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Hermite-Hadamard Type Inequalities for the Functions Whose Absolute Values of First Derivatives are p-Convex

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Abstract

In this paper, we extend some estimates of a Hermite-Hadamard type inequality for functions whose absolute values of the first derivatives are p-convex. By means of the obtained inequalities, some bound functions involving beta functions and hypergeometric functions are derived as applications. Also, we suggest an upper bound for error in numerical integration of p-convex functions via composite trapezoid rule.

1. Introduction

Some features of sets and functions make them more important than the others in mathematics, so this kind of sets and functions attract great interest, especially, if they are useful to handle optimization problems. One of them is convexity. Since the discovery of the convex sets and functions, it has been so extended and generalized in many ways that a lof of convexity types have been defined, from quasiconvexity to B-convexity, B^{-1}-convexity, p-convexity etc (See [1]- [12] and the references therein). On the other hand, in researching new types of convexity, many inequalities valid for convex functions and on convex sets such as Jensen, Ostrowski, and Hermite-Hadamard are adapted to new convexity types such as s-convex functions, p-convex function [13] -[16]. In this study, we focus on p-convex functions and Hermite-Hadamard type inequalities.
Some of the studies on p-convex sets and their properties can be seen in [17] -[21]. p-convex functions are shortly introduced in [17] and its main characteristics are given in [22].

Definition 1.1. *[17] Let $U \subseteq \mathbb{R}$ and $0 < p \leq 1$. If for each $x, y \in U$, $\lambda, \mu \geq 0$ such that $\lambda^p + \mu^p = 1$, $\lambda x + \mu y \in U$, then U is called a p-convex set in $\mathbb{R}$.*

It is clear that any interval of real numbers including zero or accepting zero as a boundary point is a p-convex set. Using Theorem 3.2 in [22], we can give the following definition of p-convex function:

Definition 1.2. *Let $U \subseteq \mathbb{R}$ a p-convex set and let $f : U \to \mathbb{R}$ be a function. f is said to be a p-convex function if the following inequality*

$$f(\lambda x + \mu y) \leq \lambda f(x) + \mu f(y)$$

is satisfied for all $\lambda, \mu \geq 0$ such that $\lambda^p + \mu^p = 1$ and for each $x, y \in U$.

Although the definition of p-convexity coincides with the classic convexity for $p = 1$, there are some cases that distinguish it from the classical convexity for $0 < p < 1$; for example, the single point set is convex but this is not p-convex. On the other hand, any open or closed interval is convex, but in order to be p-convex, it must be in the form of any interval of real numbers including zero or accepting zero as a boundary point.

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Let $U \subseteq \mathbb{R}$ be a p-convex set and $k \in \mathbb{R}$. If we define $f, g, h : U \to \mathbb{R}$ such that $f(x) = |x|$, $g(x) = kx$ and $h(x) = kx^2$ then f, g and the derivative of h are p-convex functions.
Hermite-Hadamard inequality is well-known inequality that is given by Hermite, ten years later obtained by Hadamard as follows:
Let $f : [a,b] \to \mathbb{R}$ be a convex function. Then

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2}. \tag{1.1}$$

This theorem says that the average integral of a convex function interpolates between the image of the average of endpoints and the average of the images of the endpoints. It is obtained for p-convex functions in [23] as follows:

Theorem 1.3. *Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be an integrable p-convex function. For $a,b \in \mathbb{R}_+$ with $a < b$, the following inequality holds:*

$$2^{\frac{1}{p}-1} f\left(\frac{a+b}{2^{\frac{1}{p}}}\right)(b-a) \le \int_a^b f(x)dx \le \frac{1}{2p}\left\{p\left[bf(b)+af(a)\right]+\left[bf(a)+af(b)\right]B\left(\frac{1}{p},\frac{1}{p}\right)\right\}.$$

In this paper, we obtain some bounds for the difference between the average integral and left expression and for the difference between the average integral and right expression in the inequality (1.1).
Also, let us state the necessary inequalities and formulas to be used throughout the paper. The Beta function is defined as follows:

$$B(\alpha_1,\alpha_2) = \int_0^1 t^{\alpha_1-1}(1-t)^{\alpha_2-1}dt \quad \text{for } \alpha_1,\alpha_2 > 0,$$

and $B(\alpha_1,\alpha_2)$ satisfies the properties below:

$$B(\alpha_1,\alpha_2) = B(\alpha_2,\alpha_1) \ \text{ and } \ B(\alpha_1+1,\alpha_2) = \frac{\alpha_1}{\alpha_1+\alpha_2}B(\alpha_1,\alpha_2).$$

