\lambda)).\end{aligned}$$

Then, $K_{\mathscr{G},\mathscr{B},\mathscr{Y}}(\rho,\lambda)\in\mathscr{I}$, which implies that $\emptyset\neq K^c_{\mathscr{G},\mathscr{B},\mathscr{Y}}(\rho,\lambda)\in\mathscr{F}(\mathscr{I})$. If $r\in K^c_{\mathscr{G},\mathscr{B},\mathscr{Y}}(\rho,\lambda)$, then we have three possible cases. That is, $r\in\left(K^c_{\mathscr{G}1}(\rho,\lambda)\cap K^c_{\mathscr{G}2}(\rho,\lambda)\right)$, $r\in\left(K^c_{\mathscr{B}1}(\rho,\lambda)\cap K^c_{\mathscr{B}2}(\rho,\lambda)\right)$ or $r\in\left(K^c_{\mathscr{Y}1}(\rho,\lambda)\cap K^c_{\mathscr{Y}2}(\rho,\lambda)\right)$. First, think that $r\in\left(K^c_{\mathscr{G}1}(\rho,\lambda)\cap K^c_{\mathscr{G}2}(\rho,\lambda)\right)$. Then, we obtain

$$\frac{1}{h_r}\sum_{k\in I_r}\mathscr{G}\left(x_k-\xi_1,\frac{\lambda}{2}\right)>1-\rho \text{ and } \frac{1}{h_r}\sum_{k\in I_r}\mathscr{G}\left(x_k-\xi_2,\frac{\lambda}{2}\right)>1-\rho.$$

Now, obviously, we get a $m\in\mathbb{N}$ such that

$$\begin{aligned}\mathscr{G}\left(x_m-\xi_1,\frac{\lambda}{2}\right)&>\frac{1}{h_r}\sum_{k\in I_r}\mathscr{G}\left(x_k-\xi_1,\frac{\lambda}{2}\right)>1-\rho,\\ \mathscr{G}\left(x_m-\xi_2,\frac{\lambda}{2}\right)&>\frac{1}{h_r}\sum_{k\in I_r}\mathscr{G}\left(x_k-\xi_2,\frac{\lambda}{2}\right)>1-\rho\end{aligned}$$

(e.g., consider $\max\left\{\mathscr{G}\left(x_k-\xi_1,\frac{\lambda}{2}\right),\mathscr{G}\left(x_k-\xi_2,\frac{\lambda}{2}\right):k\in I_r\right\}$ and select that k as m for which the maximum occurs).
Then, we get

$$\begin{aligned}\mathscr{G}(\xi_1-\xi_2,\lambda) \quad &\geq\mathscr{G}\left(x_m-\xi_1,\frac{\lambda}{2}\right)*\mathscr{G}\left(x_m-\xi_2,\frac{\lambda}{2}\right)\\ &>(1-\rho)*(1-\rho)>1-\varepsilon.\end{aligned}$$

Since $\varepsilon>0$ is arbitrary, we get $\mathscr{G}(\xi_1-\xi_2,\lambda)=1$ for all $\lambda>0$, which yields that $\xi_1=\xi_2$. On the other hand, if we take $r\in\left(K^c_{\mathscr{B}1}(\rho,\lambda)\cup K^c_{\mathscr{B}2}(\rho,\lambda)\right)$, then we can write

$$\mathscr{B}(\xi_1-\xi_2,\lambda)\leq\mathscr{B}\left(x_m-\xi_1,\frac{\lambda}{2}\right)\diamond\mathscr{B}\left(x_m-\xi_2,\frac{\lambda}{2}\right)\leq\rho\diamond\rho<\varepsilon.$$

Therefore, we can see that $\mathscr{B}(\xi_1-\xi_2,\lambda)<\varepsilon$. For all $\lambda>0$, we obtain $\mathscr{B}(\xi_1-\xi_2,\lambda)=0$, which implies that $\xi_1=\xi_2$. Again, for the situation $r\in\left(K^c_{\mathscr{Y}1}(\rho,\lambda)\cap K^c_{\mathscr{Y}2}(\rho,\lambda)\right)$, then, utilizing a same method, it can be proved that $\mathscr{Y}(\xi_1-\xi_2,\lambda)<\varepsilon$ for all $\lambda>0$ and arbitrary $\varepsilon>0$, and thus $\xi_1=\xi_2$. Hence, in all cases, we conclude that the $\mathscr{I}_\theta^{(\mathscr{G},\mathscr{B},\mathscr{Y})}$-limit is unique. □

