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Quasi-Hemi-Slant Conformal Submersions from Almost Hermitian Manifolds

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Abstract. In this study, we introduce some geometric properties of quasi-hemi-slant conformal submersions from an almost Hermitian manifold to a Riemannian manifold. We give an explicit example for this type submersions and obtain integrability conditions for certain distributions. Lastly, we search totally geodesicity on base manifold of the map.

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1. Introduction

The theory of Riemannian submersions between Riemannian manifolds was initially studied by O'Neill [12] and Gray [7]. Then, this theory was expansed to almost Hermitian submersions between almost Hermitian manifolds [23]. After these studies, this theory was widely studied in [6, 19]. Şahin defined various types of Riemannian submersions from an almost Hermitian manifold onto a Riemannian manifold such as anti-invariant submersions [15], semi-invariant submersions [17] and slant submersions [16], see also [9,13,20]. Also, Şahin gave some main results about Riemannian submersions and an application on robotic theory [18]. Therefore, a new vision on submersions by applying conformality conditions was presented by Akyol and Şahin [2]- [4], see also [8, 14]. Riemannian submersions have many applications as texture mapping, remeshing and simulation [10], computer graphics and medical imaging fields [21], brain mapping research [22].

In this study, in Section 2, we give some basic notions to be used along this study. In Section 3, we define quasi-hemi-slant conformal submersion from an almost Hermitian manifold onto a Riemannian manifold which is the expansion of conformal semi-slant submersions [1], conformal semi-invariant submersions [3], conformal hemi-slant submersions [8]. We introduce some geometric properties for quasi-hemi-slant conformal submersions. In Section 4, we give some conditions for certain distributions to define totally geodesic foliation on base manifold.

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2. Preliminaries

In this section, we give several definitions and results to be used throughout the study for quasi-hemi-slant conformal Riemannian submersions.

An even-dimensional Riemannian manifold (M, g_M, J) is called an almost Hermitian manifold if there exists a tensor field J of type $(1,1)$ on M such that $J^2 = -I$ where I denotes the identity transformation of TM and

$$g_M(X, Y) = g_M(JX, JY), \forall X, Y \in \Gamma(TM). \tag{2.1}$$

Let (M, g_M, J) be an almost Hermitian manifold and its Levi-Civita connection is ∇ with respect to g_M. If J is parallel with respect to ∇, i.e.

$$(\nabla_X J)Y = 0, \tag{2.2}$$

we say M is a Kähler manifold [24].

Let $\Phi : (M, g_{_M}) \longrightarrow (N, g_{_N})$ be a smooth map between Riemannian manifolds. The second fundamental form of Φ is defined by

$$(\nabla\Phi_*)(X, Y) = \overset{N}{\nabla}{}^{\Phi}_X \Phi_*(Y) - \Phi_*(\overset{M}{\nabla}_X Y) \tag{2.3}$$

for $X, Y \in \Gamma(TM)$. The second fundamental form $\nabla\Phi_*$ is symmetric [11]. Here, Φ_* is differential map of Φ from tangent space of M at a point $x \in M$ to tangent space of N at $\Phi(x)$ such that $\Phi_* : T_x M \longrightarrow T_{\Phi(x)} N$.

A smooth map $\Phi : (M^m, g_{_M}) \longrightarrow (N^n, g_{_N})$ between Riemannian manifolds is called a Riemannian submersion if Φ has maximal rank and the differential Φ_* preserves the lengths of horizontal vectors. On the other hand, let $\Phi : (M, g_{_M}) \longrightarrow (N, g_{_N})$ be a smooth map between Riemannian manifolds and $p \in M$. Then, Φ is called horizontally weakly conformal at p if either (i) $\Phi_{*p} = 0$ or (ii) Φ_{*p} is surjective and there exists a number $\wedge(p) \neq 0$ such that

$$g_N(\Phi_{*p}(X), \Phi_{*p}(Y)) = \wedge(p) g_M(X, Y)$$

for $X, Y \in \Gamma((ker\,\Phi_*)^\perp)$. We call the point p is of type (i) as a critical point and we shall call the point p a regular point if it satisfied the type (ii). At a critical point, $rank(\Phi_{*p}) = 0$, at a regular point, Φ_{*p} has rank n and Φ is a submersion. Additionally, the positive number $\wedge(p)$ is called the square dilation of Φ at p. The map Φ is called horizontally weakly conformal or semi conformal on M if it is horizontally weakly conformal at every point of M and it has no critical point, then we call it as a horizontally conformal submersion [5].

If a vector field X on M is related to a vector field X' on N, we say X is a projectable vector field. If X is both a horizontal and a projectable vector field, we say X is a basic vector field on M. From now on, when we mention a horizontal vector field, we always consider a basic vector field [5].

Then, O'Neill's tensor fields T and A for Riemannian submersions are defined as

$$A_X Y = h\overset{M}{\nabla}_{hX} vY + v\overset{M}{\nabla}_{hX} hY, \tag{2.4}$$
$$T_X Y = h\overset{M}{\nabla}_{vX} vY + v\overset{M}{\nabla}_{vX} hY, \tag{2.5}$$

for $X, Y \in \Gamma(TM)$ with the Levi-Civita connection $\overset{M}{\nabla}$ of g_M. For any $X \in \Gamma(TM)$, T_X and A_X are skew-symmetric operators on $(\Gamma(TM), g)$ reversing the horizontal and the vertical distributions. Also, T is vertical, $T_X = T_{vX}$, and A is horizontal, $A_X = A_{hX}$. Note that the tensor field T is symmetric on the vertical distribution [12]. In addition, from (2.4) and (2.5) we have

$$\overset{M}{\nabla}_U V = T_U V + \hat{\nabla}_U V, \tag{2.6}$$
$$\overset{M}{\nabla}_U X = h\overset{M}{\nabla}_U X + T_U X, \tag{2.7}$$
$$\overset{M}{\nabla}_X V = A_X V + v\overset{M}{\nabla}_X V, \tag{2.8}$$
$$\overset{M}{\nabla}_X Y = h\overset{M}{\nabla}_X Y + A_X Y \tag{2.9}$$

for $X, Y \in \Gamma((ker\Phi_*)^\perp)$ and $U, V \in \Gamma(ker\Phi_*)$, where $\hat{\nabla}_U V = v\overset{M}{\nabla}_U V$ [6].

From [5], we have the following lemma.

