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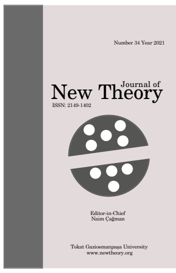

On Transmuted Power Function Distribution: Characterization, Risk Measures, and Estimation

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Abstract − Transmuted power function distribution is generated using the quadratic rank transmutation method based on the mixture of the distributions of two order statistics. The distributions generating via Quadratic rank transmutation map are more flexible than the baseline ones since they have a potential to model various dataset. In this study, we provide some distributional properties and statistical inferences of transmuted power function distribution. We describe several previously unexamined properties, such as density shape, hazard shape, and the transmuted power function distribution measures. We also tackle the problem of point estimation for transmuted power function distribution. In this regard, maximum likelihood, least-squares, weighted least-squares, Anderson-Darling method, and Crámer–Von-Mises method are considered to estimate the two parameters of transmuted power function distribution. A comprehensive Monte Carlo simulation study is performed to compare these methods via bias and mean-squared errors.

Keywords *− Transmuted power function distribution, power function distribution, point estimation, risk measures, Monte Carlo simulation*

Mathematics Subject Classification (2020) − 62F10, 62P05

1. Introduction

Many authors very commonly discuss the point estimation and various characterization of the statistical distributions. Describing the distributions' statistical properties in detail is very significant to illustrate the usefulness of the distributions. Another critical point is the parameter estimation problem for the statistical distributions. It is well-known that the maximum likelihood method is very popular for point estimation. However, many researchers studied various alternative methods to the maximum likelihood method. In the last decade, there are many papers on the characterization and estimation of the distributions. Mahmoud and Mandouh [1] described some distributional properties of transmuted Fréchet distribution. Hamedani [2] examined some characteristics of transmuted complementary Weibull geometric distribution. Ahmad et al. [3] provided the characterization of transmuted Kumaraswamy distribution. Ahmad et al. [4] focused on a number of statistical properties and point estimation for transmuted Rayleigh distribution. Bhatti et al. [5] studied a couple of characterizations of transmuted Dagum distribution. Bhatti et al. [6] discussed several distributional properties of transmuted modified Burr II distribution. Bhatti et al. [7] examined some statistical properties of the transmuted geometric-quadratic hazard rate distribution. Tanış et al. [8] considered a comparison of the approximate Bayes and maximum likelihood estimation methods for log-Dagum distribution. Tanış and

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Saraçoğlu [9] compared the methods of estimation for log-Kumaraswamy distribution. Hanif et al. [10] discussed several estimation methods for Rician distribution. Anas et al. [11] performed partial characterisation of extreme value distribution. Hanif et al. [12] tackled the estimation of parameters' discrete inverse Weibull distribution using ranked set sampling. Hanif et al. [13] focused on the estimation of parameters' generalized exponential distribution. Karakaya and Tanış [14] compared the estimation methods for Akash distribution. Tanış and Saraçoğlu [15] provided a comparison of the methods of estimation for transmuted record type Weibull distribution. Karakaya and Tanış [16] discussed the estimation problem of Xgamma-Weibull distribution. Tanış et al. [17] described the estimation methods for transmuted lower record type Fréchet distribution.

The purpose of this paper is to examine some distributional properties and compare five estimation methods such as maximum likelihood, least-squares, weighted least-squares, Anderson-Darling, and Crámer–Von-Mises for transmuted power function distribution [18]. The paper is organized as follows: In Section 2, the transmuted power function distribution and distributional properties are described, such as density and hazard shapes with theorems. Then, some risk measures are defined for transmuted power function distribution in Section 3. Section 4 presents five methods of estimation for point estimation. Section 5 provides an extensive Monte Carlo simulation study to compare these estimation methods. Finally, the conclusions are presented in Section 6.