2. Main results

2.1. Hermite-Hadamard type inequalities

For the sake of clarity, throughout this section $D[a,b]$ denotes the class of real valued differentiable functions for $a,b \in \mathbb{R}$ with $a < b$.
An upper bound for the right Hermite-Hadamard inequality for p-convex functions will be found by means of the lemma below:

Lemma 2.1. *Let $p \in (0,1]$ and $f \in D[a,b]$. If $f' \in L[a,b]$, then the following equality holds:*

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx = \frac{1}{2p(a-b)}\int_0^1 \left[a+b-2(t^{\frac{1}{p}}b+(1-t)^{\frac{1}{p}}a)\right] f'(t^{\frac{1}{p}}b+(1-t)^{\frac{1}{p}}a)\left[t^{\frac{1}{p}-1}b-(1-t)^{\frac{1}{p}-1}a\right]dt.$$

Proof. If we apply the partial integration formula and change the variable as $x = t^{\frac{1}{p}}b+(1-t)^{\frac{1}{p}}a$, we get the desired result as follows:

$$\frac{1}{2p(a-b)}\int_0^1 \left[a+b-2(t^{\frac{1}{p}}b+(1-t)^{\frac{1}{p}}a)\right] f'(t^{\frac{1}{p}}b+(1-t)^{\frac{1}{p}}a)\left[t^{\frac{1}{p}-1}b-(1-t)^{\frac{1}{p}-1}a\right]dt$$

$$= \frac{1}{2(a-b)}\left[a+b-2(t^{\frac{1}{p}}b+(1-t)^{\frac{1}{p}}a)\right] f\left(t^{\frac{1}{p}}b+(1-t)^{\frac{1}{p}}a\right)\Big|_0^1 + \frac{1}{p(a-b)}\int_0^1 f\left(t^{\frac{1}{p}}b+(1-t)^{\frac{1}{p}}a\right)\left[t^{\frac{1}{p}-1}b-(1-t)^{\frac{1}{p}-1}a\right]dt$$

$$= \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx.$$

□

Theorem 2.2. *Let $f \in D[a,b]$ such that $|f'| \in L[a,b]$ and p-convex function on $\mathbb{R}$. Then,*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{3}{2(p+1)(b-a)}\left(|a|+|b|\right)^2\left(\left|f'(a)\right|+\left|f'(b)\right|\right). \tag{2.1}$$

Proof. From Lemma 2.1, triangle inequality and the p-convexity of $|f'|$,

$$\begin{aligned}\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| &\le \frac{1}{2p(b-a)}\int_0^1\left|a+b-2(t^{\frac{1}{p}}b+(1-t)^{\frac{1}{p}}a)\right|\left|f'(t^{\frac{1}{p}}b+(1-t)^{\frac{1}{p}}a)\right|\left|t^{\frac{1}{p}-1}b-(1-t)^{\frac{1}{p}-1}a\right|dt\\ &\le \frac{1}{2p(b-a)}\int_0^1\left|a+b-2(t^{\frac{1}{p}}b+(1-t)^{\frac{1}{p}}a)\right|\left|t^{\frac{1}{p}-1}b-(1-t)^{\frac{1}{p}-1}a\right|(t^{\frac{1}{p}}|f'(b)|+(1-t)^{\frac{1}{p}}|f'(a)|)dt\\ &\le \frac{1}{2p(b-a)}3(|a|+|b|)(|a|+|b|)\left(\int_0^1 t^{\frac{1}{p}}|f'(b)|dt+\int_0^1(1-t)^{\frac{1}{p}}|f'(a)|dt\right)\\ &\le \frac{3}{2(p+1)(b-a)}(|a|+|b|)^2(|f'(a)|+|f'(b)|).\end{aligned}$$

□

Surely, the sharper versions for the inequality (2.1) and next inequalities to be presented throughout the paper can be obtained. To exemplify, we present only the following two theorems as sharper version for only the theorem above.