Lemma 2.1. *Suppose that $\Phi : (M^m, g_M) \longrightarrow (N^n, g_N)$ is a horizontally conformal submersion. Then, we have*

$$(\nabla\Phi_*)(X, Y) = X(\ln\lambda)\Phi_*(Y) + Y(\ln\lambda)\Phi_*(X) - g_M(X, Y)\Phi_*(grad(\ln\lambda)), \quad (2.10)$$

$$(\nabla\Phi_*)(U, V) = -\Phi_*(T_U V) \quad (2.11)$$

$$(\nabla\Phi_*)(X, V) = -\Phi_*(A_X V). \quad (2.12)$$

for any horizontal vector fields X, Y and vertical vector fields U, V [5]. Here, λ is the dilation of Φ at a point $x \in M$ and it is a continuous function as $\lambda : M \longrightarrow [0, \infty)$.

3. Quasi-Hemi-Slant Conformal Riemannian Submersions

Firstly, we give definition of quasi-hemi-slant Riemannian submersions from almost Hermitian manifolds to Riemannian manifolds.

Definition 3.1. Let $\Phi : (M, g_M, J) \longrightarrow (N, g_N)$ be a conformal submersion such that its vertical distribution $ker\Phi_*$ admits three orthogonal distributions D, D_θ and $D_\perp$ which are invariant ($J(D) = D$), slant (the angle θ between D_θ and $J(D_\theta)$ is a constant) and anti-invariant ($J(D_\perp) \subseteq (ker\Phi_*)^\perp$), respectively, i.e.

$$ker\Phi_* = D \oplus D_\theta \oplus D_\perp. \quad (3.1)$$

Then, we say Φ is a quasi-hemi-slant conformal submersion and the angle θ is called the quasi-hemi-slant angle of the map.

Here, we have some particular cases;

i) If the distribution $D = \{0\}$ then the map Φ is a conformal hemi-slant submersion [8].
ii) If the distribution $D_\theta = \{0\}$ then the map Φ is a conformal semi-invariant submersion [3].
iii) If the distribution $D_\perp = \{0\}$ then the map Φ is a conformal semi-slant submersion [1].

Hence, quasi-hemi-slant conformal submersions are generalization of conformal hemi-slant submersions, conformal semi-invariant submersions and conformal semi-slant submersions.

Let $\Phi : (M, g_M, J) \longrightarrow (N, g_N)$ be a quasi-hemi-slant conformal submersion. Then we have

$$TM = ker\Phi_* \oplus (ker\Phi_*)^\perp. \quad (3.2)$$

A vertical vector field U can written as

$$U = \tilde{P}U + \tilde{Q}U + \tilde{R}U \quad (3.3)$$

where $\tilde{P}$, $\tilde{Q}$ and $\tilde{R}$ are projections onto D, D_θ and $D_\perp$, respectively. We get

$$JU = \phi U + \psi U \quad (3.4)$$

where $\phi U \in \Gamma(ker\Phi_*)$ and $\psi U \in \Gamma((ker\Phi_*)^\perp)$. From (3.3), (3.4) and Definition 3.1, we obtain $\psi\tilde{P}U = 0$, $\phi\tilde{R}U = 0$ and

$$JU = \phi\tilde{P}U + \phi\tilde{Q}U + \psi\tilde{Q}U + \psi\tilde{R}X. \quad (3.5)$$

Hence, we can write

$$J(ker\Phi_*) = D \oplus \phi D_\theta \oplus \psi D_\theta \oplus J(D_\perp). \quad (3.6)$$

Using (3.6), we have

$$(ker\Phi_*)^\perp = \psi D_\theta \oplus J(D_\perp) \oplus \mu \quad (3.7)$$

where μ is the orthogonal complement distributions of $\psi D_\theta \oplus J(D_\perp)$ in $(ker\Phi_*)^\perp$ and μ is the invariant with respect to J. Lastly, for a horizontal vector field X, we have

$$JX = BX + CX \quad (3.8)$$

where $BX \in \Gamma(\psi D_\theta \oplus J(D_\perp))$ and $CX \in \Gamma(\mu)$.

Here that one can easily see from (3.1) - (3.7);

$$\phi D_\theta = D_\theta, \quad \phi D_\perp = \{0\}, \quad B\psi D_\theta = D_\theta, \quad B\psi D_\perp = D_\perp, \quad \psi D = \{0\}, \quad (3.9)$$

$$\phi^2 + B\psi = -\mathbb{I}, \quad \psi\phi + C\psi = 0, \quad \psi B + C^2 = -\mathbb{I}, \quad \phi B + BC = 0 \quad (3.10)$$

where $\mathbb{I}$ is the identity operator on the total space of Φ.

Now, we give an example to understand quasi-hemi-slant conformal submersions better.

Example 3.2. A map $\Phi : \mathbb{R}^8 \longrightarrow \mathbb{R}^4$ is defined by

$$(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) \longrightarrow \left(\frac{x_1 + x_3}{\sqrt{2}}, x_6, x_5, x_8\right).$$

Then, we get the horizontal distribution

$$(ker\Phi_*)^\perp = \left\{Z_1 = \frac{1}{\sqrt{2}}(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_3}), Z_2 = \frac{\partial}{\partial x_6}, Z_3 = \frac{\partial}{\partial x_5}, Z_4 = \frac{\partial}{\partial x_8}\right\},$$

and the vertical distribution

$$ker\Phi_* = \left\{V_1 = \frac{\partial}{\partial x_2}, V_2 = \frac{\partial}{\partial x_4}, V_3 = \frac{\partial}{\partial x_7}, V_4 = \frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_3}\right\}.$$

Hence, using complex structure $J = (-x_2, x_1, -x_4, x_3, -x_6, x_5, -x_8, x_7)$ of $\mathbb{R}^8$ on the distributions, we obtain

$$J(Z_1) = \frac{1}{\sqrt{2}}(V_1 + V_2), \quad J(Z_2) = -Z_3, \quad J(Z_3) = Z_2, \quad J(Z_4) = -V_3,$$

$$J(V_1) = -\frac{\sqrt{2}}{2}Z_1 - \frac{1}{2}V_4, \quad J(V_2) = -\frac{\sqrt{2}}{2}Z_1 + \frac{1}{2}V_4, \quad J(V_3) = Z_4, \quad J(V_4) = V_1 - V_2.$$

Therefore, we get $D = sp\{V_4\}$, $D_\theta = sp\{V_1, V_2\}$, $D_\perp = sp\{V_3\}$, $\mu = sp\{Z_2, Z_3\}$, $J(D_\perp) = sp\{Z_4\}$ and $\psi D_\theta = sp\{Z_1\}$. One can see that Φ is a quasi-hemi-slant conformal submersion with $\wedge = \lambda^2 = 1$ and quasi-hemi-slant angle $\theta = \frac{\pi}{2}$.