2. Transmuted Power Function Distribution

Transmuted power function distribution is proposed by Shahzad and Asghar [18] via a quadratic transmutation map (QRTM). The relationship between baseline distribution and transmuted distribution obtained by using QRTM are summarized by

$$F(x) = G(x)\big[1 + \lambda\big(1 - G(x)\big)\big] \tag{1}$$

where $|\lambda| \leq 1$, $G(x)$ denotes the cumulative distribution function (CDF) of baseline distribution, and $F(x)$ refers to the CDF of transmuted distribution newly generated by the QTRM. Consider the baseline distribution power function distribution with CDF $G(x;\beta) = x^{\beta}$ and the probability density function (PDF) $g(x;\beta) = \beta x^{\beta-1}$ then, the PDF and CDF of transmuted power function distribution are as follows:

$$F(x;\beta,\lambda) = x^{\beta}\big\{1 + \lambda\big(1 - x^{\beta}\big)\big\} \tag{2}$$

and

$$f(x;\beta,\lambda) = \beta x^{\beta-1}\big\{1 + \lambda - 2\lambda x^{\beta}\big\} \tag{3}$$

respectively, where $\beta > 0$ is a shape parameter and $-1 \leq \lambda \leq 1$ [18]. Transmuted power function distribution can model the datasets in many fields, such as engineering, economics, hydrology, and social and behavioural sciences. Some statistical properties include mean, mode, median, variance, quantile function, reliability function, hazard function, order statistics, and generalized TL-moments with its special cases L-, TL-, LL LH-moments are described for transmuted power function distribution in [18]. In this paper, transmuted power function distribution is briefly denoted by $TPF(\beta,\lambda)$. Recently, many papers have produced about power function distribution in the literature. Some of these studies are listed as follows: Akhter [19] studied the estimation methods for power function distribution. Tahir et al. [20] proposed a new statistical distribution called Weibull-power function distribution. Okorie et al. [21] introduced the modified power function distribution. Bursa and Özel [22] provided a new extension of power function distribution, called exponentiated Kumaraswamy-power function distribution. Hassan and Salwa [23] proposed a new statistical distribution called exponentiated Weibull-power function distribution. Haq et al. [24] suggested the transmuted Weibull power function distribution. The cubic transmuted power function distribution was introduced by [25]. Arshad et al. [26] suggested the exponentiated power function distribution. Jabenn and Zaka [27] tackled the problem of percentile estimation for power function distribution.

2.1. Density and Hazard Shapes

In this subsection, we discuss the possible shapes of density and hazard for $TPF(\beta,\lambda)$ distribution with some theorems.

Theorem 2.1. PDF of $TPF(\beta,\lambda)$ distribution is unimodal for $\beta > 2$.

Proof. $T_1(x)$ and $T_2(x)$ denote the first and second derivatives of $\log(f_{TPF}(x;\beta,\lambda))$, respectively. They are defined as follows:

$$T_1(x) = \frac{d}{dx}\log(f_{TPF}(x;\beta,\lambda)) = \frac{2(2\beta-1)\lambda x^\beta - (1+\lambda)(\beta-1)}{x(2\lambda x^\beta - \lambda - 1)}$$

and

$$T_2(x) = \frac{d^2}{dx^2}\log(f_{TPF}(x;\beta,\lambda)) = \frac{-8\lambda^2 x^{2\beta}\left(\beta-\frac{1}{2}\right) - (1+\lambda)^2(\beta-1)}{x^2(2\lambda x^\beta - \lambda - 1)^2} - \frac{2\lambda(1+\lambda)(\beta-1)(\beta-2)x^\beta}{x^2(2\lambda x^\beta - \lambda - 1)^2}$$

It is observed that $T_2(x) < 0$ for $\beta > 2$. Then, the density of $TPF(\beta,\lambda)$ distribution is log-concave and unimodal for $\beta > 2$.

□

Figure 1 illustrates the possible shapes of the density of $TPF(\beta,\lambda)$ distribution.