Theorem 2.7. *If $(\mathscr{G},\mathscr{B},\mathscr{Y})^\theta-\lim x=\xi$, then $\mathscr{I}_\theta^{(\mathscr{G},\mathscr{B},\mathscr{Y})}-\lim x=\xi$.*

Proof. Let $(\mathscr{G},\mathscr{B},\mathscr{Y})^\theta-\lim x=\xi$. Then, for every $\lambda>0$ and $\varepsilon\in(0,1)$, there is $r_0\in\mathbb{N}$ such that

$$\frac{1}{h_r}\sum_{k\in I_r}\mathscr{G}(x_k-\xi,\lambda)>1-\varepsilon \text{ and } \frac{1}{h_r}\sum_{k\in I_r}\mathscr{B}(x_k-\xi,\lambda)<\varepsilon,\ \frac{1}{h_r}\sum_{k\in I_r}\mathscr{Y}(x_k-\xi,\lambda)<\varepsilon$$

for all $r\geq r_0$. Therefore, we obtain

$$\begin{aligned}T=&\left\{\begin{array}{l} r\in\mathbb{N}:\frac{1}{h_r}\sum_{k\in I_r}\mathscr{G}(x_k-\xi,\lambda)\leq 1-\varepsilon\\ \text{or } \frac{1}{h_r}\sum_{k\in I_r}\mathscr{B}(x_k-\xi,\lambda)\geq\varepsilon,\ \frac{1}{h_r}\sum_{k\in I_r}\mathscr{Y}(x_k-\xi,\lambda)\geq\varepsilon\end{array}\right\}\\ &\subseteq\{1,2,...,k_0-1\}.\end{aligned}$$

If we accept $\mathscr{I}$ as admissible ideal, we get $T\in\mathscr{I}$. Hence, $\mathscr{I}_\theta^{(\mathscr{G},\mathscr{B},\mathscr{Y})}-\lim x=\xi$. □

Theorem 2.8. *If $(\mathscr{G},\mathscr{B},\mathscr{Y})^{\theta}-\lim x=\xi$, then there is a subsequence (x_{p_k}) of x such that $(\mathscr{G},\mathscr{B},\mathscr{Y})^{\theta}-\lim x_{p_k}=\xi$.*

Proof. Take $(\mathscr{G},\mathscr{B},\mathscr{Y})^{\theta}-\lim x=\xi$. Then, for every $\lambda>0$ and $\varepsilon\in(0,1)$, there is $r_0\in\mathbb{N}$ such that

$$\frac{1}{h_r}\sum_{k\in I_r}\mathscr{G}(x_k-\xi,\lambda)>1-\varepsilon \text{ and } \frac{1}{h_r}\sum_{k\in I_r}\mathscr{B}(x_k-\xi,\lambda)<\varepsilon,\ \frac{1}{h_r}\sum_{k\in I_r}\mathscr{Y}(x_k-\xi,\lambda)<\varepsilon$$

for all $r\geq r_0$. Obviously, for each $r\geq r_0$, we choose $p_k\in I_r$ such that

$$\begin{aligned}\mathscr{G}\left(x_{p_k}-\xi,\lambda\right) &> \tfrac{1}{h_r}\sum_{k\in I_r}\mathscr{G}(x_k-\xi,\lambda)>1-\varepsilon,\\ \mathscr{B}\left(x_{p_k}-\xi,\lambda\right) &< \tfrac{1}{h_r}\sum_{k\in I_r}\mathscr{B}(x_k-\xi,\lambda)<\varepsilon,\\ \mathscr{Y}\left(x_{p_k}-\xi,\lambda\right) &< \tfrac{1}{h_r}\sum_{k\in I_r}\mathscr{Y}(x_k-\xi,\lambda)<\varepsilon.\end{aligned}$$

It follows that $(\mathscr{G},\mathscr{B},\mathscr{Y})^{\theta}-\lim x_{p_k}=\xi$. □

Definition 2.9. *Take an NNS V. A sequence $x=(x_k)$ is named to be lacunary Cauchy with regards to the NN $\mathscr{N}$ (LCa-NN) if, for every $\varepsilon\in(0,1)$ and $\lambda>0$, there are $r_0,p\in\mathbb{N}$ satisfying*

$$\frac{1}{h_r}\sum_{k\in I_r}\mathscr{G}\left(x_k-x_p,\lambda\right)>1-\varepsilon \text{ and } \frac{1}{h_r}\sum_{k\in I_r}\mathscr{B}\left(x_k-x_p,\lambda\right)<\varepsilon,\ \frac{1}{h_r}\sum_{k\in I_r}\mathscr{Y}\left(x_k-x_p,\lambda\right)<\varepsilon$$

for all $r\geq r_0$.

