



Spectrum and the Jost Solution of Discrete Impulsive Klein-Gordon Equation with Hyperbolic Eigenparameter

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Highlights

- This paper focuses on discrete impulsive Klein-Gordon equation with hyperbolic eigenparameter.
- Several spectral properties of the spectrum and the scattering function are investigated.
- An example about the scattering function and the existence of eigenvalues is given.

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Abstract

Let L denote the quadratic pencil of difference operator with boundary and impulsive conditions generated in $\ell_2(\mathbb{N})$ by

$$\Delta (a_{n-1} \Delta y_{n-1}) + (q_n + 2\lambda p_n + \lambda^2)y_n = 0, \quad n \in \mathbb{N} \setminus \{k-1, k, k+1\},$$

$$y_0 = 0,$$

$$\begin{pmatrix} y_{k+1} \\ \Delta y_{k+1} \end{pmatrix} = \theta \begin{pmatrix} y_{k-1} \\ \nabla y_{k-1} \end{pmatrix}; \quad \theta = \begin{pmatrix} \theta_1 & \theta_2 \\ \theta_3 & \theta_4 \end{pmatrix}, \{\theta_i\}_{i=1,2,3,4} \in \mathbb{R}$$

where $\{a_n\}_{n \in \mathbb{N}}$, $\{p_n\}_{n \in \mathbb{N}}$, $\{q_n\}_{n \in \mathbb{N}}$ are real sequences, $\lambda = 2 \cosh\left(\frac{z}{2}\right)$ is a hyperbolic eigenparameter and Δ , ∇ are respectively forward and backward operators. In this paper, the spectral properties of L such as the spectrum, the eigenvalues, the scattering function and their properties are investigated. Moreover, an example about the scattering function and the existence of eigenvalues is given in the special cases, if $\sum_{n=1}^{\infty} n(|1 - a_n| + |p_n| + |q_n|) < \infty$.

1. INTRODUCTION

Spectral theory of the difference equations related with some operators and their inverses is one of the basic branches of modern functional analysis. It is very important in terms of helping to understand many daily events about biology, economics, engineering and physics, especially quantum tunneling and resonances. Because of that, spectrum of the difference equations with spectral singularities have been widely debated by many mathematicians from the recent past to the present ([1-10]).

Let consider the Klein-Gordon problem (KGP)

$$\begin{aligned} y'' + [\lambda - p(x)]^2 y &= f(x), \quad x \in \mathbb{R}_+ = [0, \infty] \\ \alpha y'(0) - \beta y(0) &= 0 \end{aligned} \tag{1}$$

for a particle zero mass with static complex potential p and complex-valued function f where $\alpha, \beta \in \mathbb{C}$ with $|\alpha| + |\beta| \neq 0$ and λ is a spectral parameter. The Jost solution and the spectral properties of the KGP (1) has been investigated in [11] and it is shown that it has finite number of eigenvalues and spectral singularities with finite multiplicities. In addition, the spectral analysis of a quadratic pencil of Schrödinger operator has been investigated in [12,13]. In that studies, the authors worked on the spectrum, principal functions corresponding to spectral singularities and an eigenfunction expansion of it. Many other studies related to these equation can be seen in [14-19]. Furthermore, the impulsive difference equations, which

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constitute a model for many problems that arise with momentary changes caused by external factors in fields such as pharmacokinetics, ecology, control theory and mechanics, have been the subject of many researchers to understand the nature of such dynamics. The studies of the spectral theory of such equations has become one of the fastest developing branches of mathematics in terms of difference equations ([20-25]).