Theorem 2.3. *Let $f\in D[a,b]$ such that $|f'|\in L[a,b]$ and p-convex function on $\mathbb{R}$. Then,*

$$\begin{aligned}\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \le{}& \frac{1}{12(b-a)}\left[|b|(7|b|+3|a|)\left|f'(b)\right|+|a|(7|a|+3|b|)\left|f'(a)\right|\right]\\ &+\frac{1}{4p(b-a)}(|a|+|b|)\left(\left|af'(b)\right|+\left|bf'(a)\right|\right)B(\frac{1}{p},\frac{1}{p})\\ &+\frac{1}{3p(b-a)}\left[|a|(|a|+3|b|)\left|f'(b)\right|+|b|(|b|+3|a|)\left|f'(a)\right|\right]B(\frac{1}{p},\frac{2}{p}).\end{aligned}\tag{2.2}$$

Proof. Let $g(t)=t^{\frac{1}{p}-1}b-(1-t)^{\frac{1}{p}-1}a$ and $h(t)=a+b-2(t^{\frac{1}{p}}b+(1-t)^{\frac{1}{p}}a)$, then

$$\begin{aligned}\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| &\le \frac{1}{2p(b-a)}\int_0^1|h(t)g(t)|\left|t^{\frac{1}{p}}f'(b)+(1-t)^{\frac{1}{p}}f'(a)\right|dt\\ &\le \frac{1}{2p(b-a)}\int_0^1|h(t)g(t)|\left(t^{\frac{1}{p}}|f'(b)|+(1-t)^{\frac{1}{p}}|f'(a)|\right)dt.\end{aligned}$$

Using triangle inequality, we have

$$\begin{aligned}|h(t)g(t)| &= \left|(ab+b^2)t^{\frac{1}{p}-1}-(a^2+ab)(1-t)^{\frac{1}{p}-1}-2b^2t^{\frac{2}{p}-1}+2ab(t^{\frac{1}{p}}(1-t)^{\frac{1}{p}-1}-t^{\frac{1}{p}-1}(1-t)^{\frac{1}{p}})+2a^2(1-t)^{\frac{2}{p}-1}\right|\\ &\le \left(|ab|+b^2\right)t^{\frac{1}{p}-1}+\left(a^2+|ab|\right)(1-t)^{\frac{1}{p}-1}+2b^2t^{\frac{2}{p}-1}+2|ab|(t^{\frac{1}{p}}(1-t)^{\frac{1}{p}-1}+t^{\frac{1}{p}-1}(1-t)^{\frac{1}{p}})+2a^2(1-t)^{\frac{2}{p}-1}.\end{aligned}\tag{2.3}$$

If we multiply (2.3) with $\left(t^{\frac{1}{p}}|f'(b)|+(1-t)^{\frac{1}{p}}|f'(a)|\right)$ then expand and integrate on $[0,1]$ with respect to t, we get

$$\begin{aligned}\int_0^1|h(t)g(t)|\left(t^{\frac{1}{p}}|f'(b)|+(1-t)^{\frac{1}{p}}|f'(a)|\right)dt &\le \tfrac{1}{6}p\left[|b|(7|b|+3|a|)|f'(b)|+|a|(7|a|+3|b|)|f'(a)|\right]\\ &+\tfrac{1}{2}(|a|+|b|)(|a||f'(b)|+|b||f'(a)|)B(\tfrac{1}{p},\tfrac{1}{p})\\ &+\tfrac{2}{3}\left[|a|(|a|+3|b|)|f'(b)|+|b|(|b|+3|a|)|f'(a)|\right]B(\tfrac{1}{p},\tfrac{2}{p}).\end{aligned}$$

When this inequality is used in the first inequality of the proof, (2.2) is obtained. □

Theorem 2.4. *Let $f\in D[a,b]$ such that $|f'|\in L[a,b]$ and p-convex function on $\mathbb{R}$. Then,*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|\le\frac{1}{2(p+1)(b-a)}\max\{|g(0)|,|g(1)|,|g(t_1)|\}\max\{|h(0)|,|h(1)|,|h(t_2)|\}\left(\left|f'(a)\right|+\left|f'(b)\right|\right)$$

where $g(t)=t^{\frac{1}{p}-1}b-(1-t)^{\frac{1}{p}-1}a$ and $h(t)=a+b-2(t^{\frac{1}{p}}b+(1-t)^{\frac{1}{p}}a)$ as in the proof of the above theorem and for $a\neq 0$,

$$t_1=\left(1+\left(\left|\frac{b}{a}\right|\right)^{\frac{p}{1-2p}}\right)^{-1}\quad\text{and}\quad t_2=\left(1+\left(\left|\frac{b}{a}\right|\right)^{\frac{p}{1-p}}\right)^{-1}$$

for $a=0$, t_1,t_2 equal to 0 or 1.