Definition 2.10. *Let V be an NNS. A sequence $x=(x_k)$ is called to be lacunary $\mathscr{I}$-Cauchy ($\mathscr{I}_\theta$-Cauchy) with regards to the NN $\mathscr{N}$ ($\mathscr{I}_\theta$Ca-NN) if, for every $\varepsilon\in(0,1)$ and $\lambda>0$, there is $p\in\mathbb{N}$ satisfying*

$$\left\{\begin{array}{l} r\in\mathbb{N}:\frac{1}{h_r}\sum_{k\in I_r}\mathscr{G}\left(x_k-x_p,\lambda\right)>1-\varepsilon\\ \text{and } \frac{1}{h_r}\sum_{k\in I_r}\mathscr{B}\left(x_k-x_p,\lambda\right)<\varepsilon,\ \frac{1}{h_r}\sum_{k\in I_r}\mathscr{Y}\left(x_k-x_p,\lambda\right)<\varepsilon\end{array}\right\}\in\mathscr{F}(\mathscr{I}).$$

Definition 2.11. *Take an NNS V. A sequence $x=(x_k)$ is named to be $\mathscr{I}_\theta^*$-Cauchy with regards to the NN $\mathscr{N}$ if there is a set $M=\{p_1<p_2<\ldots<p_k<..\}$ of $\mathbb{N}$ such that the set $M'=\{r\in\mathbb{N}:p_k\in I_r\}\in\mathscr{F}(\mathscr{I})$ and the subsequence $\left(x_{p_k}\right)$ is a lacunary Cauchy sequence with regards to the NN $\mathscr{N}$.*

The following theorems are similar of previous theorems, so the proof follows easily.

Theorem 2.12. *If a sequence $x=(x_k)$ in NNS is lacunary Cauchy with regards to NN $\mathscr{N}$, then it is $\mathscr{I}_\theta$-Cauchy with regards to the same.*

Theorem 2.13. *If a sequence $x=(x_k)$ in NNS is lacunary Cauchy with regards to NN $\mathscr{N}$, then there is a subsequence of x which is ordinary Cauchy with regards to the same.*

Theorem 2.14. *If a sequence $x=(x_k)$ in NNS is $\mathscr{I}_\theta^*$-Cauchy with regards to NN $\mathscr{N}$, then it is $\mathscr{I}_\theta$-Cauchy as well.*

Theorem 2.15. *If a sequence $x=(x_k)$ in NNS is $\mathscr{I}_\theta$-convergent with regards to NN $\mathscr{N}$, then it is $\mathscr{I}_\theta$-Cauchy with regards to NN $\mathscr{N}$.*

Proof. Let $\mathscr{I}_\theta^{(\mathscr{G},\mathscr{B},\mathscr{Y})}-\lim x=\xi$. Select $\varepsilon>0$. Then, for a given $\rho\in(0,1)$, $(1-\rho)*(1-\rho)>1-\varepsilon$ and $\rho\Diamond\rho<\varepsilon$. Then, for $\lambda>0$, we get,

$$K(\rho,\lambda)=\left\{\begin{array}{l} r\in\mathbb{N}:\frac{1}{h_r}\sum_{k\in I_r}\mathscr{G}(x_k-\xi,\lambda)\leq 1-\rho\\ \text{or } \frac{1}{h_r}\sum_{k\in I_r}\mathscr{B}(x_k-\xi,\lambda)\geq\rho,\ \frac{1}{h_r}\sum_{k\in I_r}\mathscr{Y}(x_k-\xi,\lambda)\geq\rho\end{array}\right\}\in\mathscr{I} \tag{2.1}$$

which gives that

$$\emptyset\neq K^c(\rho,\lambda)=\left\{\begin{array}{l} r\in\mathbb{N}:\frac{1}{h_r}\sum_{k\in I_r}\mathscr{G}(x_k-\xi,\lambda)>1-\rho\\ \text{and } \frac{1}{h_r}\sum_{k\in I_r}\mathscr{B}(x_k-\xi,\lambda)<\rho,\ \frac{1}{h_r}\sum_{k\in I_r}\mathscr{Y}(x_k-\xi,\lambda)<\rho\end{array}\right\}\in\mathscr{F}(\mathscr{I}).$$