Now, the quadratic pencil difference operator L in $\ell_2(\mathbb{N})$ denoted by

$$\Delta (a_{n-1} \Delta y_{n-1}) + (q_n + 2\lambda p_n + \lambda^2)y_n = 0, \quad n \in \mathbb{N} \setminus \{k-1, k, k+1\}, \quad (2)$$

with boundary and impulsive conditions

$$y_0 = 0, \quad (3)$$

$$\begin{pmatrix} y_{k+1} \\ \Delta y_{k+1} \end{pmatrix} = \theta \begin{pmatrix} y_{k-1} \\ \nabla y_{k-1} \end{pmatrix}; \quad \theta = \begin{pmatrix} \theta_1 & \theta_2 \\ \theta_3 & \theta_4 \end{pmatrix}, \{\theta_i\}_{i=1,2,3,4} \in \mathbb{R} \quad (4)$$

where \mathbb{N} is set of natural numbers, $\{a_n\}_{n \in \mathbb{N}}$, $\{p_n\}_{n \in \mathbb{N}}$, $\{q_n\}_{n \in \mathbb{N}}$ are real sequences, $a_n \neq 0$ for all $n \in \mathbb{N} \cup \{0\}$, $\det \theta \neq 0$, λ is a hyperbolic eigenparameter and Δ , ∇ are respectively forward and backward operators. The situation that makes this paper, which is one of the articles have applicability in a lot of branches of bursting rhythm models in medicine and biology, optimal control models in economics, pharmacokinetics and frequency modulated systems, different from other studies in the literature, is the paper of an discrete impulsive Klein-Gordon equation with hyperbolic eigenparameter $2 \cosh\left(\frac{z}{2}\right)$. In this study, various spectral properties of L ; i.e., the spectrum, the scattering function and their properties are investigated if

$$\sum_{n=1}^{\infty} n(|1 - a_n| + |p_n| + |q_n|) < \infty. \quad (5)$$

2. SCATTERING FUNCTION OF L

Let two semi-strips $T_- = \{z \in \mathbb{C}: z = \xi + i\tau, \xi < 0, \tau \in [0, 4\pi]\}$ and $T = T_- \cup T_0$ are defined in the closure of $\mathbb{C}_{left} = \{z \in \mathbb{C}: \operatorname{Re} z < 0\}$ where $T_0 = \{z \in \mathbb{C}: z = i\tau, \tau \in [0, 4\pi]\}$. Clearly, $T = \{z \in \mathbb{C}: z = \xi + i\tau, \xi \leq 0, \tau \in [0, 4\pi]\}$. Under the Condition (5), Equation (2) has the solution

$$f_n(z) = a_n e^{nz} \left(1 + \sum_{m=1}^{\infty} K_{nm} e^{m\frac{z}{2}}\right), \quad n \in \{k+1, k+2, \dots\},$$

satisfying the condition $\lim_{n \rightarrow \infty} e^{-nz} f_n(z) = 1$, $z \in T_-$ for $\lambda = 2 \cosh\left(\frac{z}{2}\right)$, and α_n, K_{nm} are expressed in terms of $\{a_n\}_{n \in \mathbb{N}}$, $\{p_n\}_{n \in \mathbb{N}}$, and $\{q_n\}_{n \in \mathbb{N}}$ as

$$\alpha_n = \left[\prod_{\xi=n}^{\infty} (-a_{\xi})\right]^{-1}$$

$$K_{n,1} = 2 \sum_{\xi=n+1}^{\infty} p_{\xi}$$

$$K_{n,2} = \sum_{\xi=n+1}^{\infty} (2 - a_{\xi} - a_{\xi-1}) + 2 \sum_{\xi=n+1}^{\infty} p_{\xi} K_{\xi,1}$$

$$K_{n,3} = \sum_{\xi=n+1}^{\infty} (2 - a_{\xi} - a_{\xi-1}) K_{\xi,1} + 2 \sum_{\xi=n+1}^{\infty} p_{\xi} (K_{\xi,2} + 1)$$

$$K_{n,4} = \sum_{\xi=n+1}^{\infty} [(1 - a_{\xi}^2) + (2 - a_{\xi} - a_{\xi-1}) K_{\xi,2}] + 2 \sum_{\xi=n+1}^{\infty} p_{\xi} (K_{\xi,1} + K_{\xi,3})$$

$$K_{n,m+4} = K_{nm} + \sum_{\xi=n+1}^{\infty} [(1 - a_{\xi}^2) K_{\xi+1,m} + (2 - a_{\xi} - a_{\xi-1}) K_{\xi,m+2}] + 2 \sum_{\xi=n+1}^{\infty} p_{\xi} (K_{\xi,m+1} + K_{\xi,m+3})$$