Proof. From Lemma 2.1, as in the proof of Theorem 2.2, we have

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{1}{2p(b-a)}\int_0^1 |h(t)g(t)|\left|t^{\frac{1}{p}}f'(b)+(1-t)^{\frac{1}{p}}f'(a)\right|dt$$

Let $a\neq 0$. In search of extremum points of $g(t)$ and $h(t)$ it is seen that while $\frac{b}{a}<0$ and $\frac{b}{a}>0$ $g(t)$ and $h(t)$ have one extremum point in [0,1], i.e., $g(t)$ and $h(t)$ are unimodal functions on [0,1], respectively. In other cases $g(t)$ and $h(t)$ will be monotone functions. So $g(t)$ and $h(t)$ take extremum values either at the points $t_1=\left(1+\left(\frac{-b}{a}\right)^{\frac{p}{1-2p}}\right)^{-1}$ and $t_2=\left(1+\left(\frac{b}{a}\right)^{\frac{p}{1-p}}\right)^{-1}$ for proper values of a,b, respectively, or at the points $t=0$ or $t=1$ in common.
If we take $\left|\frac{b}{a}\right|$ in the expression of t_1 and t_2, we can express the largest values that can be reached in the [0,1] interval, regardless of the sign of $\frac{b}{a}$ as follows. Thus, $|g(t)|\le\max\{|g(0)|,|g(1)|,|g(t_1)|\}$ and $|h(t)|\le\max\{|h(0)|,|h(1)|,|h(t_2)|\}$ is derived. For the case $a=0$, extremum values are obtained for $t=0$, $t=1$, which is included in the inequality above. In a similar way in the proof of Theorem 2.2, by using the p-convexity of $|f'|$, we get the desired result. □

By making use of the Hölder inequality, some kind of extensions of the above theorems can be obtained as in the following theorems.

Theorem 2.5. *Let $s>1$, $f\in D[a,b]$ such that $|f'|^s\in L[a,b]$ and p-convex function on $\mathbb{R}$. Then,*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{3}{2p(b-a)}\left(\frac{p}{p+1}\right)^{\frac{1}{s}}(|a|+|b|)^2\left(\left|f'(a)\right|^s+\left|f'(b)\right|^s\right)^{\frac{1}{s}}.$$

Proof. From Lemma 2.1, triangle and Hölder inequality and the p-convexity of $|f'|^s$,

$$\begin{aligned}\left|\frac{f(a)+f(b)}{2}\right. & \left.-\frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{1}{2p(b-a)}\int_0^1\left|a+b-2(t^{\frac{1}{p}}b+(1-t)^{\frac{1}{p}}a)\right|\left|t^{\frac{1}{p}-1}b-(1-t)^{\frac{1}{p}-1}a\right|\left|f'(t^{\frac{1}{p}}b+(1-t)^{\frac{1}{p}}a)\right|dt\\ &= \frac{1}{2p(b-a)}\left(\int_0^1\left(\left|a+b-2t^{\frac{1}{p}}b-2(1-t)^{\frac{1}{p}}a)\right|\left|t^{\frac{1}{p}-1}b-(1-t)^{\frac{1}{p}-1}a\right|\right)^{\frac{s}{s-1}}dt\right)^{\frac{s-1}{s}}\left(\int_0^1\left|f'(t^{\frac{1}{p}}b+(1-t)^{\frac{1}{p}}a)\right|^s dt\right)^{\frac{1}{s}}\\ &\le \frac{1}{2p(b-a)}\left(\int_0^1(|a|+|b|+2|b|+2|a|)^{\frac{s}{s-1}}(|b|+|a|)^{\frac{s}{s-1}}dt\right)^{\frac{s-1}{s}}\left(\int_0^1\left(t^{\frac{1}{p}}|f'(b)|^s+(1-t)^{\frac{1}{p}}|f'(a)|^s\right)dt\right)^{\frac{1}{s}}\\ &\le \frac{3}{2p(b-a)}\left(\frac{p}{p+1}\right)^{\frac{1}{s}}(|a|+|b|)^2\left(|f'(a)|^s+|f'(b)|^s\right)^{\frac{1}{s}}.\end{aligned}$$