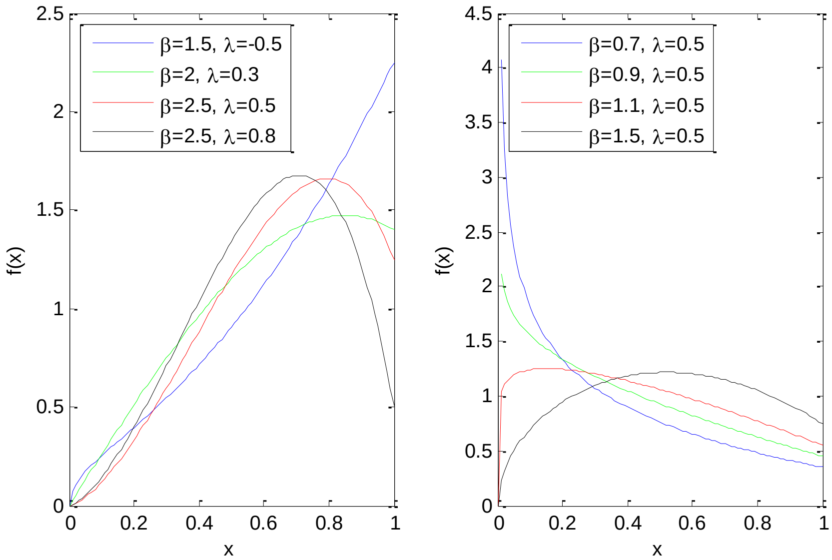

Fig. 1. The density plots of $TPF(\beta,\lambda)$ distribution for selected parameters

Theorem 2.2. The hazard function (HF) of $TPF(\beta,\lambda)$ distribution increases for $\beta > 2$.

Proof.

$$\eta(x) = -\frac{f'(x)}{f(x)} = \frac{2(1-2\beta)\lambda x^\beta + (1+\lambda)(\beta-1)}{x(2\lambda x^\beta - \lambda - 1)}$$

and the first derivative of $\eta(x)$ is defined by

$$\eta'(x) = \frac{d}{dx}\eta(x) = \frac{8\left(\beta-\frac{1}{2}\right)\lambda^2 x^{2\beta} + (1+\lambda)^2(\beta-1)}{x^2(2\lambda x^\beta - \lambda - 1)^2} + \frac{2\lambda(1+\lambda)(\beta-1)(\beta-2)x^\beta}{x^2(2\lambda x^\beta - \lambda - 1)^2}$$

We notice that $\eta(x) > 0$ for $\beta > 2$, and it can be concluded that the HF of $TPF(\beta,\lambda)$ distribution is increasing for $\beta > 2$ according to Glaser [28]. □

Figure 2 shows that the possible shapes of HF of $TPF(\beta,\lambda)$ distribution.

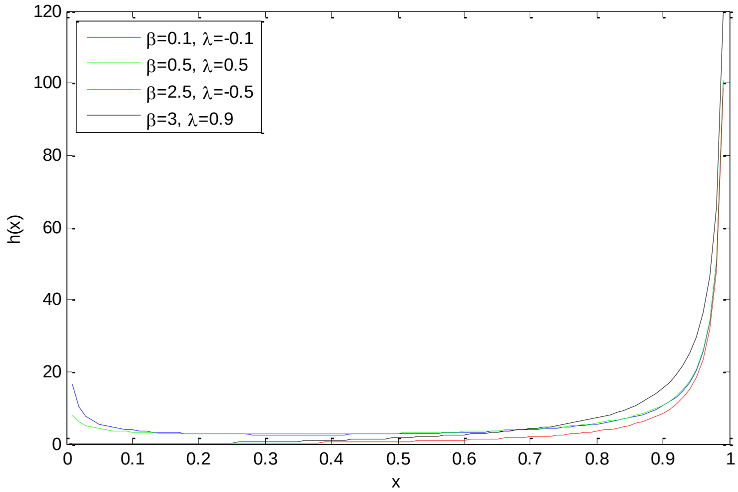

Fig. 2. The hazard plots of $TPF(\beta,\lambda)$ distribution for selected parameters

From Figure 2, we observe that the shape of HF tends to increase. Shahzad and Asghar [18] mention that the $TPF(\beta,\lambda)$ distribution is more flexible than power function distribution since it has an increasing and bathtub-shaped hazard rate.