In the rest of this study, we assume that $\Phi : (M, g_M, J) \longrightarrow (N, g_N)$ is a quasi-hemi-slant conformal submersion from a Kähler manifold (M, g_M, J) to a Riemannian manifold (N, g_N). We have the following lemma which has the same proof for quasi-hemi-slant submersions.

Lemma 3.3. *Let $\Phi : (M, g_M, J) \longrightarrow (N, g_N)$ be a slant submersion. Then,*

$$-\phi^2 X = cos^2\theta X, \tag{3.11}$$

$$g_M(\phi X, \phi Y) = \cos^2\theta g_M(X, Y), \tag{3.12}$$

$$g_M(\psi X, \psi Y) = \sin^2\theta g_M(X, Y) \tag{3.13}$$

for $X, Y \in \Gamma(D_\theta)$ [16].

Throughout this section, we give necessary and sufficient conditions to be integrability for distributions.

Theorem 3.4. *Let $\Phi : (M, g_M, J) \longrightarrow (N, g_N)$ be a quasi-hemi-slant conformal submersion. Then, the distribution D_θ is integrable if and only if*

$$g_N((\nabla\Phi_*)(V_2, \phi\tilde{P}\xi), \Phi_*(\psi V_1)) - g_N((\nabla\Phi_*)(V_1, \phi\tilde{P}\xi), \Phi_*(\psi V_2)) = \lambda^2\{g_M(\hat{\nabla}_{V_2}\phi\tilde{P}\xi + T_{V_2}\psi\tilde{R}\xi, \phi V_1) - g_M(h\overset{M}{\nabla}_{V_1}\psi\tilde{R}\xi, \psi V_2)$$
$$+ g_M(h\overset{M}{\nabla}_{V_2}\psi\tilde{R}\xi, \psi V_1) - g_M(\hat{\nabla}_{V_1}\phi\tilde{P}\xi + T_{V_1}\psi\tilde{R}\xi, \phi V_2)\}$$

for $V_1, V_2 \in \Gamma(D_\theta)$ and $\xi \in \Gamma(D \oplus D_\perp)$.

Proof. Since M is a Kähler manifold, we have $g_M(\overset{M}{\nabla}_{V_1}V_2, \xi) = -g_M(\overset{M}{\nabla}_{V_1}J\xi, JV_2)$ for $V_1, V_2 \in \Gamma(D_\theta)$ and $\xi \in \Gamma(D \oplus D_\perp)$. So, we get from (2.6), (2.7), (3.3) and (3.4)

$$\begin{aligned} -g_M(\overset{M}{\nabla}_{V_1}J\xi, JV_2) &= -g_M(\overset{M}{\nabla}_{V_1}\phi\tilde{P}\xi + \overset{M}{\nabla}_{V_1}\psi\tilde{R}\xi, \phi V_2 + \psi V_2) \\ &= -g_M(\hat{\nabla}_{V_1}\phi\tilde{P}\xi + T_{V_1}\psi\tilde{R}\xi, \phi V_2) - g_M(T_{V_1}\phi\tilde{P}\xi + h\overset{M}{\nabla}_{V_1}\psi\tilde{R}\xi, \psi V_2). \end{aligned} \tag{3.14}$$

Changing the roles of V_1 and V_2 in (3.14), we have second part of $g_M([V_1, V_2], \xi)$. Hence, from (2.11) we obtain

$$\begin{aligned} g_M([V_1, V_2], \xi) = {}& g_M(\hat{\nabla}_{V_2}\phi\tilde{P}\xi + T_{V_2}\psi\tilde{R}\xi, \phi V_1) - g_M(\hat{\nabla}_{V_1}\phi\tilde{P}\xi + T_{V_1}\psi\tilde{R}\xi, \phi V_2) \\ &+ g_M(h\overset{M}{\nabla}_{V_2}\psi\tilde{R}\xi, \psi V_1) - g_M(h\overset{M}{\nabla}_{V_1}\psi\tilde{R}\xi, \psi V_2) \\ &+ \frac{1}{\lambda^2}\{g_N((\nabla\Phi_*)(V_1, \phi\tilde{P}\xi), \Phi_*(\psi V_2)) - g_N((\nabla\Phi_*)(V_2, \phi\tilde{P}\xi), \Phi_*(\psi V_1))\}. \end{aligned} \tag{3.15}$$

The proof is completed from (3.15). □

In a similar way, we have the following theorem.

Theorem 3.5. *Let $\Phi : (M, g_M, J) \longrightarrow (N, g_N)$ be a quasi-hemi-slant conformal submersion. Then, the distribution D is integrable if and only if*

$$\tilde{P}(\hat{\nabla}_{U_1}\phi\tilde{Q}\xi + T_{U_1}\psi\xi) = 0$$

for $U_1, U_2 \in \Gamma(D)$ and $\xi \in \Gamma(D_\theta \oplus D_\perp)$.

Proof. Using (2.2), (2.6), (2.7) and (3.5), we have

$$\begin{aligned} g_M(\overset{M}{\nabla}_{U_1}U_2, \xi) &= -g_M(\overset{M}{\nabla}_{U_1}\phi\tilde{Q}\xi + \psi\tilde{Q}\xi + \psi\tilde{R}\xi, JU_2) \\ &= -g_M(\hat{\nabla}_{U_1}\phi\tilde{Q}\xi + T_{U_1}\psi\tilde{Q}\xi + T_{U_1}\psi\tilde{R}\xi, JU_2) \end{aligned} \tag{3.16}$$

for $U_1, U_2 \in \Gamma(D)$ and $\xi \in \Gamma(D_\theta \oplus D_\perp)$. Now, since $\psi(\tilde{Q}\xi + \tilde{R}\xi) = \psi\xi$ and from (3.16) we obtain

$$g_M([U_1, U_2], \xi) = g_M(\hat{\nabla}_{U_2}\phi\tilde{Q}\xi + T_{U_2}\psi\xi, JU_1) - g_M(\hat{\nabla}_{U_1}\phi\tilde{Q}\xi + T_{U_1}\psi\xi, JU_2). \tag{3.17}$$

Since D is an invariant distribution, we have $JU_1, JU_2 \in \Gamma(D)$. Therefore, we obtain the proof from (3.17). □

Here, integrability condition of the anti-invariant distribution $D_\perp$ is same as the condition for hemi-slant submersions in [20]. In addition, we know that the vertical distribution of a submersion is always integrable. Hence, we lastly give integrability condition for the horizontal distribution $(ker\Phi_*)^\perp$.