Let $m\in K^c(\rho,\lambda)$. But then, for every $\lambda>0$ we have, $\mathscr{G}(x_m-\xi,\lambda)>1-\rho$ and $\mathscr{B}(x_m-\xi,\lambda)<\rho$, $\mathscr{Y}(x_m-\xi,\lambda)<\rho$. If we take

$$B(\rho,\lambda)=\left\{\begin{array}{l} r\in\mathbb{N}:\frac{1}{h_r}\sum_{k\in I_r}\mathscr{G}(x_k-x_m,\lambda)\leq 1-\varepsilon\\ \text{or} \frac{1}{h_r}\sum_{k\in I_r}\mathscr{B}(x_k-x_m,\lambda)\geq\varepsilon,\ \frac{1}{h_r}\sum_{k\in I_r}\mathscr{Y}(x_k-x_m,\lambda)\geq\varepsilon\end{array}\right\},$$

then to demonstrate the result it is sufficient to prove $B(\rho,\lambda)$ is included in $K(\rho,\lambda)$. Let $k\in B(\rho,\lambda)$, then we get $\mathscr{G}\left(x_k-x_m,\frac{\lambda}{2}\right)\leq 1-\varepsilon$ or $\mathscr{B}\left(x_k-x_m,\frac{\lambda}{2}\right)\geq\varepsilon$, $\mathscr{Y}\left(x_k-x_m,\frac{\lambda}{2}\right)\geq\varepsilon$, for $\lambda>0$. We have three possible cases.

Case (i) We first think that $\mathscr{G}(x_k - x_m, \lambda) \leq 1-\varepsilon$. Then, we have $\mathscr{G}\left(x_k - \xi, \frac{\lambda}{2}\right) \leq 1-\rho$ and therefore, $k \in K(\rho,\lambda)$. As otherwise i.e., if $\mathscr{G}\left(x_k - \xi, \frac{\lambda}{2}\right) > 1-\rho$, then we get

$$1-\varepsilon \geq \mathscr{G}(x_k - x_m, \lambda) \geq \mathscr{G}\left(x_k - \xi, \frac{\lambda}{2}\right) * \mathscr{G}\left(x_m - \xi, \frac{\lambda}{2}\right)$$

$$> (1-\rho) * (1-\rho) > 1-\varepsilon$$

which is not possible. So, $B(\rho,\lambda) \subset K(\rho,\lambda)$.
Case (ii) If $\mathscr{B}(x_k - x_m, \lambda) \geq \varepsilon$, then we get $\mathscr{B}\left(x_k - \xi, \frac{\lambda}{2}\right) > \rho$ and therefore $k \in K(\rho,\lambda)$. As otherwise i.e., if $\mathscr{B}\left(x_k - \xi, \frac{\lambda}{2}\right) < \rho$, then we obtain

$$\varepsilon \leq \mathscr{B}\left(x_k - x_m, \frac{\lambda}{2}\right) \geq \mathscr{B}\left(x_k - \xi, \frac{\lambda}{2}\right) \lozenge \mathscr{B}\left(x_m - \xi, \frac{\lambda}{2}\right)$$

$$< \rho \lozenge \rho < \varepsilon;$$

which is not possible. Hence, $B(\rho,\lambda) \subset K(\rho,\lambda)$. The last case, again we get $B(\rho,\lambda) \subset K(\rho,\lambda)$. Thus, in all cases we obtain $B(\rho,\lambda) \subset K(\rho,\lambda)$. By 2.1, $B(\rho,\lambda) \in \mathscr{I}$. This shows that (x_k) is $\mathscr{I}_\theta$-Cauchy sequence with regards to NN $\mathscr{N}$. □

Definition 2.16. *Let V be an NNS and take $x = (x_k)$ in NNS.*

(a) An element $\xi \in F$ is named to be lacunary $\mathscr{I}$-limit point of $x = (x_k)$ if there is set $M = \{p_1 < p_2 < ... < p_k < ..\} \subset \mathbb{N}$ such that the set

$$M' = \{r \in \mathbb{N} : p_k \in I_r\} \notin \mathscr{I}$$

and $(\mathscr{G},\mathscr{B},\mathscr{Y})^\theta - \lim x_{p_k} = \xi$.