3. Risk Measures

In this section, we discuss the theoretical and computational aspects of some essential risk measures such as value at risk (VaR), tail value at risk (TVaR), tail variance (TV), and tail variance premium (TVP) for the $TPF(\beta,\lambda)$ distribution.

3.1. VaR Measure

The VaR is a well-known measure of the risk of loss for investments. It is also called quantile risk measure. Firms and regulators generally use the VaR in the financial sector to determine the number of assets required to cover potential losses. The VaR of a random variable X is the q^{th} quantile of its CDF, denoted by VaR_q, and it is defined by $VaR_q = Q(q)$ [29,30].

Let X be a random variable from $TPF(\beta,\lambda)$ distribution, then its VaR can be obtained by

$$VaR_q = \left[\frac{\lambda + 1 - \sqrt{(1+\lambda)^2 - 4\lambda q}}{2\lambda}\right]^{\frac{1}{\beta}} \tag{4}$$

where $q \in (0,1)$.

3.2. TVaR Measure

TVaR, also known as conditional tail expectation, is a significant risk measure. It measures the expected value of the loss given that an event outside a given probability level has occurred. The TVaR of $TPF(\beta,\lambda)$ distribution is

$$TVaR_q = \frac{1}{1-q}\int_{VaR_q}^{1} xf(x)\,dx = \frac{1}{1-q}\left[\frac{(1+\lambda)\beta\left(1 - {VaR_q}^{\beta+1}\right)}{\beta+1} + \frac{2\lambda\beta\left({VaR_q}^{\beta+1} - 1\right)}{2\beta+1}\right] \tag{5}$$

where VaR_q is defined in (4).

3.3. TV Measure

The TV is one of the most significant risk-quantifying measures, which pay attention to the tail variance beyond the VaR. TV was suggested by Landsman [31]. The TV of $TPF(\beta,\lambda)$ distribution is given by

$$\begin{aligned} TV_q(X) &= E\left(X^2 \mid X > x_q\right) - \left\{TVaR_q\right\}^2 \\ &= \frac{1}{1-q}\int_{VaR_q}^{1} x^2 f(x)\,dx - \left\{TVaR_q\right\}^2 \\ &= \frac{1}{1-q}\left[\frac{(1+\lambda)\left(1-{VaR_q}^{\beta+2}\right)}{\beta+2} + \frac{\lambda\beta\left({VaR_q}^{2\beta+2}-1\right)}{\beta+1}\right] - \left\{TVaR_q\right\}^2 \end{aligned} \tag{6}$$

where $TVaR_q$ is defined in (5).

3.4. TVP Measure

The TVP is one of the most used risk measures, which essentially plays a role in insurance sciences. The TVP of $TPF(\beta,\lambda)$ distribution is given as follows:

$$TVP_q = TVaR_q + \mu TV_q \tag{7}$$

where $0 < \mu < 1$, $TVaR_q$, and TV_q are defined in (5) and (6), respectively. Tables 1-2 provide the VaR, TVaR, TV, and TVP of the $TPF(\beta,\lambda)$ distribution for some parameters.

Table 1. VaR, TVaR, TV, and TVP of the $TPF(\beta,\lambda)$ distribution for selected parameters