Theorem 3.6. *Let $\Phi : (M, g_M, J) \longrightarrow (N, g_N)$ be a quasi-hemi-slant conformal submersion. Then, the distribution $(ker\Phi_*)^\perp$ is integrable if and only if*

$$\begin{aligned} &g_N((\nabla\Phi_*)(Z_1, BZ_2) - (\nabla\Phi_*)(Z_2, BZ_1) + \overset{N}{\nabla}{}^{\Phi}_{Z_2}\Phi_*(CZ_1) - \overset{N}{\nabla}{}^{\Phi}_{Z_1}\Phi_*(CZ_2), \Phi_*(\psi\xi)) \\ &= \lambda^2 g_M(v\overset{M}{\nabla}_{Z_1}BZ_2 - v\overset{M}{\nabla}_{Z_2}BZ_1, \phi\xi) + \lambda^2 g_M(A_{Z_1}CZ_2 - A_{Z_2}CZ_1, \phi\xi) - CZ_2(\ln\lambda)g_M(Z_1, \psi\xi) \\ &\quad + \psi\xi(\ln\lambda)g_M(Z_1, CZ_2) + CZ_1(\ln\lambda)g_M(Z_2, \psi\xi) - \psi\xi(\ln\lambda)g_M(Z_2, CZ_1) \end{aligned}$$

for $Z_1, Z_2 \in \Gamma((ker\Phi_*)^\perp)$ and $\xi \in \Gamma(ker\Phi_*)$.

Proof. Firstly, from (2.8), (2.9), (3.4) and (3.8), we have

$$g_M(\overset{M}{\nabla}_{Z_1}Z_2, \xi) = g_M(A_{Z_1}BZ_2 + h\overset{M}{\nabla}_{Z_1}CZ_2, \psi\xi) + g_M(v\overset{M}{\nabla}_{Z_1}BZ_2 + A_{Z_1}CZ_2, \phi\xi) \tag{3.18}$$

for $Z_1, Z_2 \in \Gamma((ker\Phi_*)^\perp)$ and $\xi \in \Gamma(ker\Phi_*)$. Now, changing the roles of Z_1 and Z_2 in (3.18), we get

$$\begin{aligned} g_M([Z_1, Z_2], \xi) &= g_M(A_{Z_1}BZ_2 + h\overset{M}{\nabla}_{Z_1}CZ_2 - A_{Z_2}BZ_1 - h\overset{M}{\nabla}_{Z_2}CZ_1, \psi\xi) \\ &\quad + g_M(v\overset{M}{\nabla}_{Z_1}BZ_2 + A_{Z_1}CZ_2 - v\overset{M}{\nabla}_{Z_2}BZ_1 - A_{Z_2}CZ_1, \phi\xi) \end{aligned} \tag{3.19}$$

Hence, using equations (2.3), (2.10), (2.12) in (3.19) and since μ is orthogonal to $\psi D_\theta \oplus J(D_\perp)$, we obtain

$$\begin{aligned} 0 = {} & g_M(v\overset{M}{\nabla}_{Z_1}BZ_2 + A_{Z_1}CZ_2 - v\overset{M}{\nabla}_{Z_2}BZ_1 - A_{Z_2}CZ_1, \phi\xi) + \frac{1}{\lambda^2}g_N((\nabla\Phi_*)(Z_2, BZ_1) - (\nabla\Phi_*)(Z_1, BZ_2), \Phi_*(\psi\xi)) \\ & + \frac{1}{\lambda^2}g_N(\overset{N}{\nabla}{}^{\Phi}_{Z_1}\Phi_*(CZ_2) - \overset{N}{\nabla}{}^{\Phi}_{Z_2}\Phi_*(CZ_1), \Phi_*(\psi\xi)) - CZ_2(\ln\lambda)g_M(Z_1, \psi\xi) + \psi\xi(\ln\lambda)g_M(Z_1, CZ_2) \\ & + CZ_1(\ln\lambda)g_M(Z_2, \psi\xi) - \psi\xi(\ln\lambda)g_M(Z_2, CZ_1). \end{aligned} \tag{3.20}$$

One can see the proof from (3.20). □

4. Totally Geodesicness on Distributions

In this section, we present conditions for certain distributions and the map Φ to define totally geodesic foliations on M.

Theorem 4.1. *Let $\Phi:(M,g_M,J)\longrightarrow(N,g_N)$ be a quasi-hemi-slant conformal submersion. Then, the distribution D defines totally geodesic foliations on M if and only if*

i) $\lambda^2 g_M(\hat{\nabla}_{U_1}JU_2,\phi\tilde{Q}\xi)=g_N((\nabla\Phi_*)(U_1,JU_2),\Phi_*(\psi\xi))$,

ii) $\lambda^2 g_M(\hat{\nabla}_{U_1}JU_2,BZ)=g_N((\nabla\Phi_*)(U_1,JU_2),\Phi_*(CZ))$

are provided for $U_1,U_2\in\Gamma(D)$, $\xi\in\Gamma(D_\theta\oplus D_\perp)$ and $Z\in\Gamma((ker\Phi_*)^\perp)$.

Proof. Firstly, from (2.6), (2.11) and (3.4) we have

$$\begin{aligned} g_M(\overset{M}{\nabla}_{U_1}U_2,\xi) &= g_M(T_{U_1}JU_2,\psi\xi)+g_M(\hat{\nabla}_{U_1}JU_2,\phi\xi)\\ &= -\frac{1}{\lambda^2}g_N((\nabla\Phi_*)(U_1,JU_2),\Phi_*(\psi\xi))+g_M(\hat{\nabla}_{U_1}JU_2,\phi\xi) \end{aligned}\tag{4.1}$$

for $U_1,U_2\in\Gamma(D)$ and $\xi\in\Gamma(D_\theta\oplus D_\perp)$. On the other hand, from (2.6), (2.11) and (3.8) we have

$$\begin{aligned} g_M(\overset{M}{\nabla}_{U_1}U_2,Z) &= g_M(T_{U_1}JU_2,CZ)+g_M(\hat{\nabla}_{U_1}JU_2,BZ)\\ &= -\frac{1}{\lambda^2}g_N((\nabla\Phi_*)(U_1,JU_2),\Phi_*(CZ))+g_M(\hat{\nabla}_{U_1}JU_2,BZ). \end{aligned}\tag{4.2}$$