for $m = 1, 2, \dots$ by following up [26]. Moreover

$$|K_{nm}| \leq \delta \sum_{k=n+\lfloor \frac{m}{2} \rfloor}^{\infty} (|1 - a_k| + |p_k| + |q_k|)$$

holds, where $\delta > 0$ is constant and $\lfloor \frac{m}{2} \rfloor$ is the integer part of $\frac{m}{2}$. Hence, $4\pi i$ periodic function $f_n(z)$ is analytic with respect to z in \mathbb{C}_{left} and continuous on the real axis.

Also, the elementary solutions of (2) $\{\varphi_n(z)\}$ and $\{\psi_n(z)\}$, $n = 0, 1, \dots, k-1$ are considered for $z \in T$ subject to the initial conditions

$$\begin{aligned} \varphi_0(z) &= 0, & \varphi_1(z) &= 1, \\ \psi_0(z) &= a_0^{-1}, & \psi_1(z) &= 0. \end{aligned} \quad (6)$$

These functions are entire for $z \in \mathbb{C}$ and the Wronskian of them can be found as

$$\begin{aligned} W[\varphi_n(z), \psi_n(z)] &= a_n[\varphi_n(z)\psi_{n+1}(z) - \varphi_{n+1}(z)\psi_n(z)] \\ &= a_0[\varphi_0(z)\psi_1(z) - \varphi_1(z)\psi_0(z)] \\ &= -1. \end{aligned}$$

In here, the Jost solution of L is defined by using $f_n(z)$, $\varphi_n(z)$ and $\psi_n(z)$

$$J_n(z) = \begin{cases} p^{(1)}(z)\varphi_n(z) + p^{(2)}(z)\psi_n(z) & ; n = 0, 1, \dots, k-1 \\ f_n(z) & ; n = k+1, k+2, \dots \end{cases} \quad (7)$$

for $z \in T$ with

$$p^{(1)}(z) = -\frac{a_{k-2}}{\det(\theta B^{-1})}, \quad p^{(2)}(z) = \frac{a_{k-2}}{\det(\theta C^{-1})}$$

where

$$B = \begin{pmatrix} f_{k+1}(z) & \Delta f_{k+1}(z) \\ \theta_1 \psi_{k-1}(z) + \theta_2 \nabla \psi_{k-1}(z) & \theta_3 \psi_{k-1}(z) + \theta_4 \nabla \psi_{k-1}(z) \end{pmatrix},$$

$$C = \begin{pmatrix} f_{k+1}(z) & \Delta f_{k+1}(z) \\ \theta_1 \varphi_{k-1}(z) + \theta_2 \nabla \varphi_{k-1}(z) & \theta_3 \varphi_{k-1}(z) + \theta_4 \nabla \varphi_{k-1}(z) \end{pmatrix}$$

obtained from (4). Moreover,

$$\begin{aligned} W[f_n(z), f_n(-z)] &= \lim_{n \rightarrow \infty} \{a_n[f_n(z)f_{n+1}(-z) - f_{n+1}(z)f_n(-z)]\} \\ &= \lim_{n \rightarrow \infty} \{a_n[e^{-z}f_n(z)e^{-nz}f_{n+1}(-z)e^{(n+1)z} - e^z f_{n+1}(z)e^{-(n+1)z}f_n(-z)e^{nz}]\} \\ &= e^{-z} - e^z \\ &= -2 \sinh z \end{aligned} \quad (8)$$

for $z \in T_0 \setminus \{k\pi i: k = 0, 1, 2, 3, 4\}$. If

$$F_n(z) = \begin{cases} \varphi_n(z) & ; n = 0, 1, \dots, k-1 \\ q^{(1)}(z)f_n(z) + q^{(2)}(z)f_n(-z) & ; n = k+1, k+2, \dots \end{cases} \quad (9)$$

be the another solution of L for $z \in T_0 \setminus \{k\pi i: k = 0, 1, 2, 3, 4\}$ with