Parameters	μ	**Significance Level**	**VaR**	**TVaR**	**TV**	**TVP**
		0.7	0.226137	0.460796	0.035095	0.478343
		0.75	0.276244	0.502876	0.031448	0.5186
		0.8	0.337432	0.552153	0.027091	0.565698
$\beta = 0.5, \lambda = 0.9$	0.5	0.85	0.41415	0.61146	0.021889	0.622404
		0.9	0.514985	0.686193	0.015657	0.694022
		0.95	0.661611	0.789652	0.008135	0.79372
		0.99	0.876849	0.930279	0.001204	0.930881
		0.7	0.885733	0.945493	0.001083	0.945818
		0.75	0.907125	0.955287	0.000717	0.955502
		0.8	0.927441	0.964767	0.000438	0.964898
$\beta = 2, \lambda = -0.5$	0.3	0.85	0.946797	0.973958	0.000236	0.974028
		0.9	0.965289	0.982881	0.0001	0.982911
		0.95	0.982999	0.991556	2.41×10^{-5}	0.991563
		0.99	0.996654	0.998329	9.33×10^{-7}	0.998329
		0.7	0.92527	0.964136	0.000463	0.96446
		0.75	0.939076	0.970518	0.000308	0.970733
		0.8	0.952255	0.976718	0.00019	0.976851
$\beta = 5, \lambda = 0.1$	0.7	0.85	0.96488	0.982754	0.000103	0.982826
		0.9	0.977013	0.988638	4.4×10^{-5}	0.988669
		0.95	0.988704	0.994383	1.06×10^{-5}	0.994391
		0.99	0.997771	0.998886	4.14×10^{-7}	0.998887
		0.7	0.398807	0.633135	0.026904	0.649277
		0.75	0.458578	0.674146	0.022134	0.687427
		0.8	0.526709	0.719719	0.017187	0.730031
$\beta = 0.7, \lambda = 0.7$	0.6	0.85	0.605911	0.771214	0.012135	0.778495
		0.9	0.700959	0.830911	0.007135	0.835192
		0.95	0.821919	0.903249	0.002592	0.904804
		0.99	0.955898	0.977402	0.000162	0.977499

Table 2. VaR, TVaR, TV, and TVP of the $TPF(\beta, \lambda)$ distribution for selected parameters

Parameters	μ	Significance Level	VaR	TVaR	TV	TVP
$\beta = 1, \lambda = -0.9$	0.4	0.7	0.82811	0.916594	0.002456	0.917576
		0.75	0.859004	0.931184	0.001654	0.931845
		0.8	0.888889	0.945473	0.001028	0.945884
		0.85	0.917856	0.959482	0.000562	0.959707
		0.9	0.945986	0.97323	0.000243	0.973327
		0.95	0.973348	0.986731	5.92×10^{-5}	0.986754
		0.99	0.994724	0.997364	2.32×10^{-6}	0.997365
$\beta = 15, \lambda = 0.95$	0.9	0.7	0.950062	0.970539	0.000164	0.970687
		0.75	0.956434	0.973997	0.000125	0.97411
		0.8	0.96285	0.977589	9.05×10^{-5}	0.97767
		0.85	0.96947	0.981407	6.12×10^{-5}	0.981462
		0.9	0.976577	0.985628	3.62×10^{-5}	0.98566
		0.95	0.984855	0.990692	1.55×10^{-5}	0.990706
		0.99	0.994485	0.996725	2.24×10^{-6}	0.996727
$\beta = 0.3, \lambda = 0.05$	0.1	0.7	0.289297	0.592721	0.041935	0.596915
		0.75	0.36717	0.645842	0.03329	0.649171
		0.8	0.459168	0.70432	0.024338	0.706754
		0.85	0.566811	0.768554	0.015625	0.770116
		0.9	0.691699	0.838956	0.007918	0.839747
		0.95	0.835509	0.915956	0.002255	0.916181
		0.99	0.96536	0.982606	10^{-4}	0.982616
$\beta = 3, \lambda = 0.5$	0.8	0.7	0.833017	0.913352	0.002269	0.915167
		0.75	0.859061	0.926815	0.001624	0.928114
		0.8	0.885264	0.940485	0.001081	0.941349
		0.85	0.911932	0.954465	0.00064	0.954977
		0.9	0.93947	0.968894	0.000304	0.969136
		0.95	0.968481	0.983968	8.26×10^{-5}	0.984034
		0.99	0.993418	0.996695	3.61×10^{-6}	0.996698

4. Point Estimation

In this section, we consider five estimation methods to estimate the parameters of the $TPF(\beta, \lambda)$ distribution including maximum likelihood, least squares, weighted least squares, Anderson-Darling method, and Cramér-Von Mises method.