We obtain (i) and (ii) from (4.1) and (4.2), respectively. □

Theorem 4.2. *Let $\Phi:(M,g_M,J)\longrightarrow(N,g_N)$ be a quasi-hemi-slant conformal submersion. Then, the distribution D_θ defines totally geodesic foliations on M if and only if*

$$\begin{aligned} i)\quad -\lambda^2\{\cos^2\theta g_M(\hat{\nabla}_{V_1}\tilde{Q}V_2,\xi)+g_M(h\overset{M}{\nabla}_{V_1}\psi\tilde{Q}V_2,\psi\tilde{R}\xi)\} &= g_N((\nabla\Phi_*)(V_1,\xi),\Phi_*(\psi\phi\tilde{Q}V_2))\\ &\quad+g_N((\nabla\Phi_*)(V_1,\phi\tilde{P}\xi),\Phi_*(\psi\tilde{Q}V_2)), \end{aligned}$$

$$\begin{aligned} ii)\quad \lambda^2\{g_M(h\overset{M}{\nabla}_{V_1}\psi\phi\tilde{Q}V_2,Z)+g_M(h\overset{M}{\nabla}_{V_1}\psi\tilde{Q}V_2,CZ)\} &= \cos^2\theta g_N((\nabla\Phi_*)(V_1,\tilde{Q}V_2),\Phi_*(Z))\\ &\quad-g_N((\nabla\Phi_*)(V_1,BZ),\Phi_*(\psi\tilde{Q}V_2)) \end{aligned}$$

are provided for $V_1,V_2\in\Gamma(D_\theta)$, $\xi\in\Gamma(D\oplus D_\perp)$ and $Z\in\Gamma((ker\Phi_*)^\perp)$.

Proof. From equations (2.6), (2.7), (2.11), (3.11) and skew-symmetry properties of T we have

$$\begin{aligned} g_M(\overset{M}{\nabla}_{V_1}V_2,\xi) &= \cos^2\theta g_M(\overset{M}{\nabla}_{V_1}\tilde{Q}V_2,\xi)+g_M(T_{V_1}\psi\phi\tilde{Q}V_2,\xi)+g_M(T_{V_1}\psi\tilde{Q}V_2,\phi\tilde{P}\xi)+g_M(h\overset{M}{\nabla}_{V_1}\psi\tilde{Q}V_2,\psi\tilde{R}\xi)\\ &= \cos^2\theta g_M(\hat{\nabla}_{V_1}\tilde{Q}V_2,\xi)-g_M(T_{V_1}\xi,\psi\phi\tilde{Q}V_2)-g_M(T_{V_1}\phi\tilde{P}\xi,\psi\tilde{Q}V_2)+g_M(h\overset{M}{\nabla}_{V_1}\psi\tilde{Q}V_2,\psi\tilde{R}\xi)\\ &= \cos^2\theta g_M(\hat{\nabla}_{V_1}\tilde{Q}V_2,\xi)+g_M(h\overset{M}{\nabla}_{V_1}\psi\tilde{Q}V_2,\psi\tilde{R}\xi)\\ &\quad+\frac{1}{\lambda^2}g_N((\nabla\Phi_*)(V_1,\xi),\Phi_*(\psi\phi\tilde{Q}V_2))+\frac{1}{\lambda^2}g_N((\nabla\Phi_*)(V_1,\phi\tilde{P}\xi),\Phi_*(\psi\tilde{Q}V_2)) \end{aligned}\tag{4.3}$$

for $V_1,V_2\in\Gamma(D_\theta)$ and $\xi\in\Gamma(D\oplus D_\perp)$. In a similar way, from (3.8) we have

$$\begin{aligned} g_M(\overset{M}{\nabla}_{V_1}V_2,Z) &= \cos^2\theta g_M(\overset{M}{\nabla}_{V_1}\tilde{Q}V_2,Z)+g_M(h\overset{M}{\nabla}_{V_1}\psi\phi\tilde{Q}V_2,Z)-g_M(T_{V_1}BZ,\psi\tilde{Q}V_2)+g_M(h\overset{M}{\nabla}_{V_1}\psi\tilde{Q}V_2,CZ)\\ &= \cos^2\theta g_M(T_{V_1}\tilde{Q}V_2,Z)+g_M(h\overset{M}{\nabla}_{V_1}\psi\phi\tilde{Q}V_2,Z)+\frac{1}{\lambda^2}g_N((\nabla\Phi_*)(V_1,BZ),\Phi_*(\psi\tilde{Q}V_2))+g_M(h\overset{M}{\nabla}_{V_1}\psi\tilde{Q}V_2,CZ)\\ &= -\cos^2\theta\frac{1}{\lambda^2}g_N((\nabla\Phi_*)(V_1,\tilde{Q}V_2),\Phi_*(Z))+g_M(h\overset{M}{\nabla}_{V_1}\psi\phi\tilde{Q}V_2,Z)\\ &\quad+\frac{1}{\lambda^2}g_N((\nabla\Phi_*)(V_1,BZ),\Phi_*(\psi\tilde{Q}V_2))+g_M(h\overset{M}{\nabla}_{V_1}\psi\tilde{Q}V_2,CZ) \end{aligned}\tag{4.4}$$

for $V_1,V_2\in\Gamma(D_\theta)$ and $Z\in\Gamma((ker\Phi_*)^\perp)$. We obtain (i) and (ii) from (4.3) and (4.4), respectively. □

Theorem 4.3. *Let $\Phi : (M, g_{_M}, J) \longrightarrow (N, g_{_N})$ be a quasi-hemi-slant conformal submersion. Then, the distribution $D_\perp$ defines totally geodesic foliations on M if and only if*

i) $-\lambda^2 g_{_M}(h\overset{M}{\nabla}_{W_1} JW_2, \psi\tilde{Q}\xi) = g_{_N}((\nabla\Phi_*)(W_1, \phi\xi), \Phi_*(JW_2)),$

ii) $-\lambda^2 g_{_M}(h\overset{M}{\nabla}_{W_1} CZ, JW_2) = g_{_N}((\nabla\Phi_*)(W_1, BZ), \Phi_*(JW_2))$

are provided for $W_1, W_2 \in \Gamma(D_\perp)$, $\xi \in \Gamma(D \oplus D_\theta)$ and $Z \in \Gamma((ker\Phi_*)^\perp)$.