(b) An element $\xi \in F$ is called to be lacunary $\mathscr{I}$-cluster point of $x = (x_k)$ if, for every $\lambda > 0$ and $\varepsilon \in (0,1)$, we get

$$\left\{ \begin{array}{l} r \in \mathbb{N} : \frac{1}{h_r} \sum\limits_{k \in I_r} \mathscr{G}(x_k - \xi, \lambda) > 1-\varepsilon \\ \text{and } \frac{1}{h_r} \sum\limits_{k \in I_r} \mathscr{B}(x_k - \xi, \lambda) < \varepsilon, \ \frac{1}{h_r} \sum\limits_{k \in I_r} \mathscr{Y}(x_k - \xi, \lambda) < \varepsilon \end{array} \right\} \notin \mathscr{I}.$$

Let $\Lambda^{\mathscr{I}_\theta}_{(\mathscr{G},\mathscr{B},\mathscr{Y})}(x)$ demonstrate the set of all lacunary $\mathscr{I}$-limit points and $\Gamma^{\mathscr{I}_\theta}_{(\mathscr{G},\mathscr{B},\mathscr{Y})}(x)$ indicate the set of all lacunary $\mathscr{I}$-cluster points in NNS, respectively.

Theorem 2.17. *For each sequence $x = (x_k)$ in NNS, we have $\Lambda^{\mathscr{I}_\theta}_{(\mathscr{G},\mathscr{B},\mathscr{Y})}(x) \subseteq \Gamma^{\mathscr{I}_\theta}_{(\mathscr{G},\mathscr{B},\mathscr{Y})}(x)$.*

Proof. Let $\xi \in \Lambda^{\mathscr{I}_\theta}_{(\mathscr{G},\mathscr{B},\mathscr{Y})}(x)$. So, there is a set $M \subset \mathbb{N}$ such that the set $M' \notin \mathscr{I}$, where M and M' are as in Definition 2.16, satisfies $(\mathscr{G},\mathscr{B},\mathscr{Y})^\theta - \lim x_{p_k} = \xi$. Hence, for every $\lambda > 0$ and $\varepsilon \in (0,1)$, there is $r_0 \in \mathbb{N}$ such that

$$\frac{1}{h_r}\sum_{k \in I_r} \mathscr{G}\left(x_{p_k} - \xi, \lambda\right) > 1-\varepsilon \text{ and } \frac{1}{h_r}\sum_{k \in I_r} \mathscr{B}\left(x_{p_k} - \xi, \lambda\right) < \varepsilon, \ \frac{1}{h_r}\sum_{k \in I_r} \mathscr{Y}\left(x_{p_k} - \xi, \lambda\right) < \varepsilon$$

for all $r \geq r_0$. Therefore,

$$B = \left\{ \begin{array}{l} r \in \mathbb{N} : \frac{1}{h_r} \sum\limits_{k \in I_r} \mathscr{G}(x_k - \xi, \lambda) > 1-\varepsilon \\ \text{and } \frac{1}{h_r} \sum\limits_{k \in I_r} \mathscr{B}(x_k - \xi, \lambda) < \varepsilon, \ \frac{1}{h_r} \sum\limits_{k \in I_r} \mathscr{Y}(x_k - \xi, \lambda) < \varepsilon \end{array} \right\}$$

$$\supseteq M' \setminus \left\{p_1, p_2, ..., p_{k_0}\right\}.$$

Now, with $\mathscr{I}$ being admissible, we must have $M' \setminus \left\{p_1, p_2, ..., p_{k_0}\right\} \notin \mathscr{I}$ and as such $B \notin \mathscr{I}$. Hence, $\xi \in \Gamma^{\mathscr{I}_\theta}_{(\mathscr{G},\mathscr{B},\mathscr{Y})}(x)$. □