Let $X_1, X_2, \cdots, X_n$ be a random sample from the $TPF(\beta,\lambda)$ distribution and $X_{(1)} < X_{(2)} < \cdots < X_{(n)}$ denote the corresponding order statistics. Further, $x_{(i)}$ refers to the observed value of $X_{(i)}$. In this regard, the log-likelihood function of the $TPF(\beta,\lambda)$ distribution is

$$\ell(\theta) = n\log(\beta) + (\beta-1)\sum_{i=1}^{n}\log\left(1+x_i^2\right) + \sum_{i=1}^{n}\log\left(1+\lambda-2\lambda x_i^{\beta}\right) \tag{8}$$

where $\theta = (\beta,\lambda)$ is a parameter vector. Then, the maximum likelihood estimator (MLE) of θ is given as follows:

$$\hat{\theta}_{MLE} = \underset{\theta}{\operatorname{argmax}}\{\ell(\theta)\} \tag{9}$$

Let us define the following four functions, which are used to obtain the different type of estimates:

$$Q_{LS}(\theta) = \sum_{i=1}^{n}\left(\left[x_{(i)}{}^{\beta}\left\{1+\lambda\left(1-x_{(i)}{}^{\beta}\right)\right\}\right] - \frac{i}{n+1}\right)^2,$$

$$Q_{WLS}(\theta) = \sum_{i=1}^{n}\frac{(n+2)(n+1)^2}{i(n-i+1)}\left(\left[x_{(i)}{}^{\beta}\left\{1+\lambda\left(1-x_{(i)}{}^{\beta}\right)\right\}\right] - \frac{i}{n+1}\right)^2,$$

$$Q_{CvM}(\theta) = \frac{1}{12n} + \sum_{i=1}^{n}\left(\left[x_{(i)}{}^{\beta}\left\{1+\lambda\left(1-x_{(i)}{}^{\beta}\right)\right\}\right] - \frac{2i-1}{2n}\right)^2,$$

and

$$Q_{AD}(\theta) = -n - \frac{1}{n}\sum_{i=1}^{n}\left((2i-1)\log\left[x_{(i)}{}^{\beta}\left\{1+\lambda\left(1-x_{(i)}{}^{\beta}\right)\right\}\right]\right) + \frac{1}{n}\sum_{i=1}^{n}\left(\log\left(1-\left[x_{(i)}{}^{\beta}\left\{1+\lambda\left(1-x_{(i)}{}^{\beta}\right)\right\}\right]\right)\right)$$

The least squares estimators (LSEs), weighted least squares estimators (WLSEs), Cramér–von Mises estimators (CvMEs), and Anderson-Darling estimators (ADEs) of the parameters $\theta = (\beta,\lambda)$ are given, respectively, by

$$\hat{\theta}_{LSE} = \underset{\theta}{\operatorname{argmin}}\{Q_{LS}(\theta)\} \tag{10}$$

$$\hat{\theta}_{WLSE} = \underset{\theta}{\operatorname{argmin}}\{Q_{WLS}(\theta)\} \tag{11}$$

$$\hat{\theta}_{CvME} = \underset{\theta}{\operatorname{argmin}}\{Q_{CvM}(\theta)\} \tag{12}$$

$$\hat{\theta}_{ADE} = \underset{\theta}{\operatorname{argmin}}\{Q_{AD}(\theta)\} \tag{13}$$

The estimators given in (9)-(13) can be obtained by **optim ()** function in R with the BFGS algorithm.

5. Simulation Study

In this section, we perform a comprehensive Monte Carlo simulation study to compare the performances of MLEs, LSEs, WLSEs, CvMEs, and ADEs of β and λ according to biases and MSEs. The simulation study is performed based on 1000 repetitions. We consider the sample size 50, 100, 250, 500, 1000, and four-parameter settings as follows:

$$(\beta = 2, \lambda = 0.5), (\beta = 0.9, \lambda = -0.5), (\beta = 1, \lambda = 0.7), (\beta = 1.5, \lambda = -0.7)$$

BFGS algorithm is performed to get the five estimates given in (9)-(13). Tables 3 and 4 provide the biases and MSEs of five estimators for selected parameters and sample sizes.

