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Research Article

A generalization of reverse Hölder's inequality via the diamond-α integral on time scales

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Abstract

In this paper, we give a generalization of the reverse Hölder's diamond-α inequality on time scales by introducing two parameters. We note that many inequalities related to the Hölder's inequality can be obtained via this inequality.

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1. Introduction

Hölder's inequality is one of the most important inequalities of pure and applied mathematics. It is the key for resolving many problems in social and natural sciences. Hölder's inequality in time scales is given in the following theorem (see [2, Theorem 1.1.11] and [3, Theorem 6.13]).

Theorem 1.1. *Let $h, f, g \in \mathcal{C}_{rd}([a,b]_{\mathbb{T}}, [0,+\infty))$. If $\frac{1}{p}+\frac{1}{p'}=1$ with $p>1$, then*

$$\int_a^b h(t)f(t)g(t)\triangle t \leq \left(\int_a^b h(t)f^p(t)\triangle t\right)^{\frac{1}{p}} \left(\int_a^b g(t)f^{p'}(t)\triangle t\right)^{\frac{1}{p'}}. \tag{1.1}$$

The reverse Hölder's inequality has been explored by a number of scientists. The famous ones are [2], [9].

Theorem 1.2. *Let $p>1$, $\frac{1}{p}+\frac{1}{p'}=1$ and $a,b \in \mathbb{T}$ with $a<b$, f and g be two positive functions defined on the interval $[a,b]_{\mathbb{T}}$ if $0<m \leq \dfrac{f^p(t)}{g^{p'}(t)} \leq M$. Then*

$$\left(\int_a^b f^p(t)\Delta t\right)^{\frac{1}{p}} \left(\int_a^b g^{p'}(t)\Delta t\right)^{\frac{1}{p'}} \leq \left(\frac{M}{m}\right)^{\frac{1}{pp'}} \int_a^b f(t)g(t)\Delta t. \tag{1.2}$$

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Theorem 1.3. *Let $\mathbb{T}$ be a time scale, $a, b \in \mathbb{T}$ with $a < b$, and f and g be two positive functions satisfying $0 < m \leq \dfrac{f^p(t)}{g^{p'}(t)} \leq M$, on the set $[a, b]$. If $\frac{1}{p} + \frac{1}{p'} = 1$ with $p > 1$, then*

$$\left(\int_a^b f^p(t)\Diamond_\alpha t\right)^{\frac{1}{p}} \left(\int_a^b g^{p'}(t)\Diamond_\alpha t\right)^{\frac{1}{p'}} \leq \left(\frac{M}{m}\right)^{\frac{1}{pp'}} \int_a^b f(t)g(t)\Diamond_\alpha t. \tag{1.3}$$

In 2020, Benaissa and Budak [1] give the following result:

Theorem 1.4. *(Theorem 2.1, [1]) Let $\alpha, \beta > 0$, $p > 1$, $\frac{1}{p} + \frac{1}{p'} = 1$ and $f, g > 0$ integrable functions on $[a, b]$, w a weight function (measurable and positive) on $[a, b]$. If*

$$0 < m \leq \frac{f^\alpha(x)}{g^\beta(x)} \leq M \text{ for all } x \in [a, b], \tag{1.4}$$

then

$$\left(\int_a^b f^\alpha(x)w(x)dx\right)^{\frac{1}{p}} \left(\int_a^b g^\beta(x)w(x)dx\right)^{\frac{1}{p'}} \leq \left(\frac{M}{m}\right)^{\frac{1}{pp'}} \int_a^b f^{\frac{\alpha}{p}}(x)g^{\frac{\beta}{p'}}(x)w(x)dx. \tag{1.5}$$

In [10] the authors provide a version of the above inequality in time scales by the following theorem.