Theorem 2.18. *For each sequence $x = (x_k)$ in NNS, the set $\Gamma^{\mathscr{I}_\theta}_{(\mathscr{G},\mathscr{B},\mathscr{Y})}(x)$ is closed in NNS with regards to the usual topology induced by the NN $\mathscr{N}$.*

Proof. Let $y \in \overline{\Gamma^{\mathscr{I}_\theta}_{(\mathscr{G},\mathscr{B},\mathscr{Y})}(x)}$. Take $\lambda > 0$ and $\varepsilon \in (0,1)$. Then, there is $\xi_0 \in \Gamma^{\mathscr{I}_\theta}_{(\mathscr{G},\mathscr{B},\mathscr{Y})}(x) \cap B(y,\varepsilon,\lambda)$. Select $\delta > 0$ such that $B(\xi_0,\delta,\lambda) \subseteq B(y,\varepsilon,\lambda)$. We obtain

$$G = \left\{ \begin{array}{l} r \in \mathbb{N} : \frac{1}{h_r} \sum\limits_{k \in I_r} \mathscr{G}(x_k - y, \lambda) > 1-\varepsilon \\ \text{and } \frac{1}{h_r} \sum\limits_{k \in I_r} \mathscr{B}(x_k - y, \lambda) < \varepsilon, \ \frac{1}{h_r} \sum\limits_{k \in I_r} \mathscr{Y}(x_k - y, \lambda) < \varepsilon \end{array} \right\}$$

$$\supseteq \left\{ \begin{array}{l} r \in \mathbb{N} : \frac{1}{h_r} \sum\limits_{k \in I_r} \mathscr{G}(x_k - \xi_0, \lambda) > 1-\delta \\ \text{and } \frac{1}{h_r} \sum\limits_{k \in I_r} \mathscr{B}(x_k - \xi_0, \lambda) < \delta, \ \frac{1}{h_r} \sum\limits_{k \in I_r} \mathscr{Y}(x_k - \xi_0, \lambda) < \delta \end{array} \right\} = H.$$

Thus, $H \notin \mathscr{I}$, and so $G \notin \mathscr{I}$. Hence, $y \in \Gamma^{\mathscr{I}_\theta}_{(\mathscr{G},\mathscr{B},\mathscr{Y})}(x)$. □

Theorem 2.19. *The following situations are equivalent.*

(a) $\xi \in \Lambda^{\mathscr{I}_\theta}_{(\mathscr{G},\mathscr{B},\mathscr{Y})}(x)$.

(b) There are two sequences $y=(y_k)$ and $z=(z_k)$ in NNS such that $x=y+z$ and $(\mathscr{G},\mathscr{B},\mathscr{Y})^\theta - \lim y = \xi$ and

$$\{r \in \mathbb{N} : k \in I_r,\, z_k \neq \theta\} \in \mathscr{I},$$

where θ indicates zero element of NNS.

Proof. Presume that (a) holds. Then there are M and M' are as above such that $M' \notin \mathscr{I}$ and $(\mathscr{G},\mathscr{B},\mathscr{Y})^\theta - \lim x_{p_k} = \xi$. Take the sequences y and z as follows:

$$y_k = \begin{cases} x_k, & \text{if } k \in I_r \text{ such that } r \in M' \\ \xi, & \text{otherwise} \end{cases}$$

and

$$z_k = \begin{cases} \theta, & \text{if } k \in I_r \text{ such that } r \in M' \\ x_k - \xi, & \text{otherwise.} \end{cases}$$

It sufficies to think the case $k \in I_r$ such that $r \in \mathbb{N} \setminus M'$. For each $\lambda > 0$ and $\varepsilon \in (0,1)$, we get $\mathscr{G}(y_k - \xi, \lambda) = 1 > 1-\varepsilon$ and $\mathscr{B}(y_k - \xi, \lambda) = 0 < \varepsilon$, $\mathscr{Y}(y_k - \xi, \lambda) = 0 < \varepsilon$. Thus, in this statement,

$$\frac{1}{h_r}\sum_{k\in I_r}\mathscr{G}(y_k - \xi,\lambda) = 1 > 1-\varepsilon \text{ and } \frac{1}{h_r}\sum_{k\in I_r}\mathscr{B}(y_k - \xi,\lambda) = 0 < \varepsilon,\ \frac{1}{h_r}\sum_{k\in I_r}\mathscr{Y}(y_k - \xi,\lambda) = 0 < \varepsilon.$$