Table 3. Average biases of MLEs, LSEs, WLSEs, ADEs, and CvMEs of β and λ parameters

				$\hat{\beta}$					$\hat{\lambda}$		
Parameters	$\boldsymbol{n}$	**MLE**	**LSE**	**WLSE**	**ADE**	**CvME**	**MLE**	**LSE**	**WLSE**	**ADE**	**CvME**
	50	0.0777	-0.0300	-0.0071	0.0178	0.0650	-0.0592	-0.1386	-0.1235	-0.1015	-0.0573
	100	0.0518	0.0068	0.0256	0.0317	0.0550	-0.0338	-0.0655	-0.0515	-0.0456	-0.0234
$\beta = 2, \lambda = 0.5$	250	0.0388	0.0189	0.0273	0.0293	0.0370	-0.0245	-0.0388	-0.0334	-0.0316	-0.0234
	500	0.0332	0.0217	0.0274	0.0270	0.0305	-0.0136	-0.0221	-0.0174	-0.0177	-0.0145
	1000	0.0167	0.0088	0.0125	0.0123	0.0132	-0.0160	-0.0216	-0.0188	-0.0189	-0.0179
	50	0.1899	0.0331	0.1018	0.0337	0.0205	0.1487	-0.1357	-0.0043	-0.1118	-0.1762
	100	0.1330	0.0128	0.0317	0.0335	0.0027	0.0853	-0.1309	-0.0827	-0.0789	-0.1574
$\beta = 0.9, \lambda = -0.5$	250	0.0688	-0.0191	0.0354	0.0051	-0.0255	0.0321	-0.1272	-0.0313	-0.0770	-0.1418
	500	0.0496	-0.0152	0.0171	0.0138	-0.0207	0.0227	-0.0959	-0.0377	-0.0430	-0.1066
	1000	0.0222	-00022	0.0040	0.0248	-0.0032	-0.0043	-00576	-0.0410	-0.0092	-0.0603
	50	0.0234	-0.0179	-0.0082	-0.0003	0.0237	-0.1228	-0.1806	-0.1685	-0.1531	-0.1043
	100	0.0215	0.0006	0.0092	0.0098	0.0197	-0.0652	-0.0958	-0.0818	-0.0820	-0.0608
$\beta = 1, \lambda = 0.7$	250	0.0226	0.0156	0.0185	0.0185	0.0229	-0.0238	-0.0337	-0.0302	-0.0293	-0.0199
	500	0.0134	0.0091	0.0111	0.0108	0.0128	-0.0210	-0.0269	-0.0237	-0.0244	-0.0201
	1000	0.0094	0.0081	0.0090	0.0088	0.0099	-0.0137	-0.0157	-0.0142	-0.0146	-0.0123
	50	0.3421	0.2711	0.2529	0.2517	0.2234	0.1521	0.0240	0.0402	0.0493	-0.0327
	100	0.2050	0.1831	0.1782	0.1740	0.1507	0.0780	0.0069	0.0259	0.0278	-0.0303
$\beta = 1.5, \lambda = -0.7$	250	0.1492	0.1341	0.1300	0.1271	0.1134	0.0590	0.0192	0.0281	0.0272	-0.0025
	500	0.0923	0.1046	0.0884	0.0860	0.0908	0.0287	0.0204	0.0158	0.0141	0.0065
	1000	0.0712	0.0796	0.0924	0.0684	0.0693	0.0268	0.0220	0.0372	0.0183	00122

Table 4. Average MSEs of MLEs, LSEs, WLSEs, ADEs, and CvMEs of β and λ parameters