Theorem 1.5. *Let $\alpha \in [0, 1]$, β, $\lambda > 0$, $p > 1$, $\frac{1}{p} + \frac{1}{p'} = 1$ and $\mathbb{T}$ is a time scale, $a, b \in \mathbb{T}$ with $a < b$. Let $f, g \in \mathcal{C}([a, b]_{\mathbb{T}}, [0, +\infty))$, w a weight function (measurable and positive) on $[a, b]_{\mathbb{T}}$. If*

$$0 < m \leq \frac{f^\beta(t)}{g^\lambda(t)} \leq M \text{ for all } t \in [a, b]_{\mathbb{T}}, \tag{1.6}$$

then

$$\left(\int_a^b f^\beta(t)w(t)\Diamond_\alpha t\right)^{\frac{1}{p}} \left(\int_a^b g^\lambda(t)w(t)\Diamond_\alpha t\right)^{\frac{1}{p'}} \leq \left(\frac{M}{m}\right)^{\frac{1}{pp'}} \int_a^b f^{\frac{\beta}{p}}(t)g^{\frac{\lambda}{p'}}(t)w(t)\Diamond_\alpha t. \tag{1.7}$$

The reverse Hölder's inequalities play an important role in many areas of pure and applied mathematics. A large number of generalizations, refinements, variations and applications of these inequalities have been investigated in the literature (see [1, 4, 5, 8]).

2. Preliminaries

We introduce the diamond-α dynamic derivative and diamond-α dynamic integration. The comprehensive development of the calculus of the diamond-α derivative and diamond-α integration is given in [6], [7]. Let $\mathbb{T}$ be a time scale and $f(t)$ be differentiable on $\mathbb{T}$ in the Δ and ∇ sense. For $t \in \mathbb{T}$, we define the diamond-α derivative $f^{\Diamond_\alpha}(t)$ by

$$f^{\Diamond_\alpha}(t) = \alpha f^\Delta(t) + (1 - \alpha)f^\nabla(t).$$

Thus f is diamond-α differentiable if and only if f is Δ and ∇ differentiable.

Theorem 2.1. *Let $0 \leq \alpha \leq 1$. If f is both Δ and ∇ differentiable at $t \in \mathbb{T}$, then f is $\Diamond_\alpha$ differentiable at t and*

$$f^{\Diamond_\alpha}(t) = \alpha f^\Delta(t) + (1 - \alpha)f^\nabla(t).$$

Definition 2.2. Let $a, b \in \mathbb{T}$, and $f : \mathbb{T} \to \mathbb{R}$. Then, the diamond-$\alpha$ integral from a to b of f is defined by

$$\int_a^b f(s)\Diamond_\alpha s = \alpha \int_a^b f(s)\Delta s + (1 - \alpha) \int_a^b f(s)\nabla s, \quad 0 \leq \alpha \leq 1,$$

provided that there exist Δ and ∇ integrals of f on $\mathbb{T}$.

It is clear that the diamond-α integral of f exists when f is a continuous function. Let a, b, $c \in \mathbb{T}$, $\lambda, \beta \in \mathbb{R}$ and f, g be continuous functions on $[a, b] \cap \mathbb{T} = [a, b]_{\mathbb{T}}$. Then the following properties hold:

(1) $\displaystyle\int_a^b (\lambda f(s) + \beta g(s)) \diamondsuit_\alpha s = \lambda \int_a^b f(s)\diamondsuit_\alpha s + \beta \int_a^b g(s)\diamondsuit_\alpha s.$

(2) $\displaystyle\int_a^b f(s)\diamondsuit_\alpha s = -\int_b^a f(s)\diamondsuit_\alpha s, \quad \int_a^a f(s)\diamondsuit_\alpha s = 0.$

(3) $\displaystyle\int_a^b f(s)\diamondsuit_\alpha s = \int_a^c f(s)\diamondsuit_\alpha s + \int_c^b f(s)\diamondsuit_\alpha s.$

(4) If $f(s) \geq 0$ for all $s \in [a, b]_{\mathbb{T}}$, then $\displaystyle\int_a^b f(s)\diamondsuit_\alpha s \geq 0.$