□

Theorem 2.6. *Let $s>1$, $f\in D[a,b]$ such that $|f'|^s\in L[a,b]$ and p-convex function on $\mathbb{R}$. Then,*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|\le\frac{1}{2(p+1)(b-a)}\max\{|g(0)|,|g(1)|,|g(t_1)|\}\max\{|h(0)|,|h(1)|,|h(t_2)|\}\left(|f'(a)|^s+|f'(b)|^s\right)^{\frac{1}{s}}$$

where $g(t)=t^{\frac{1}{p}-1}b-(1-t)^{\frac{1}{p}-1}a$, $h(t)=a+b-2(t^{\frac{1}{p}}b+(1-t)^{\frac{1}{p}}a)$ and for $a\neq 0$, $t_1=\left(1+\left(\left|\frac{b}{a}\right|\right)^{\frac{p}{1-2p}}\right)^{-1}$, $t_2=\left(1+\left(\left|\frac{b}{a}\right|\right)^{\frac{p}{1-p}}\right)^{-1}$ for $a=0$, t_1,t_2 equal to 0 or 1.

Proof. By applying the Hölder inequality as in the proof of Theorem 2.5, and then using the findings about the maximum of $h(t)$ and $g(t)$ from the proof of Theorem 2.4, the desired inequality is obtained. □

An upper bound for the left Hermite-Hadamard inequality for p-convex functions will be found using the following lemma.

Lemma 2.7. *Let $p\in(0,1]$ and $f\in D[a,b]$. If $f'\in L[a,b]$, then the following equality holds:*

$$\begin{aligned}&f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\\ &=\frac{1}{p(b-a)}\int_0^1\left[t^{\frac{1}{p}}\frac{a+b}{2}+((1-t)^{\frac{1}{p}}-1)a\right]f'(t^{\frac{1}{p}}\frac{a+b}{2}+(1-t)^{\frac{1}{p}}a)\left[t^{\frac{1}{p}-1}\frac{a+b}{2}-(1-t)^{\frac{1}{p}-1}a\right]dt\\ &+\frac{1}{p(b-a)}\int_0^1\left[b(t^{\frac{1}{p}}-1)+(1-t)^{\frac{1}{p}}\frac{a+b}{2}\right]f'(t^{\frac{1}{p}}b+(1-t)^{\frac{1}{p}}\frac{a+b}{2})\left[t^{\frac{1}{p}-1}b-(1-t)^{\frac{1}{p}-1}\frac{a+b}{2}\right]dt.\end{aligned}\tag{2.4}$$

Proof. If we apply partial integration to the integrals on the right side of equality (2.4) and make the necessary variable substitution, we get equality (2.4).

$$\frac{1}{p(b-a)}\int_0^1\left[t^{\frac{1}{p}}\frac{a+b}{2}+((1-t)^{\frac{1}{p}}-1)a\right]f'(t^{\frac{1}{p}}\frac{a+b}{2}+(1-t)^{\frac{1}{p}}a)\left[t^{\frac{1}{p}-1}\frac{a+b}{2}-(1-t)^{\frac{1}{p}-1}a\right]dt$$

$$+\frac{1}{p(b-a)}\int_0^1\left[b(t^{\frac{1}{p}}-1)+(1-t)^{\frac{1}{p}}\frac{a+b}{2}\right]f'(t^{\frac{1}{p}}b+(1-t)^{\frac{1}{p}}\frac{a+b}{2})\left[t^{\frac{1}{p}-1}b-(1-t)^{\frac{1}{p}-1}\frac{a+b}{2}\right]dt$$

$$=\frac{1}{(b-a)}\left[\left(\frac{a+b}{2}-a\right)f(\tfrac{a+b}{2})-\int_a^{\frac{a+b}{2}}f(x)dx\right]+\frac{1}{(b-a)}\left[\left(b-\frac{a+b}{2}\right)f(\tfrac{a+b}{2})-\int_{\frac{a+b}{2}}^b f(x)dx\right]$$

$$=f(\tfrac{a+b}{2})-\frac{1}{b-a}\int_a^b f(x)dx.$$

□

Theorem 2.8. *Let $f\in D[a,b]$ such that $|f'|\in L[a,b]$ and p-convex function on $\mathbb{R}$. Then,*

$$\left|f(\tfrac{a+b}{2})-\frac{1}{b-a}\int_a^b f(x)dx\right|\le\frac{1}{(p+1)(b-a)}\left[\left(\frac{3|a|+|b|}{2}\right)^2|f'(a)|+\frac{5a^2+6|ab|+5b^2}{2}\left|f'(\tfrac{a+b}{2})\right|+\left(\frac{|a|+3|b|}{2}\right)^2|f'(b)|\right].$$