$$q^{(1)}(z) = \frac{a_{k+1}}{2 \sinh z} \det D, \quad q^{(2)}(z) = -\frac{a_{k+1}}{2 \sinh z} \det C$$

where

$$D = \begin{pmatrix} f_{k+1}(-z) & \Delta f_{k+1}(-z) \\ \theta_1 \varphi_{k-1}(z) + \theta_2 \nabla \varphi_{k-1}(z) & \theta_3 \varphi_{k-1}(z) + \theta_4 \nabla \varphi_{k-1}(z) \end{pmatrix},$$

then $q^{(2)}(z) = q^{(1)}(-z) = \overline{q^{(1)}(z)}$ due to $\varphi_n(-z) = \varphi_n(z)$. So, the following result can be expressed:

Lemma 2.1.

$$W[J_n(z), F_n(z)] = \begin{cases} p^{(2)}(z) & ; n = 0, 1, \dots, k-1 \\ \det \theta \frac{a_{k+1}}{a_{k-2}} p^{(2)}(z) & ; n = k+1, k+2, \dots \end{cases}$$

for $z \in T_0 \setminus \{k\pi i: k = 0, 1, 2, 3, 4\}$.

Proof. From (6)-(9),

$$\begin{aligned} W[J_n(z), F_n(z)] &= a_0 [J_0(z)F_1(z) - J_1(z)F_0(z)] \\ &= p^{(2)}(z) \end{aligned}$$

for $n = 0, 1, \dots, k-1$ and

$$\begin{aligned} W[J_n(z), F_n(z)] &= a_{k+1} [J_{k+1}(z)F_{k+2}(z) - J_{k+2}(z)F_{k+1}(z)] \\ &= a_{k+1} q^{(2)}(z) [f_{k+1}(z)f_{k+2}(-z) - f_{k+2}(z)f_{k+1}(-z)] \\ &= q^{(2)}(z) (-2 \sinh z) \\ &= \det \theta \frac{a_{k+1}}{a_{k-2}} p^{(2)}(z) \end{aligned}$$

for $n = k+1, k+2, \dots$ because $q^{(2)}(z) = -\frac{a_{k+1}}{a_{k-2}} \frac{\det \theta}{2 \sinh z} p^{(2)}(z)$.

In addition, if $\tilde{f}_n(z)$ is unbounded solution of Equation (2) for $n = k+1, k+2, \dots$ with $\lim_{n \rightarrow \infty} e^{nz} \tilde{f}_n(z) = 1, z \in \overline{\mathbb{C}}_{left}$, then

$$W[f_n(z), \tilde{f}_n(z)] = -2 \sinh z$$

in $z \in T \setminus \{k\pi i: k = 0, 1, 2, 3, 4\}$, and the another solution of L

$$G_n(z) = \begin{cases} \varphi_n(z) & ; n = 0, 1, \dots, k-1 \\ r^{(1)}(z)f_n(z) + r^{(2)}(z)\tilde{f}_n(z) & ; n = k+1, k+2, \dots \end{cases} \quad (10)$$

can be defined for $z \in T$ with

$$r^{(1)}(z) = \frac{a_{k+1}}{2 \sinh z} \det E, \quad r^{(2)}(z) = -\frac{a_{k+1}}{2 \sinh z} \det C$$

where

$$E = \begin{pmatrix} \tilde{f}_{k+1}(z) & \Delta \tilde{f}_{k+1}(z) \\ \theta_1 \varphi_{k-1}(z) + \theta_2 \nabla \varphi_{k-1}(z) & \theta_3 \varphi_{k-1}(z) + \theta_4 \nabla \varphi_{k-1}(z) \end{pmatrix}.$$

Also, the function $G_n(z)$ is unbounded solution of L and

$$r^{(2)}(z) = q^{(2)}(z) = \overline{q^{(1)}(z)} = -\frac{a_{k+1}}{a_{k-2}} \frac{\det \theta}{2 \sinh z} p^{(2)}(z) \quad (11)$$

for $z \in T_0 \setminus \{k\pi i: k = 0,1,2,3,4\}$.