(5) If $f(s) \leq g(s)$ for all $s \in [a, b]_{\mathbb{T}}$, then $\displaystyle\int_a^b f(s)\diamondsuit_\alpha s \leq \int_a^b g(s)\diamondsuit_\alpha s.$

(6) If $f(s) \geq 0$ for all $s \in [a, b]_{\mathbb{T}}$, then $f(s) = 0$ if only if $\displaystyle\int_a^b f(s)\diamondsuit_\alpha s = 0.$

Lemma 2.3. ([2, Theorem 1.1.21]). Let $\mathbb{T}$ be a time scale, $a, b \in \mathbb{T}$ with $a < b$, and h, ϕ be two positive functions. If $\frac{1}{p} + \frac{1}{p'} = 1$ with $p < 1$, then

$$\int_a^b h(\tau)\phi(\tau)\diamondsuit_\alpha \tau \geq \left(\int_a^b h^p(\tau)\diamondsuit_\alpha \tau\right)^{\frac{1}{p}} \left(\int_a^b \phi^{p'}(\tau)\diamondsuit_\alpha \tau\right)^{\frac{1}{p'}}. \tag{2.1}$$

3. Main results

In this section we give our results by using a simple proof method to generalize the inequality (1.3).

Lemma 3.1. *Let $1 < q \leq p < \infty$, $\frac{1}{p}+\frac{1}{p'} = \frac{1}{q}+\frac{1}{q'} = 1$ and ϕ, w be non-negative continuous functions on $[a, b]_{\mathbb{T}}$. We suppose that $0 < \int_a^b \phi^s(t)w(t)\diamondsuit_\alpha t < \infty$, for $s > 1$, then*

$$\int_a^b \phi^p(t)w(t)\diamondsuit_\alpha t \geq \left(\int_a^b w(t)\diamondsuit_\alpha t\right)^{\frac{q-p}{q}} \left(\int_a^b \phi^q(t)w(t)\diamondsuit_\alpha t\right)^{\frac{p}{q}}, \tag{3.1}$$

$$\int_a^b \phi^{q'}(t)w(t)\diamondsuit_\alpha t \geq \left(\int_a^b w(t)\diamondsuit_\alpha t\right)^{\frac{p'-q'}{p'}} \left(\int_a^b \phi^{p'}(t)w(t)\diamondsuit_\alpha t\right)^{\frac{q'}{p'}}. \tag{3.2}$$

Proof. If $p = q$, then we have equality and for $p \neq q$, we use Hölder's integral inequality (2.1) with $\dfrac{q}{p} < 1$. We get

$$\begin{aligned} \int_a^b \phi^p(t)w(t)\diamondsuit_\alpha t &= \int_a^b \left(w^{\frac{q-p}{q}}(t)\right)\left(\phi^p(t)w^{\frac{p}{q}}(t)\right)\diamondsuit_\alpha t \\ &\geq \left(\int_a^b t(t)\diamondsuit_\alpha t\right)^{\frac{q-p}{q}} \left(\int_a^b \phi^q(t)w(t)\diamondsuit_\alpha t\right)^{\frac{p}{q}}. \end{aligned}$$

The proof of the second inequality is similar to the first one. We have

$$1 < q \leq p < \infty \Longrightarrow 1 < p' \leq q' < \infty.$$

□

Theorem 3.2. *Let $\alpha \in [0, 1]$, β, $\lambda > 0$, $1 < q \leq p < \infty$, $\frac{1}{p} + \frac{1}{p'} = \frac{1}{q} + \frac{1}{q'} = 1$ and $\mathbb{T}$ is a time scale, a, $b \in \mathbb{T}$ with $a < b$. Let $f, g > 0$ be continuous functions on $[a, b]_{\mathbb{T}}$, w a* *weight function on* $[a,\, b]_{\mathbb{T}}$*. If*