Proof. From Lemma 2.7, triangle inequality and the p-convexity of $|f'|$,

$$\begin{aligned}\left|f(\tfrac{a+b}{2})-\frac{1}{b-a}\int_a^b f(x)dx\right| &\le\frac{1}{p(b-a)}\int_0^1\left|t^{\frac{1}{p}}\frac{a+b}{2}+((1-t)^{\frac{1}{p}}-1)a\right|\left|t^{\frac{1}{p}-1}\frac{a+b}{2}-(1-t)^{\frac{1}{p}-1}a\right|\left|f'(t^{\frac{1}{p}}\frac{a+b}{2}+(1-t)^{\frac{1}{p}}a)\right|dt\\ &+\frac{1}{p(b-a)}\int_0^1\left|b(t^{\frac{1}{p}}-1)+(1-t)^{\frac{1}{p}}\frac{a+b}{2}\right|\left|t^{\frac{1}{p}-1}b-(1-t)^{\frac{1}{p}-1}\frac{a+b}{2}\right|\left|f'(t^{\frac{1}{p}}b+(1-t)^{\frac{1}{p}}\frac{a+b}{2})\right|dt\\ &\le\frac{1}{p(b-a)}\int_0^1\left(\frac{|a|+|b|}{2}+|a|\right)^2(t^{\frac{1}{p}}\left|f'(\tfrac{a+b}{2})\right|+(1-t)^{\frac{1}{p}}|f'(a)|)dt\\ &+\frac{1}{p(b-a)}\int_0^1\left(\frac{|a|+|b|}{2}+|b|\right)^2(t^{\frac{1}{p}}|f'(b)|+(1-t)^{\frac{1}{p}}\left|f'(\tfrac{a+b}{2})\right|)dt\\ &\le\frac{1}{(p+1)(b-a)}\left[\left(\frac{3|a|+|b|}{2}\right)^2|f'(a)|+\frac{5a^2+6|ab|+5b^2}{2}\left|f'(\tfrac{a+b}{2})\right|+\left(\frac{|a|+3|b|}{2}\right)^2|f'(b)|\right].\end{aligned}$$

□

Theorem 2.9. *Let $f\in D[a,b]$ such that $|f'|\in L[a,b]$ and p-convex function on $\mathbb{R}$. Let*

$$g_1(t)=t^{\frac{1}{p}}\frac{a+b}{2}+((1-t)^{\frac{1}{p}}-1)a\ ,\ g_2(t)=b(t^{\frac{1}{p}}-1)+(1-t)^{\frac{1}{p}}\frac{a+b}{2},$$

$$h_1(t)=t^{\frac{1}{p}-1}\frac{a+b}{2}-(1-t)^{\frac{1}{p}-1}a\ \text{ and }\ h_2(t)=t^{\frac{1}{p}-1}b-(1-t)^{\frac{1}{p}-1}\frac{a+b}{2}.$$

Then,

$$\left|f(\frac{a+b}{2})-\frac{1}{b-a}\int_a^b f(x)dx\right|\le\frac{1}{(p+1)(b-a)}\left(w_1|f'(a)|+(w_1+w_2)f'(\frac{a+b}{2})+w_2|f'(b)|\right)$$

where

$$w_1=\max\{|g_1(0)|,|g_1(1)|,|g_1(t_1)|\}\cdot\max\{|h_1(0)|,|h_1(1)|,|h_1(s_1)|\},$$
$$w_2=\max\{|g_2(0)|,|g_2(1)|,|g_2(t_2)|\}\cdot\max\{|h_2(0)|,|h_2(1)|,|h_2(s_2)|\}$$

and for a,b which makes t_1,t_2,s_1,s_2 defined,

$$t_1=\left(1+\left(\frac{a+b}{2a}\right)^{\frac{p}{p-1}}\right)^{-1},\quad t_2=\left(1+\left(\frac{a+b}{2b}\right)^{\frac{p}{p-1}}\right)^{-1},\ s_1=\left(1+\left(\frac{a+b}{2a}\right)^{\frac{p}{2p-1}}\right)^{-1},\ s_2=\left(1+\left(\frac{a+b}{2b}\right)^{\frac{p}{2p-1}}\right)^{-1}$$

for a,b which makes any of t_1,t_2,s_1,s_2 undefined, that one will be zero or one.