Theorem 2.2. For all z in $T_0 \setminus \{k\pi i: k = 0,1,2,3,4\}$, $p^{(2)}(z) \neq 0$.

Proof. If $p^{(2)}(z_0) = 0$ for at least $z_0 \in T_0 \setminus \{k\pi i: k = 0,1,2,3,4\}$, then $q^{(1)}(z_0) = q^{(2)}(z_0) = 0$ from (11) and hence $F_n(z_0) = 0, n \in \mathbb{N} \cup \{0\}$ is trivial by using the Impulsive Conditions (4) which is a contradiction.

Definiton 2.1. The scattering function of L is defined by

$$S(z) = \frac{\overline{J_0(z)}}{J_0(z)}$$

with respect to the Jost solution of L .

From the definiton of $S(z)$, it can be written that

$$S(z) = \frac{\overline{p^{(2)}(z)}}{p^{(2)}(z)} = \frac{\overline{a_0 J_0(z)}}{a_0 J_0(z)} = \frac{J_0(-z)}{J_0(z)} = \frac{p^{(2)}(-z)}{p^{(2)}(z)} \quad (12)$$

since $\{a_n\}_{n \in \mathbb{N}}$ is nonzero real sequence for all $n \in \mathbb{N} \cup \{0\}$ and $\overline{J_n(z)} = J_n(-z)$ for $z \in T_0 \setminus \{k\pi i: k = 0,1,2,3,4\}$. Therefore,

$$S(z) = \frac{f_{k+1}(-z)[\theta_3 \varphi_{k-1}(z) + \theta_4 \nabla \varphi_{k-1}(z)] - [\theta_1 \varphi_{k-1}(z) + \theta_2 \nabla \varphi_{k-1}(z)] \Delta f_{k+1}(-z)}{f_{k+1}(z)[\theta_3 \varphi_{k-1}(z) + \theta_4 \nabla \varphi_{k-1}(z)] - [\theta_1 \varphi_{k-1}(z) + \theta_2 \nabla \varphi_{k-1}(z)] \Delta f_{k+1}(z)} \quad (13)$$

and so $\lim_{z \rightarrow 0} S(z) = S(0) = 1$. Additionally,

$$S(-z) = \frac{J_0(z)}{J_0(-z)} = S^{-1}(z) = \overline{S(z)}$$

for $z \in T_0 \setminus \{k\pi i: k = 0,1,2,3,4\}$ by considering the Equalities (12).

Theorem 2.2. Let $\sigma_d(L)$ and $\sigma_{ss}(L)$ are the sets of eigenvalues and spectral singularities of L , respectively. Then,

$$\sigma_d(L) = \left\{ \lambda \in \mathbb{C}: \lambda = 2 \cosh \frac{z}{2}, z \in T_-, p^{(2)}(z) = 0 \right\},$$

$$\sigma_{ss}(L) = \emptyset.$$

Proof. From (7), the Jost solution $J_n(z) \in \ell_2(\mathbb{N})$. Because, the first part of it consists of a finite number of elements and $f_n(z)$ is in $\ell_2(\mathbb{N})$, too. Furthermore, by using the Condition (3),

$$0 = J_0(z) = p^{(1)}(z)\varphi_0(z) + p^{(2)}(z)\psi_0(z) = \frac{p^{(2)}(z)}{a_0}$$

and then $p^{(2)}(z) = 0$. So, from the definition of spectral singularities and eigenvalues in [27] and Theorem 2.2.,

$$\sigma_d(L) = \left\{ \lambda \in \mathbb{C}: \lambda = 2 \cosh \frac{z}{2}, z \in T_-, p^{(2)}(z) = 0 \right\}$$

and

$$\sigma_{ss}(L) = \left\{ \lambda \in \mathbb{C}: \lambda = 2 \cosh \frac{z}{2}, z \in T_0 \setminus \{k\pi i: k = 0,1,2,3,4\}, p^{(2)}(z) = 0 \right\}$$

$$= \emptyset.$$

Note that, the Wronskian of $J_n(z)$ and $G_n(z)$ can be written from (7) and (10) as

$$W[J_n(z), G_n(z)] = \begin{cases} p^{(2)}(z) & ; n = 0, 1, \dots, k-1 \\ \det \theta \frac{a_{k+1}}{a_{k-2}} p^{(2)}(z) & ; n = k+1, k+2, \dots \end{cases}$$

for $z \in T$. For this reason, since the last theorem is taken into account, the quantitative properties of zeros of $p^{(2)}(z)$ in T_- are required to investigate the quantitative properties of the eigenvalues of L .