$$0 < m \leq \frac{f^{\beta}(t)}{g^{\lambda}(t)} \leq M \quad \textit{for all } t \in [a,\, b]_{\mathbb{T}}, \tag{3.3}$$

then

$$\begin{aligned} &\left(\int_a^b f^{\frac{q\beta}{p}}(t)w(t)\diamondsuit_\alpha t\right)^{\frac{1}{q}} \left(\int_a^b g^{\frac{p'\lambda}{q'}}(t)w(t)\diamondsuit_\alpha t\right)^{\frac{1}{p'}} \\ &\leq M^{\frac{1}{pp'}}(\frac{1}{m})^{\frac{1}{qq'}} \left(\int_a^b w(t)\diamondsuit_\alpha t\right)^{\frac{2}{q}-\frac{2}{p}} \left(\int_a^b f^{\frac{\beta}{p}}(t)g^{\frac{\lambda}{p'}}(t)w(t)\diamondsuit_\alpha t\right)^{\frac{1}{p}} \left(\int_a^b f^{\frac{\beta}{q}}(t)g^{\frac{\lambda}{q'}}(t)w(t)\diamondsuit_\alpha t\right)^{\frac{1}{q'}} \end{aligned} \tag{3.4}$$

Proof. From the assumption (3.3) we have

$$f^{-\frac{\beta}{p'}} g^{\frac{\lambda}{p'}} \geq M^{-\frac{1}{p'}}$$

yielding

$$f^{\beta} \leq M^{\frac{1}{p'}} f^{\frac{\beta}{p}} g^{\frac{\lambda}{p'}}.$$

Multiplying the above inequality by $w(t)$ and integrating on $[a, b]_{\mathbb{T}}$, we obtain

$$\left(\int_a^b f^{\beta}(t)w(t)\diamondsuit_\alpha t\right)^{\frac{1}{p}} \leq M^{\frac{1}{pp'}} \left(\int_a^b f^{\frac{\beta}{p}}(t)g^{\frac{\lambda}{p'}}(t)w(t)\diamondsuit_\alpha t\right)^{\frac{1}{p}}. \tag{3.5}$$

Now we use the inequality (3.1) and putting $\phi = f^{\frac{\beta}{p}}$, we get

$$\int_a^b f^{\beta}(t)w(t)\diamondsuit_\alpha t \geq \left(\int_a^b w(t)\diamondsuit_\alpha t\right)^{\frac{q-p}{q}} \left(\int_a^b f^{\frac{q\beta}{p}}(t)w(t)\diamondsuit_\alpha t\right)^{\frac{p}{q}}.$$

Hence

$$\left(\int_a^b f^{\frac{q\beta}{p}}(t)w(t)\diamondsuit_\alpha t\right)^{\frac{1}{q}} \leq \left(\int_a^b w(t)\diamondsuit_\alpha t\right)^{\frac{p-q}{pq}} \left(\int_a^b f^{\beta}(t)w(t)\diamondsuit_\alpha t\right)^{\frac{1}{p}}. \tag{3.6}$$

From the inequalities (3.5) and (3.6), we deduce that

$$\left(\int_a^b f^{\frac{q\beta}{p}}(t)w(t)\diamondsuit_\alpha t\right)^{\frac{1}{q}} \leq M^{\frac{1}{pp'}} \left(\int_a^b w(t)\diamondsuit_\alpha t\right)^{\frac{p-q}{pq}} \left(\int_a^b f^{\frac{\beta}{p}}(t)g^{\frac{\lambda}{p'}}(t)w(t)\diamondsuit_\alpha t\right)^{\frac{1}{p}}. \tag{3.7}$$