Proof. When their first derivatives of these functions are investigated, it is seen that $g_1(t)$, $g_2(t)$, $h_1(t)$, $h_2(t)$ with respect to values of a,b,p are either monotonic functions or unimodal functions on $[0,1]$, the maximum values of $|g_1(t)|$, $|g_2(t)|$, $|h_1(t)|$, $|h_2(t)|$ are attained at either boundary points of $[0,1]$ or extremum points. The extremum points for these functions with respect to values of a,b making the following values defined are

$$t_1=\left(1+\left(\frac{a+b}{2a}\right)^{\frac{p}{p-1}}\right)^{-1},\ t_2=\left(1+\left(\frac{a+b}{2b}\right)^{\frac{p}{p-1}}\right)^{-1},\ s_1=\left(1+\left(\frac{a+b}{2a}\right)^{\frac{p}{2p-1}}\right)^{-1},\ s_2=\left(1+\left(\frac{a+b}{2a}\right)^{\frac{p}{2p-1}}\right)^{-1},$$

respectively. For the values of a and b that makes $\frac{a+b}{2a}$ or $\frac{a+b}{2b}$ negative, these functions will be monotone function on $[0,1]$. Therefore for $i=1,2$

$$|g_i(t)| \leq \max\{|g_i(0)|,|g_i(1)|,|g_i(t_i)|\} \text{ and } |h_i(t)| \leq \max\{|h_i(0)|,|h_i(1)|,|h_i(s_i)|\}.$$

From Lemma 2.7, we have

$$\begin{aligned}
&\left|f(\tfrac{a+b}{2})-\tfrac{1}{b-a}\int_a^b f(x)dx\right| \\
&\leq \tfrac{1}{p(b-a)}\int_0^1 |g_1(t)|\,|h_1(t)|\left|f'(t^{\frac{1}{p}}\tfrac{a+b}{2}+(1-t)^{\frac{1}{p}}a)\right|dt+\tfrac{1}{p(b-a)}\int_0^1 |g_2(t)|\,|h_2(t)|\left|f'(t^{\frac{1}{p}}b+(1-t)^{\frac{1}{p}}\tfrac{a+b}{2})\right|dt \\
&\leq \tfrac{1}{p(b-a)}\int_0^1 \max\{|g_1(0)|,|g_1(1)|,|g_1(t_1)|\}\max\{|h_1(0)|,|h_1(1)|,|h_1(s_1)|\}(t^{\frac{1}{p}}\left|f'(\tfrac{a+b}{2})\right|+(1-t)^{\frac{1}{p}}|f'(a)|)dt \\
&\quad+\tfrac{1}{p(b-a)}\int_0^1 \max\{|g_2(0)|,|g_2(1)|,|g_2(t_2)|\}\max\{|h_2(0)|,|h_2(1)|,|h_2(s_2)|\}(t^{\frac{1}{p}}|f'(b)|+(1-t)^{\frac{1}{p}}\left|f'(\tfrac{a+b}{2})\right|)dt \\
&\leq \tfrac{1}{(p+1)(b-a)}\left(w_1|f'(a)|+(w_1+w_2)f'(\tfrac{a+b}{2})+w_2|f'(b)|\right) \quad \square
\end{aligned}$$

Theorem 2.10. *Let $f\in D[a,b]$ such that $|f'|\in L[a,b]$ and p-convex function on $\mathbb{R}$. Then,*

$$\begin{aligned}
\left|f(\tfrac{a+b}{2})-\tfrac{1}{b-a}\int_a^b f(x)dx\right| &\leq \tfrac{5}{12(b-a)}\left(2a^2|f'(a)|+(a+b)^2\left|f'(\tfrac{a+b}{2})\right|+2b^2|f'(b)|\right) \\
&+\tfrac{1}{4p(b-a)}\left(|a|(|a|+|b|)|f'(a)|+2(a^2+b^2)\left|f'(\tfrac{a+b}{2})\right|+|b|(|a|+|b|)|f'(b)|\right)B(\tfrac{1}{p},\tfrac{1}{p}) \\
&+\tfrac{1}{12p(b-a)}\left((7|a|+|b|)(|a|+|b|)|f'(a)|+2(5a^2+6|a||b|+5b^2)\left|f'(\tfrac{a+b}{2})\right|+(|a|+7|b|)(|a|+|b|)|f'(b)|\right)B(\tfrac{1}{p},\tfrac{2}{p}).
\end{aligned}$$