3. AN EXAMPLE

Let L_1 denote the operator in $\ell_2(\mathbb{N})$ generated by the equation

$$y_{n-1} + y_{n+1} = (2 - \lambda^2)y_n, \quad n \in \mathbb{N} \setminus \{3, 4, 5\}, \quad (14)$$

with boundary and impulsive conditions

$$\begin{aligned} y_0 &= 0, \\ \begin{pmatrix} y_5 \\ \Delta y_5 \end{pmatrix} &= \theta \begin{pmatrix} y_3 \\ \nabla y_3 \end{pmatrix} \end{aligned} \quad (15)$$

where $\det \theta \neq 0$ for $\theta = \begin{pmatrix} \theta_1 & \theta_2 \\ \theta_3 & \theta_4 \end{pmatrix}$, $\{\theta_i\}_{i=1,2,3,4} \in \mathbb{R}$ and $\lambda = 2 \cosh\left(\frac{z}{2}\right)$ is a hyperbolic eigenparameter. Moreover, $\{\varphi_n(z)\}$ and $\{\psi_n(z)\}$, $n = 0, 1, 2, 3$ are the elementary solutions of (14) for $z \in T$ subject to the Initial Conditions (6) and clearly $f_n(z) = \alpha_n e^{nz}$ where $\alpha_{n+1} = -\alpha_n$. Also, it can be written from (14) and (15)

$$\begin{aligned} \varphi_2(\lambda) &= 2 - \lambda^2, & \varphi_3(\lambda) &= \lambda^4 - 4\lambda^2 + 3 \\ \psi_2(\lambda) &= -1, & \psi_3(\lambda) &= \lambda^2 - 2 \end{aligned}$$

and

$$\begin{aligned} p^{(1)}(z) &= \frac{1}{\det \theta} \{[\theta_1 \psi_3(z) + \theta_2 \nabla \psi_3(z)] \Delta f_5(z) - f_5(z)[\theta_3 \psi_3(z) + \theta_4 \nabla \psi_3(z)]\} \\ &= -\frac{\alpha_5}{\det \theta} e^{4z} [(\theta_1 + \theta_2)e^{3z} + (\theta_1 + 2\theta_2 + \theta_3 + \theta_4)e^{2z} + (\theta_1 + 2\theta_2 + \theta_4)e^z \\ &\quad + (\theta_1 + \theta_2 + \theta_3 + \theta_4)], \end{aligned}$$

$$\begin{aligned} p^{(2)}(z) &= -\frac{1}{\det \theta} \{[\theta_1 \varphi_3(z) + \theta_2 \nabla \varphi_3(z)] \Delta f_5(z) - f_5(z)[\theta_3 \varphi_3(z) + \theta_4 \nabla \varphi_3(z)]\} \\ &= \frac{\alpha_5}{\det \theta} e^{3z} [(\theta_1 + \theta_2)e^{5z} + (\theta_1 + \theta_3 + \theta_4)e^{4z} + (\theta_1 - \theta_4)e^{3z} + (\theta_1 + \theta_3 + \theta_4)e^{2z} \\ &\quad + (\theta_1 - \theta_4)e^z + (\theta_1 + \theta_2 + \theta_3 + \theta_4)]. \end{aligned}$$