Similarly, from the assumption (3.3) we have

$$m^{\frac{1}{q}} \leq f^{\frac{\beta}{q}} g^{-\frac{\lambda}{q}}.$$

Multiplying by g^{λ} yields

$$m^{\frac{1}{q}} g^{\lambda} \leq f^{\frac{\beta}{q}} g^{\frac{\lambda}{q'}}.$$

We deduce that

$$\left(\int_a^b g^{\lambda}(t)w(t)\diamondsuit_\alpha t\right)^{\frac{1}{q'}} \leq (\frac{1}{m})^{\frac{1}{qq'}} \left(\int_a^b f^{\frac{\beta}{q}}(t)g^{\frac{\lambda}{q'}}(t)w(t)\diamondsuit_\alpha t\right)^{\frac{1}{q'}}. \tag{3.8}$$

Again we put $\phi = g^{\frac{\lambda}{q'}}$ in the inequality (3.2). We get

$$\int_a^b g^{\lambda}(t)w(t)\diamondsuit_\alpha t \geq \left(\int_a^b w(t)\diamondsuit_\alpha t\right)^{\frac{p'-q'}{p'}} \left(\int_a^b g^{\frac{p'\lambda}{q'}}(t)w(t)\diamondsuit_\alpha t\right)^{\frac{q'}{p'}},$$

which gives

$$\left(\int_a^b g^{\frac{p'\lambda}{q'}}(t)w(t)\diamondsuit_\alpha t\right)^{\frac{1}{p'}} \leq \left(\int_a^b w(t)\diamondsuit_\alpha t\right)^{\frac{q'-p'}{q'p'}} \left(\int_a^b g^{\lambda}(t)w(t)\diamondsuit_\alpha t\right)^{\frac{1}{q'}}. \tag{3.9}$$

By the inequalities (3.8) and (3.9), we deduce that

$$\begin{aligned}\left(\int_a^b g^{\frac{p'\lambda}{q'}}(t)w(t)\diamondsuit_\alpha t\right)^{\frac{1}{p'}} &\leq (\tfrac{1}{m})^{\frac{1}{qq'}} \left(\int_a^b w(t)\diamondsuit_\alpha t\right)^{\frac{q'-p'}{q'p'}} \\ &\times \left(\int_a^b f^{\frac{\beta}{q}}(t)g^{\frac{\lambda}{q'}}(t)w(t)\diamondsuit_\alpha t\right)^{\frac{1}{q'}}.\end{aligned} \tag{3.10}$$

Finally, by multiplying the inequalities (3.7) and (3.10), we obtain the desired inequality (3.4). □

Remark 3.3. If $p = q$, we obtain the reverse Hölder's inequality (1.7) [10].

3.1. Particular cases of Theorem 3.2

If we take $\alpha = 1$, we get the following inequality for the Δ-integrable.

Corollary 3.4. *Let β, $\lambda > 0, 1 < q \leq p < \infty$, $\frac{1}{p} + \frac{1}{p'} = \frac{1}{q} + \frac{1}{q'} = 1$ and $\mathbb{T}$ is a time scale, a, $b \in \mathbb{T}$ with $a < b$. Let $f, g \in \mathcal{C}_{rd}([a,\, b]_{\mathbb{T}}, [0,\, +\infty))$, w a weight function on $[a,\, b]_{\mathbb{T}}$. If*

$$0 < m \leq \frac{f^{\beta}(t)}{g^{\lambda}(t)} \leq M \quad \text{for all } t \in [a,\, b]_{\mathbb{T}}, \tag{3.11}$$

then

$$\begin{aligned}&\left(\int_a^b f^{\frac{q\beta}{p}}(t)w(t)\Delta t\right)^{\frac{1}{q}} \left(\int_a^b g^{\frac{p'\lambda}{q'}}(t)w(t)\Delta t\right)^{\frac{1}{p'}} \\ &\leq M^{\frac{1}{pp'}}(\frac{1}{m})^{\frac{1}{qq'}} \left(\int_a^b w(t)\Delta t\right)^{\frac{2}{q}-\frac{2}{p}} \left(\int_a^b f^{\frac{\beta}{p}}(t)g^{\frac{\lambda}{p'}}(t)w(t)\Delta t\right)^{\frac{1}{p}} \left(\int_a^b f^{\frac{\beta}{q}}(t)g^{\frac{\lambda}{q'}}(t)w(t)\Delta t\right)^{\frac{1}{q'}}.\end{aligned} \tag{3.12}$$

If we take $\alpha = 0$, we get the following inequality for the ∇-integrable.