Proof. Since the distribution D is invariant from (3.3) and (3.4) we have $J\xi = \phi\xi + \psi\tilde{Q}\xi$. So, we get using skew-symmetry properties of T, (2.7) and (2.11)

$$\begin{aligned} g_{_M}(\overset{M}{\nabla}_{W_1} W_2, \xi) &= g_{_M}(T_{W_1} JW_2, \phi\xi) + g_{_M}(h\overset{M}{\nabla}_{W_1} JW_2, \psi\tilde{Q}\xi) \\ &= -g_{_M}(T_{W_1}\phi\xi, JW_2) + g_{_M}(h\overset{M}{\nabla}_{W_1} JW_2, \psi\tilde{Q}\xi) \\ &= \frac{1}{\lambda^2} g_{_N}((\nabla\Phi_*)(W_1, \phi\xi), \Phi_*(JW_2)) + g_{_M}(h\overset{M}{\nabla}_{W_1} JW_2, \psi\tilde{Q}\xi) \end{aligned} \tag{4.5}$$

for $W_1, W_2 \in \Gamma(D_\perp)$ and $\xi \in \Gamma(D \oplus D_\theta)$. Similarly, from (2.6), (2.7), (3.4) and (3.8) we get

$$\begin{aligned} g_{_M}(\overset{M}{\nabla}_{W_1} W_2, Z) &= -g_{_M}(T_{W_1} BZ + h\overset{M}{\nabla}_{W_1} CZ, JW_2) \\ &= \frac{1}{\lambda^2} g_{_N}((\nabla\Phi_*)(W_1, BZ), \Phi_*(JW_2)) + g_{_M}(h\overset{M}{\nabla}_{W_1} CZ, JW_2) \end{aligned} \tag{4.6}$$

for $W_1, W_2 \in \Gamma(D_\perp)$ and $Z \in \Gamma((ker\Phi_*)^\perp)$. We obtain (i) and (ii) from (4.5) and (4.6), respectively. □

Theorem 4.4. *Let $\Phi : (M, g_{_M}, J) \longrightarrow (N, g_{_N})$ be a quasi-hemi-slant conformal submersion. Then, the vertical distribution $ker\Phi_*$ defines totally geodesic foliations on M if and only if*

$$\begin{aligned} &\lambda^2\{g_{_M}(h\overset{M}{\nabla}_{\xi_1}\psi\tilde{Q}\xi_2 + T_{\xi_1}\psi\tilde{R}\xi_2, CZ) - g_{_M}(h\overset{M}{\nabla}_{\xi_1}\psi\phi\tilde{Q}\xi_2, Z) + g_{_M}(\hat{\nabla}_{\xi_1}\phi\tilde{P}\xi_2 + T_{\xi_1}\psi\tilde{Q}\xi_2 + v\overset{M}{\nabla}_{\xi_1}\psi\tilde{R}\xi_2, BZ)\} \\ &\quad = \cos^2\theta g_{_N}((\nabla\Phi_*)(\xi_1, \tilde{Q}\xi_2), \Phi_*(Z)) + g_{_N}((\nabla\Phi_*)(\xi_1, \phi\tilde{P}\xi_2), \Phi_*(CZ)) \end{aligned}$$

is provided for $\xi_1, \xi_2 \in \Gamma(ker\Phi_*)$ and $Z \in \Gamma((ker\Phi_*)^\perp)$.

Proof. We calculate the case of $g_{_M}(\overset{M}{\nabla}_{\xi_1}\xi_2, Z) = 0$ for $\xi_1, \xi_2 \in \Gamma(ker\Phi_*)$ and $Z \in \Gamma((ker\Phi_*)^\perp)$. So, (2.6), (2.7) and (3.5) we have

$$\begin{aligned} g_{_M}(\overset{M}{\nabla}_{\xi_1}\xi_2, Z) &= g_{_M}(\overset{M}{\nabla}_{\xi_1}\phi\tilde{P}\xi_2 + \phi\tilde{Q}\xi_2 + \psi\tilde{Q}\xi_2 + \psi\tilde{R}\xi_2, JZ) \\ &= g_{_M}(T_{\xi_1}\phi\tilde{P}\xi_2 + h\overset{M}{\nabla}_{\xi_1}\psi\tilde{Q}\xi_2 + T_{\xi_1}\psi\tilde{R}\xi_2, CZ) \\ &\quad + g_{_M}(\hat{\nabla}_{\xi_1}\phi\tilde{P}\xi_2 + T_{\xi_1}\psi\tilde{Q}\xi_2 + v\overset{M}{\nabla}_{\xi_1}\psi\tilde{R}\xi_2, BZ) \\ &\quad - g_{_M}(\overset{M}{\nabla}_{\xi_1}\phi^2\tilde{Q}\xi_2 + \overset{M}{\nabla}_{\xi_1}\psi\phi\tilde{Q}\xi_2, Z). \end{aligned} \tag{4.7}$$

Here, we use equations (2.11) and (3.11) in (4.7). Hence, we obtain

$$\begin{aligned} 0 &= g_{_M}(T_{\xi_1}\phi\tilde{P}\xi_2 + h\overset{M}{\nabla}_{\xi_1}\psi\tilde{Q}\xi_2 + T_{\xi_1}\psi\tilde{R}\xi_2, CZ) \\ &\quad + g_{_M}(\hat{\nabla}_{\xi_1}\phi\tilde{P}\xi_2 + T_{\xi_1}\psi\tilde{Q}\xi_2 + v\overset{M}{\nabla}_{\xi_1}\psi\tilde{R}\xi_2, BZ) \\ &\quad + \cos^2\theta g_{_M}(T_{\xi_1}\tilde{Q}\xi_2, Z) - g_{_M}(h\overset{M}{\nabla}_{\xi_1}\psi\phi\tilde{Q}\xi_2, Z) \\ &= -\frac{1}{\lambda^2} g_{_N}((\nabla\Phi_*)(\xi_1, \phi\tilde{P}\xi_2), \Phi_*(CZ)) + g_{_M}(h\overset{M}{\nabla}_{\xi_1}\psi\tilde{Q}\xi_2 + T_{\xi_1}\psi\tilde{R}\xi_2, CZ) \\ &\quad + g_{_M}(\hat{\nabla}_{\xi_1}\phi\tilde{P}\xi_2 + T_{\xi_1}\psi\tilde{Q}\xi_2 + v\overset{M}{\nabla}_{\xi_1}\psi\tilde{R}\xi_2, BZ) \\ &\quad - \cos^2\theta\frac{1}{\lambda^2} g_{_N}((\nabla\Phi_*)(\xi_1, \tilde{Q}\xi_2), \Phi_*(Z)) - g_{_M}(h\overset{M}{\nabla}_{\xi_1}\psi\phi\tilde{Q}\xi_2, Z) \end{aligned} \tag{4.8}$$