By following the Equality (7),

$$J_n(z) = \begin{cases} \alpha_5 e^{5z} [p^{(11)}(z)\varphi_n(z) + p^{(21)}(z)\psi_n(z)] & ; n = 0, 1, 2, 3 \\ -\alpha_n e^{nz} & ; n = 5, 6, 7, \dots \end{cases}$$

is the Jost solution of L_1 where

$$\begin{aligned} p^{(11)}(z) &= -\frac{[\theta_1(\lambda^2-2)+\theta_2(\lambda^2-1)](e^z+1)+\theta_3(\lambda^2-2)+\theta_4(\lambda^2-1)}{\det \theta}, \\ p^{(21)}(z) &= \frac{[\theta_1(\lambda^4-4\lambda^2+3)+\theta_2(\lambda^4-3\lambda^2+1)](e^z+1)+\theta_3(\lambda^4-4\lambda^2+3)+\theta_4(\lambda^4-3\lambda^2+1)}{\det \theta}. \end{aligned}$$

From (13), the scattering function of L_1 is

$$S(z) = \frac{[\theta_1\varphi_3(z)+\theta_2\nabla\varphi_3(z)]\Delta f_5(-z)-f_5(-z)[\theta_3\varphi_3(z)+\theta_4\nabla\varphi_3(z)]}{[\theta_1\varphi_3(z)+\theta_2\nabla\varphi_3(z)]\Delta f_5(z)-f_5(z)[\theta_3\varphi_3(z)+\theta_4\nabla\varphi_3(z)]}$$

$$= e^{-6z} \left[\frac{t(-z)}{t(z)} \right]$$

for $z \in T_0 \setminus \{k\pi i: k = 0,1,2,3,4\}$ where

$$t(z) = [(\theta_1 + \theta_2)e^{5z} + (\theta_1 + \theta_3 + \theta_4)e^{4z} + (\theta_1 - \theta_4)e^{3z} + (\theta_1 + \theta_3 + \theta_4)e^{2z} + (\theta_1 - \theta_4)e^z + (\theta_1 + \theta_2 + \theta_3 + \theta_4)].$$

In addition, the eigenvalues of L_1 is

$$\sigma_d(L_1) = \left\{ \lambda \in \mathbb{C}: \lambda = 2 \cosh \frac{z}{2}, z \in T_-, p^{(2)}(z) = 0 \right\},$$

so

$$t(z) = 0 \tag{16}$$

can be obtained because $\lambda = 2 \cosh \frac{z}{2}$ and $p^{(2)}(z) = 0$ in $\sigma_d(L_1)$.

Case 1: If $\theta = I_2$ where I_2 is 2×2 identity matrix, then

$$e^{5z} + 2e^{4z} + 2e^{2z} + 2 = 0$$

from using (16). After that,

$$\begin{aligned} e^z &\approx -2.40541, \\ e^z &\approx -0.370293 - i0.872969, \\ e^z &\approx -0.370293 + i0.872969, \\ e^z &\approx 0.572995 - i0.772239, \\ e^z &\approx 0.572995 + i0.772239. \end{aligned} \tag{17}$$

and $z_1 \approx -0.05312 + i4.31122$, $z_2 \approx -0.05312 + i10.59441$, $z_3 \approx -0.05312 + i8.25516$, $z_4 \approx -0.03915 + i4.07404$, $z_5 \approx -0.03915 + i10.35723$, $z_6 \approx -0.03915 + i8.49234$ are the roots of (17) in T_- , then the eigenvalues of L_1 is $\sigma_d(L_1) = \left\{ \lambda \in \mathbb{C}: \lambda = 2 \cosh \frac{z_k}{2}; k = 1,2,3,4,5,6 \right\}$.

Case 2: For $\theta = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, it can be found that

$$e^{5z} + e^{4z} + e^{2z} + 2 = 0.$$

From the last equation,

$$\begin{aligned} e^z &\approx -1.6439, \\ e^z &\approx -0.398336 - i0.97864, \\ e^z &\approx -0.398336 + i0.97864, \\ e^z &\approx 0.720286 - i0.755613, \\ e^z &\approx 0.720286 + i0.755613. \end{aligned} \tag{18}$$

Since there is no root of (18), the eigenvalues of L_1 is empty.

CONFLICTS OF INTEREST

No conflict of interest was declared by the author.

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