Hence, $(\mathscr{G},\mathscr{B},\mathscr{Y})^\theta - \lim y = \xi$. Now,

$$\{r \in \mathbb{N} : k \in I_r,\, z_k \neq \theta\} \subset \mathbb{N} \setminus M'.$$

But $\mathbb{N}\setminus M' \in \mathscr{I}$, and so

$$\{r \in \mathbb{N} : k \in I_r,\, z_k \neq \theta\} \in \mathscr{I}.$$

Now, assume that (b) holds. Let $M' = \{r \in \mathbb{N} : k \in I_r,\, z_k = \theta\}$. Then, obviously $M' \in \mathscr{F}(\mathscr{I})$ and so it is an infinite set. Construct the set $M = \{p_1 < p_2 < \ldots < p_k < \ldots\} \subset \mathbb{N}$ such that $p_k \in I_r$ and $z_{p_k} = \theta$. Since $x_{p_k} = y_{p_k}$ and $(\mathscr{G},\mathscr{B},\mathscr{Y})^\theta - \lim y = \xi$ we get $(\mathscr{G},\mathscr{B},\mathscr{Y})^\theta - \lim x_{p_k} = \xi$. □

Definition 2.20. *A mapping $T : V \to V$ is called to be continuous at $y_0 \in F$ with regards to the NN $\mathscr{N}$ if for every $\varepsilon > 0$ and $\alpha \in (0,1)$, there are $\delta > 0$ and $\beta \in (0,1)$ such that, for all $y \in F$, $\mathscr{G}(y - y_0, \delta) > 1-\beta$ and $\mathscr{B}(y-y_0,\delta) < \beta$, $\mathscr{Y}(y-y_0,\delta) < \beta$ give that $\mathscr{G}(T(y)-T(y_0),\varepsilon) > 1-\alpha$ and $\mathscr{B}(T(y)-T(y_0),\varepsilon) < \alpha$, $\mathscr{Y}(T(y)-T(y_0),\varepsilon) < \alpha$. If T is continuous on all point of V, then T is called to be continuous on V.*

Definition 2.21. *A mapping $T : V \to V$ is called to be sequentially continuous at $y_0 \in F$ with regards to the NN $\mathscr{N}$ if for any sequence $\{y_k\}$, with $(\mathscr{G},\mathscr{B},\mathscr{Y}) - \lim y_k = y_0$ implies that $(\mathscr{G},\mathscr{B},\mathscr{Y}) - \lim T(y_k) = T(y_0)$. If T is sequentially continuous at all point of V, then T is said to be sequentially continuous on V.*

Theorem 2.22. *A mapping $T : V \to V$ is continuous with regards to the NN $\mathscr{N}$ iff it is sequentially continuous with regards to the same.*

Definition 2.23. *A lineer operator $T : V \to V$ is called to preserve $\mathscr{I}_\theta^{(\mathscr{G},\mathscr{B},\mathscr{Y})}$-convergence in NNS if $\mathscr{I}_\theta^{(\mathscr{G},\mathscr{B},\mathscr{Y})} - \lim x_k = \xi$ gives that $\mathscr{I}_\theta^{(\mathscr{G},\mathscr{B},\mathscr{Y})} - \lim T(x_k) = T(\xi)$ for each sequence $x=(x_k)$ in NNS which is $\mathscr{I}_\theta^{(\mathscr{G},\mathscr{B},\mathscr{Y})}$-convergent to $\xi \in F$.*

Theorem 2.24. *A linear operator $T : V \to V$ preserves $\mathscr{I}_\theta^{(\mathscr{G},\mathscr{B},\mathscr{Y})}$-convergence in V iff T is continuous on V.*

Proof. Let $\mathscr{I}_\theta^{(\mathscr{G},\mathscr{B},\mathscr{Y})} - \lim x_k = \xi$. If T is continuous, then for every $\varepsilon > 0$ and $\alpha \in (0,1)$, there are $\delta > 0$ and $\beta \in (0,1)$ such that, for $y \in F$, if $y \in B(\xi,\beta,\delta)$, then $T(y) \in B(T(\xi),\alpha,\varepsilon)$. But then, we obtain