The proof is completed from (4.8). □

Theorem 4.5. *Let $\Phi : (M, g_{_M}, J) \longrightarrow (N, g_{_N})$ be a quasi-hemi-slant conformal submersion. Then, the horizontal distribution $(ker\Phi_*)^\perp$ defines totally geodesic foliations on M if and only if*

$$\frac{1}{\lambda^2}\{g_{_N}((\nabla\Phi_*)(Z_1, BZ_2), \Phi_*(\psi\xi)) - g_{_N}((\nabla\Phi_*)(Z_1, \phi\xi), \Phi_*(CZ_2))\} = g_{_M}(v\overset{M}{\nabla}_{Z_1} BZ_2, \phi\xi) + CZ_2(\ln\lambda)g_{_M}(Z_1, \psi\xi) - \psi\xi(\ln\lambda)g_{_M}(Z_1, CZ_2)$$

is provided for $Z_1, Z_2 \in \Gamma((ker\Phi_*)^\perp)$ and $\xi \in \Gamma(ker\Phi_*)$.

Proof. Using equations (2.2), (2.8), (2.9) and (3.4), we get

$$\begin{aligned} g_{_M}(\overset{M}{\nabla}_{Z_1} Z_2, \xi) &= g_{_M}(\overset{M}{\nabla}_{Z_1} BZ_2 + \overset{M}{\nabla}_{Z_1} CZ_2, \phi\xi + \psi\xi) \\ &= g_{_M}(A_{Z_1} BZ_2 + h\overset{M}{\nabla}_{Z_1} CZ_2, \psi\xi) + g_{_M}(v\overset{M}{\nabla}_{Z_1} BZ_2 + A_{Z_1} CZ_2, \phi\xi) \end{aligned} \tag{4.9}$$

for $Z_1, Z_2 \in \Gamma((ker\Phi_*)^\perp)$ and $\xi \in \Gamma(ker\Phi_*)$. Here, we apply (2.10), (2.12), (3.8) to (4.9) and from skew-symmetric properties of A, we obtain

$$\begin{aligned} g_{_M}(\overset{M}{\nabla}_{Z_1} Z_2, \xi) &= -\frac{1}{\lambda^2} g_{_N}((\nabla\Phi_*)(Z_1, BZ_2), \Phi_*(\psi\xi)) + CZ_2(\ln\lambda)g_{_M}(Z_1, \psi\xi) - \psi\xi(\ln\lambda)g_{_M}(Z_1, CZ_2) \\ &\quad + g_{_M}(v\overset{M}{\nabla}_{Z_1} BZ_2, \phi\xi) + \frac{1}{\lambda^2} g_{_N}((\nabla\Phi_*)(Z_1, \phi\xi), \Phi_*(CZ_2)). \end{aligned} \tag{4.10}$$

The proof is completed from (4.10) □

Note that, a horizontally conformal submersion $\Phi : (M, g_{_M}, J) \longrightarrow (N, g_{_N})$ is said to be horizontally homothetic if the gradient of its dilation λ is vertical, i.e., $h(grad\lambda) = 0$ at regular points [19]. Hence, we have the following.

Corollary 4.6. *Let $\Phi : (M, g_{_M}, J) \longrightarrow (N, g_{_N})$ be a quasi-hemi-slant conformal submersion. Then, the horizontal distribution $(ker\Phi_*)^\perp$ defines totally geodesic foliations on M if and only if*

i) *Φ is a horizontally homothetic map,*

ii) $g_{_N}((\nabla\Phi_*)(Z_1, BZ_2), \Phi_*(\psi\xi)) - g_{_N}((\nabla\Phi_*)(Z_1, \phi\xi), \Phi_*(CZ_2)) = \lambda^2 g_{_M}(v\overset{M}{\nabla}_{Z_1} BZ_2, \phi\xi)$

are provided for $Z_1, Z_2 \in \Gamma((ker\Phi_*)^\perp)$ and $\xi \in \Gamma(ker\Phi_*)$.

Proof. Because of Φ defines totally geodesic foliations on M, we have (4.10). Suppose that Φ is a horizontally homothetic map, we have from (4.10)

$$0 = CZ_2(\ln\lambda)g_{_M}(Z_1, \psi\xi) - \psi\xi(\ln\lambda)g_{_M}(Z_1, CZ_2) \tag{4.11}$$

for $Z_1, Z_2 \in \Gamma((ker\Phi_*)^\perp)$ and $\xi \in \Gamma(ker\Phi_*)$. Here, if we take $Z_1 = \psi\xi$ in (4.11) we get

$$0 = CZ_2(\ln\lambda)g_{_M}(\psi\xi, \psi\xi). \tag{4.12}$$

In (4.12), we get $0 = CZ_2(\ln\lambda)$ and it means λ is a constant on μ. Similarly, if we take $Z_1 = CZ_2$ in (4.11) we get

$$0 = -\psi\xi(\ln\lambda)g_{_M}(CZ_2, CZ_2). \tag{4.13}$$

In (4.13), we get $0 = \psi\xi(\ln\lambda)$ and it means λ is a constant on $\psi D_\theta \oplus J(D_\perp)$. Therefore, from (4.12) and (4.13) we say that λ is a constant on horizontal distribution. So, (i) is satisfied. Now, if (i) is satisfied in (4.10), we obtain

$$0 = -\frac{1}{\lambda^2} g_{_N}((\nabla\Phi_*)(Z_1, BZ_2), \Phi_*(\psi\xi)) + g_{_M}(v\overset{M}{\nabla}_{Z_1} BZ_2, \phi\xi) + \frac{1}{\lambda^2} g_{_N}((\nabla\Phi_*)(Z_1, \phi\xi), \Phi_*(CZ_2)). \tag{4.14}$$

From (4.14), (ii) is satisfied. The proof is completed. □

A horizontally conformal submersion $\Phi : (M, g_{_M}, J) \longrightarrow (N, g_{_N})$ is said to be totally geodesic if second fundamental form of the map $(\nabla\Phi_*)(X, Y) = 0$ for $X, Y \in \Gamma(TM)$ [4]. Hence, we have the next theorem.