Corollary 3.5. *Let β, $\lambda > 0, 1 < q \leq p < \infty$, $\frac{1}{p} + \frac{1}{p'} = \frac{1}{q} + \frac{1}{q'} = 1$ and $\mathbb{T}$ is a time scale, a, $b \in \mathbb{T}$ with $a < b$. Let $f, g \in \mathcal{C}_{ld}([a,\, b]_{\mathbb{T}}, [0,\, +\infty))$, w a weight function on $[a,\, b]_{\mathbb{T}}$. If*

$$0 < m \leq \frac{f^{\beta}(t)}{g^{\lambda}(t)} \leq M \quad \text{for all } t \in [a,\, b]_{\mathbb{T}}, \tag{3.13}$$

then

$$\begin{aligned}&\left(\int_a^b f^{\frac{q\beta}{p}}(t)w(t)\nabla t\right)^{\frac{1}{q}} \left(\int_a^b g^{\frac{p'\lambda}{q'}}(t)w(t)\nabla t\right)^{\frac{1}{p'}} \\ &\leq M^{\frac{1}{pp'}}(\frac{1}{m})^{\frac{1}{qq'}} \left(\int_a^b w(t)\nabla t\right)^{\frac{2}{q}-\frac{2}{p}} \left(\int_a^b f^{\frac{\beta}{p}}(t)g^{\frac{\lambda}{p'}}(t)w(t)\nabla t\right)^{\frac{1}{p}} \left(\int_a^b f^{\frac{\beta}{q}}(t)g^{\frac{\lambda}{q'}}(t)w(t)\nabla t\right)^{\frac{1}{q'}}.\end{aligned} \tag{3.14}$$

If we take $\mathbb{T} = \mathbb{R}$ in the above theorem, we get the following corollary.

Corollary 3.6. *Let $\beta,\ \lambda > 0, 1 < q \leq p < \infty,\ \frac{1}{p}+\frac{1}{p'}=\frac{1}{q}+\frac{1}{q'}=1$ and $a,\ b \in \mathbb{R}$ with $a<b$. Let $f,g \in \mathcal{C}([a,\ b],[0,\ +\infty))$, w a weight function on $[a,\ b]$. If*

$$0 < m \leq \frac{f^{\beta}(t)}{g^{\lambda}(t)} \leq M \quad \text{for all } t \in [a,\ b], \tag{3.15}$$

then

$$\begin{aligned}&\left(\int_a^b f^{\frac{q\beta}{p}}(t)w(t)dt\right)^{\frac{1}{q}}\left(\int_a^b g^{\frac{p'\lambda}{q'}}(t)w(t)dt\right)^{\frac{1}{p'}}\\ &\leq M^{\frac{1}{pp'}}(\frac{1}{m})^{\frac{1}{qq'}}\left(\int_a^b w(t)dt\right)^{\frac{2}{q}-\frac{2}{p}}\left(\int_a^b f^{\frac{\beta}{p}}(t)g^{\frac{\lambda}{p'}}(t)w(t)dt\right)^{\frac{1}{p}}\left(\int_a^b f^{\frac{\beta}{q}}(t)g^{\frac{\lambda}{q'}}(t)w(t)dt\right)^{\frac{1}{q'}}.\end{aligned} \tag{3.16}$$

Remark 3.7. The inequality (3.16) is a new generalization with two parameters of the weighted inequality given in Theorem 1.4 [1].

If we take $\beta = p$ and $\lambda = q'$, we get the following corollary.