				$\hat{\beta}$					$\hat{\lambda}$		
Parameters	$\boldsymbol{n}$	**MLE**	**LSE**	**WLSE**	**ADE**	**CvME**	**MLE**	**LSE**	**WLSE**	**ADE**	**CvME**
	50	0.1403	0.1725	0.1581	0.1431	0.1751	0.1192	0.1655	0.1556	0.1347	0.1412
	100	0.0752	0.0917	0.0794	0.0731	0.0899	0.0726	0.0837	0.0744	0.0670	0.0722
$\beta = 2, \lambda = 0.5$	250	0.0276	0.0335	0.0321	0.0302	0.0341	0.0228	0.0286	0.0295	0.0266	0.0270
	500	0.0144	0.0174	0.0149	0.0148	0.0178	0.0116	0.0142	0.0121	0.0121	0.0138
	1000	0.0069	0.0085	0.0074	0.0074	0.0086	0.0056	0.0069	0.0060	0.0060	0.0067
	50	0.1018	0.0563	0.0664	0.0506	0.0688	0.1655	0.1422	0.1289	0.1175	0.2011
	100	0.0692	0.0400	0.0380	0.0372	0.0465	0.1110	0.1164	0.0960	0.0913	0.1454
$\beta = 0.9, \lambda = -0.5$	250	0.0377	0.0273	0.0308	0.0254	0.0301	0.0768	0.0865	0.0730	0.0700	0.0982
	500	0.0258	0.0210	0.0219	0.0215	0.0224	0.0551	0.0640	0.0571	0.0563	0.0700
	1000	0.0147	0.0175	0.0149	0.0167	0.0185	0.0358	0.0460	0.0398	0.0387	0.0487
	50	0.0274	0.0354	0.0308	0.0276	0.0349	0.0951	0.1450	0.1281	0.1093	0.1141
	100	0.0129	0.0168	0.0135	0.0138	0.0172	0.0416	0.0629	0.0473	0.0513	0.0574
$\beta = 1, \lambda = 0.7$	250	0.0053	0.0064	0.0060	0.0055	0.0067	0.0149	0.0204	0.0201	0.0161	0.0194
	500	0.0025	0.0030	0.0026	0.0026	0.0031	0.0075	0.0097	0.0081	0.0080	0.0094
	1000	0.0012	0.0016	0.0013	0.0013	0.0016	0.0036	0.0050	0.0040	0.0040	0.0049
	50	0.2602	0.2566	0.2013	0.1731	0.2609	0.0762	0.0947	0.0582	0.0380	0.1220
	100	0.1212	0.1695	0.1314	0.1156	0.1741	0.0524	0.0855	0.0544	0.0414	0.1001
$\beta = 1.5, \lambda = -0.7$	250	0.0988	0.1104	0.0899	0.0838	0.1110	0.0504	0.0635	0.0454	0.0407	0.0681
	500	0.0648	0.0850	0.0638	0.0612	0.0842	0.0388	0.0515	0.0363	0.0346	0.0531
	1000	0.0481	0.0618	0.0624	0.0470	0.0604	0.0304	0.0392	0.0359	0.0285	0.0391

As a result of the simulation study, we observe that, with increasing sample sizes, the MSEs and biases decrease as expected. From Table 3 and 4, it has been general observed that the MLE has a smaller MSE compared to other estimators for both β and λ parameters. However, there are some situations that ADE and CvME have a smaller bias than MLE. As a result, we recommend MLE for point estimation of $TPF(\beta,\lambda)$ distribution. However, ADE and CvME can be good alternatives to MLE to estimate the parameters of $TPF(\beta,\lambda)$ distribution.

6. Conclusion

In this study, $TPF(\beta,\lambda)$ distribution proposed by Shahzad and Asghar [18] is studied in terms of some characteristic properties and statistical inferences. Some critical risk measures are discussed and numerically obtained for $TPF(\beta,\lambda)$ distribution. We compare five estimators of parameters of $TPF(\beta,\lambda)$ distribution, such as MLE, LSE, WSE, ADE, and CvME via Monte Carlo simulations. In the simulation study, it is seen that MLE is the best estimator among others according to MSE criteria. We recommend MLE to estimate the parameters of $TPF(\beta,\lambda)$ distribution.

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