$$\begin{aligned} C(\delta,\beta) &= \left\{ \begin{array}{l} r \in \mathbb{N} : \frac{1}{h_r}\sum\limits_{k\in I_r}\mathscr{G}(x_k - \xi,\delta) > 1-\beta \\ \text{and } \frac{1}{h_r}\sum\limits_{k\in I_r}\mathscr{B}(x_k-\xi,\delta) < \beta,\ \frac{1}{h_r}\sum\limits_{k\in I_r}\mathscr{Y}(x_k-\xi,\delta) < \beta \end{array} \right\} \\ &\subseteq \left\{ \begin{array}{l} r \in \mathbb{N} : \frac{1}{h_r}\sum\limits_{k\in I_r}\mathscr{G}(T(x_k)-T(\xi),\varepsilon) > 1-\alpha \\ \text{and } \frac{1}{h_r}\sum\limits_{k\in I_r}\mathscr{B}(T(x_k)-T(\xi),\varepsilon) < \alpha,\ \frac{1}{h_r}\sum\limits_{k\in I_r}\mathscr{Y}(T(x_k)-T(\xi),\varepsilon) < \alpha \end{array} \right\} \\ &= D(\varepsilon,\alpha). \end{aligned}$$

Since $C(\delta,\beta) \in \mathscr{F}(\mathscr{I})$, we get $D(\varepsilon,\alpha) \in \mathscr{F}(\mathscr{I})$. Hence $\mathscr{I}_\theta^{(\mathscr{G},\mathscr{B},\mathscr{Y})} - \lim T(x_k) = T(\xi)$.

To demonstrate the converse, assume T be not continuous at same $\xi \in F$. Then, there is some $\varepsilon > 0$ and $\alpha \in (0,1)$ such that $\delta > 0$ and $\beta \in (0,1)$, if $y \in B(\xi,\beta,\delta)$, then $T(y) \notin B(T(\xi),\alpha,\varepsilon)$, where $y \in F$. Now we get a sequence $x = (x_k)$ such that $(\mathscr{G},\mathscr{B},\mathscr{Y})^{\theta} - \lim x_k = \xi$ but $(\mathscr{G},\mathscr{B},\mathscr{Y})^{\theta} - \lim T(x_k) \neq T(\xi)$. Then, we obtain

$$\begin{aligned} C'(\delta,\beta) &= \left\{ \begin{array}{l} r \in \mathbb{N} : \frac{1}{h_r} \sum\limits_{k \in I_r} \mathscr{G}(x_k - \xi, \delta) > 1 - \beta \\ \text{and } \frac{1}{h_r} \sum\limits_{k \in I_r} \mathscr{B}(x_k - \xi, \delta) < \beta,\ \frac{1}{h_r} \sum\limits_{k \in I_r} \mathscr{Y}(x_k - \xi, \delta) < \beta \end{array} \right\} \\ &\subseteq \left\{ \begin{array}{l} r \in \mathbb{N} : \frac{1}{h_r} \sum\limits_{k \in I_r} \mathscr{G}(T(x_k) - T(\xi), \varepsilon) \leq 1 - \alpha \\ \text{or } \frac{1}{h_r} \sum\limits_{k \in I_r} \mathscr{B}(T(x_k) - T(\xi), \varepsilon) \geq \alpha,\ \frac{1}{h_r} \sum\limits_{k \in I_r} \mathscr{Y}(T(x_k) - T(\xi), \varepsilon) \geq \alpha \end{array} \right\} \\ &= D'(\varepsilon,\alpha). \end{aligned}$$

Now, $C'(\delta,\beta) \in \mathscr{F}(\mathscr{I})$, and as a result $D'(\varepsilon,\alpha) \in \mathscr{F}(\mathscr{I})$. Therefore $\mathscr{I}_{\theta}^{(\mathscr{G},\mathscr{B},\mathscr{Y})} - \lim T(x_k) \neq T(\xi)$. □

3. Conclusion

We have examined lacunary ideal convergence of sequences in NNS. The fundamental characteristic features of this type of convergence in NNS has been studied. The notions of lacunary $\mathscr{I}$-convergence, lacunary $\mathscr{I}$-Cauchy and lacunary $\mathscr{I}^*$-Cauchy for sequences in NNS are investigated and noteworthy results are established. The results of the paper are expected to be a source for researchers in the areas of convergence methods for sequences and applications in NNS. In future studies on this topic, it is also possible to work with the idea of "Lacunary ideal convergence in Probabilistic metric space" using neutrosophic probability.

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