Corollary 3.8. *Let $\alpha \in [0,\ 1],\ 1 < q \leq p < \infty,\ \frac{1}{p}+\frac{1}{p'}=\frac{1}{q}+\frac{1}{q'}=1$ and $\mathbb{T}$ is a time scale, $a,\ b \in \mathbb{T}$ with $a<b$. Let $f,g>0$ be continuous functions on $[a,\ b]_{\mathbb{T}}$, w a weight function on $[a,\ b]_{\mathbb{T}}$. If*

$$0 < m \leq \frac{f^{p}(t)}{g^{q'}(t)} \leq M \quad \text{for all } t \in [a,\ b]_{\mathbb{T}}, \tag{3.17}$$

then

$$\begin{aligned}&\left(\int_a^b f^{q}(t)w(t)\diamondsuit_\alpha t\right)^{\frac{1}{q}}\left(\int_a^b g^{p'}(t)w(t)\diamondsuit_\alpha t\right)^{\frac{1}{p'}}\\ &\leq M^{\frac{1}{pp'}}(\frac{1}{m})^{\frac{1}{qq'}}\left(\int_a^b w(t)\diamondsuit_\alpha t\right)^{\frac{2}{q}-\frac{2}{p}}\left(\int_a^b f(t)g^{\frac{q'}{p'}}(t)w(t)\diamondsuit_\alpha t\right)^{\frac{1}{p}}\left(\int_a^b f^{\frac{p}{q}}(t)g(t)w(t)\diamondsuit_\alpha t\right)^{\frac{1}{q'}}.\end{aligned} \tag{3.18}$$

Remark 3.9. The last inequality (3.18) is a generalization with two parameters of the inequality given in Theorem 1.3.

If we put $\beta = 1,\ g = 1$, we get the following result.

Corollary 3.10. *Let $\alpha \in [0,\ 1],\ 1 < q \leq p < \infty,\ \frac{1}{p}+\frac{1}{p'}=\frac{1}{q}+\frac{1}{q'}=1$ and $\mathbb{T}$ is a time scale, $a,\ b \in \mathbb{T}$ with $a<b$. Let $f>0$ be a continuous function on $[a,\ b]_{\mathbb{T}}$, w a weight function on $[a,\ b]_{\mathbb{T}}$. If*

$$0 < m \leq f(t) \leq M \quad \text{for all } t \in [a,\ b]_{\mathbb{T}}, \tag{3.19}$$

then

$$\begin{aligned}&\left(\int_a^b f^{\frac{q}{p}}(t)w(t)\diamondsuit_\alpha t\right)^{\frac{1}{q}}\\ &\leq M^{\frac{1}{pp'}}(\frac{1}{m})^{\frac{1}{qq'}}\left(\int_a^b w(t)\diamondsuit_\alpha t\right)^{\frac{1}{p'}-\frac{2}{q'}}\left(\int_a^b f^{\frac{1}{p}}(t)w(t)\diamondsuit_\alpha t\right)^{\frac{1}{p}}\left(\int_a^b f^{\frac{1}{q}}(t)w(t)\diamondsuit_\alpha t\right)^{\frac{1}{q'}}.\end{aligned} \tag{3.20}$$

Remark 3.11. If we put $p = q$ in the above inequality (3.20), we get for $p > 1$,

$$\int_a^b f(t)w(t)\diamondsuit_\alpha t \leq \left(\frac{M}{m}\right)^{\frac{1}{p'}}\left(\int_a^b w(t)\diamondsuit_\alpha t\right)^{1-p}\left(\int_a^b f^{\frac{1}{p}}(t)w(t)\diamondsuit_\alpha t\right)^{p}. \tag{3.21}$$

4. Conclusion

In the present article, we obtained Hölder's original inverse diamond-α inequality with two parameters. The proven inequalities generalize certain dynamic and classical inequalities known in the literature.

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