Theorem 4.7. *Let $\Phi : (M, g_{_M}, J) \longrightarrow (N, g_{_N})$ be a quasi-hemi-slant conformal submersion. Then, the map Φ is totally geodesic if and only if*

i) $\cos^2\theta T_{\xi_1}\tilde{Q}\xi_2 = h\overset{M}{\nabla}_{\xi_1}\psi\phi\tilde{Q}\xi_2 + C\{T_{\xi_1}\phi\tilde{P}\xi_2 + h\overset{M}{\nabla}_{\xi_1}\psi\xi_2\} + \psi\{\hat{\nabla}_{\xi_1}\phi\tilde{P}\xi_2 + T_{\xi_1}\psi\xi_2\},$

ii) $0 = C\{A_{Z_1}\phi\xi_1 + h\overset{M}{\nabla}_{Z_1}\psi\xi_1\} + \psi\{v\overset{M}{\nabla}_{Z_1}\phi\xi_1 + A_{Z_1}\psi\xi_1\}$

iii) Φ *is a horizontally homothetic map*

are provided for $Z_1, Z_2 \in \Gamma((ker\Phi_*)^\perp)$ *and* $\xi_1, \xi_2 \in \Gamma(ker\Phi_*)$.

Proof. Firstly, we examine $(\nabla\Phi_*)(\xi_1, \xi_2)$ for $\xi_1, \xi_2 \in \Gamma(ker\Phi_*)$. Because of $\psi\tilde{Q}\xi_2 + \psi\tilde{R}\xi_2 = \psi\xi_2$ we have from (2.2), (2.3) and (3.5)

$$(\nabla\Phi_*)(\xi_1, \xi_2) \quad = \quad \Phi_*(J\overset{M}{\nabla}_{\xi_1}\phi\tilde{P}\xi_2 + \phi\tilde{Q}\xi_2 + \psi\xi_2)$$

for $\xi_1, \xi_2 \in \Gamma(ker\Phi_*)$. Then, using equations (2.6), (2.7) and (3.11) have

$$\begin{aligned}(\nabla\Phi_*)(\xi_1, \xi_2) &= \Phi_*(JT_{\xi_1}\phi\tilde{P}\xi_2 + J\hat{\nabla}_{\xi_1}\phi\tilde{P}\xi_2)\\ &\quad +\Phi_*(\overset{M}{\nabla}_{\xi_1}\phi^2\tilde{Q}\xi_2 + \overset{M}{\nabla}_{\xi_1}\psi\phi\tilde{Q}\xi_2)\\ &\quad +\Phi_*(JT_{\xi_1}\psi\xi_2 + Jh\overset{M}{\nabla}_{\xi_1}\psi\xi_2)\\ &= \Phi_*(CT_{\xi_1}\phi\tilde{P}\xi_2 + \psi\hat{\nabla}_{\xi_1}\phi\tilde{P}\xi_2)\\ &\quad -\cos^2\theta\Phi_*(\overset{M}{\nabla}_{\xi_1}\tilde{Q}\xi_2) + \Phi_*(h\overset{M}{\nabla}_{\xi_1}\psi\phi\tilde{Q}\xi_2)\\ &\quad +\Phi_*(\psi T_{\xi_1}\psi\xi_2 + Ch\overset{M}{\nabla}_{\xi_1}\psi\xi_2)\\ &= \Phi_*(C\{T_{\xi_1}\phi\tilde{P}\xi_2 + h\overset{M}{\nabla}_{\xi_1}\psi\xi_2\} + \psi\{\hat{\nabla}_{\xi_1}\phi\tilde{P}\xi_2 + T_{\xi_1}\psi\xi_2\})\\ &\quad -\cos^2\theta\Phi_*(T_{\xi_1}\tilde{Q}\xi_2) + \Phi_*(h\overset{M}{\nabla}_{\xi_1}\psi\phi\tilde{Q}\xi_2). \end{aligned} \tag{4.15}$$

We obtain (i) from (4.15). Second fundamental form of a map is symmetric. So, we have $(\nabla\Phi_*)(\xi_1, Z_1) = (\nabla\Phi_*)(Z_1, \xi_1)$ for $Z_1 \in \Gamma((ker\Phi_*)^\perp)$ and $\xi_1 \in \Gamma(ker\Phi_*)$. From (2.3), (2.8), (2.9) and (3.4) we obtain

$$\begin{aligned}(\nabla\Phi_*)(Z_1, \xi_1) &= \Phi_*(J\overset{M}{\nabla}_{Z_1}\phi\xi_1 + J\overset{M}{\nabla}_{Z_1}\psi\xi_1)\\ &= \Phi_*(JA_{Z_1}\phi\xi_1 + Jv\overset{M}{\nabla}_{Z_1}\phi\xi_1) + \Phi_*(JA_{Z_1}\psi\xi_1 + Jh\overset{M}{\nabla}_{Z_1}\psi\xi_1)\\ &= \Phi_*(CA_{Z_1}\phi\xi_1 + \psi v\overset{M}{\nabla}_{Z_1}\phi\xi_1) + \Phi_*(\psi A_{Z_1}\psi\xi_1 + Ch\overset{M}{\nabla}_{Z_1}\psi\xi_1). \end{aligned} \tag{4.16}$$

We obtain (ii) from (4.16). Lastly, from (2.10) we have

$$(\nabla\Phi_*)(Z_1, Z_2) \quad = \quad Z_1(\ln\lambda)\Phi_*(Z_2) + Z_2(\ln\lambda)\Phi_*(Z_1) - g_M(Z_1, Z_2)\Phi_*(grad(\ln\lambda)) \tag{4.17}$$

for $Z_1, Z_2 \in \Gamma((ker\Phi_*)^\perp)$. For Z_1 in (4.17) we obtain

$$\begin{aligned} 0 &= Z_2(\ln\lambda)g_{_N}(\Phi_*(Z_1), \Phi_*(Z_1))\\ 0 &= \lambda^2 Z_2(\ln\lambda)g_{_M}(Z_1, Z_1). \end{aligned} \tag{4.18}$$

In (4.18), we get $Z_2(\ln\lambda) = 0$. It means λ is a constant on horizontal distribution. So, the map is horizontally homothetic. (iii) is satisfied. The proof is completed. □

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this article.

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