Proof. Let $g_1(t),g_2(t),h_1(t)$ and $h_2(t)$ functions as in Theorem 2.9. Using triangle inequality, we can write the followings

$$\begin{aligned}
|g_1(t)h_1(t)| &= \left|\left(t^{\frac{1}{p}}\frac{a+b}{2}+((1-t)^{\frac{1}{p}}-1)a\right)\left(t^{\frac{1}{p}-1}\frac{a+b}{2}-(1-t)^{\frac{1}{p}-1}a\right)\right| \\
&= \left|(\frac{a+b}{2})^2t^{\frac{2}{p}-1}+a(\frac{a+b}{2})(t^{\frac{1}{p}-1}(1-t)^{\frac{1}{p}}-t^{\frac{1}{p}-1}-t^{\frac{1}{p}}(1-t)^{\frac{1}{p}-1})-a^2((1-t)^{\frac{2}{p}-1}+(1-t)^{\frac{1}{p}-1})\right| \\
&\leq (\frac{a+b}{2})^2t^{\frac{2}{p}-1}+|a|\left|\frac{a+b}{2}\right|(t^{\frac{1}{p}-1}(1-t)^{\frac{1}{p}}+t^{\frac{1}{p}-1}+t^{\frac{1}{p}}(1-t)^{\frac{1}{p}-1})+a^2((1-t)^{\frac{2}{p}-1}+(1-t)^{\frac{1}{p}-1}) \qquad (2.5)
\end{aligned}$$

and

$$\begin{aligned}
|g_2(t)h_2(t)| &= \left|\left(b(t^{\frac{1}{p}}-1)+(1-t)^{\frac{1}{p}}\frac{a+b}{2}\right)\left(t^{\frac{1}{p}-1}b-(1-t)^{\frac{1}{p}-1}\frac{a+b}{2}\right)\right| \\
&= \left|b^2(t^{\frac{2}{p}-1}-t^{\frac{1}{p}-1})+b(\frac{a+b}{2})(t^{\frac{1}{p}-1}(1-t)^{\frac{1}{p}}+(1-t)^{\frac{1}{p}-1}-t^{\frac{1}{p}}(1-t)^{\frac{1}{p}-1})-(\frac{a+b}{2})^2(1-t)^{\frac{2}{p}-1}\right| \\
&\leq b^2(t^{\frac{2}{p}-1}+t^{\frac{1}{p}-1})+|b|\left|\frac{a+b}{2}\right|(t^{\frac{1}{p}-1}(1-t)^{\frac{1}{p}}+(1-t)^{\frac{1}{p}-1}+t^{\frac{1}{p}}(1-t)^{\frac{1}{p}-1})+(\frac{a+b}{2})^2(1-t)^{\frac{2}{p}-1}. \qquad (2.6)
\end{aligned}$$

If we multiply (2.5) and (2.6) inequalities with $\left(t^{\frac{1}{p}}\left|f'(\frac{a+b}{2})\right|+(1-t)^{\frac{1}{p}}|f'(a)|\right)$ and $\left(t^{\frac{1}{p}}|f'(b)|+(1-t)^{\frac{1}{p}}\left|f'(\frac{a+b}{2})\right|\right)$, respectively and integrate on [0,1], then, use Lemma 2.7 and the p-convexity of $|f'|$, we have the following

$$\begin{aligned}
\left|f(\tfrac{a+b}{2})-\tfrac{1}{b-a}\int_a^b f(x)dx\right| &\leq \tfrac{1}{p(b-a)}\int_0^1 |g_1(t)h_1(t)|\left(t^{\frac{1}{p}}\left|f'(\tfrac{a+b}{2})\right|+(1-t)^{\frac{1}{p}}|f'(a)|\right)dt\\
&+\tfrac{1}{p(b-a)}\int_0^1 |g_2(t)h_2(t)|\left(t^{\frac{1}{p}}|f'(b)|+(1-t)^{\frac{1}{p}}\left|f'(\tfrac{a+b}{2})\right|\right)dt\\
&\leq \tfrac{1}{p(b-a)}\int_0^1\Big[\left|\tfrac{a+b}{2}\right|^2 t^{\frac{2}{p}-1}+|a|\left|\tfrac{a+b}{2}\right|(t^{\frac{1}{p}-1}(1-t)^{\frac{1}{p}}+t^{\frac{1}{p}-1}+t^{\frac{1}{p}}(1-t)^{\frac{1}{p}-1})+a^2((1-t)^{\frac{2}{p}-1}+(1-t)^{\frac{1}{p}-1})\Big]\\
&\times\left(t^{\frac{1}{p}}\left|f'(\tfrac{a+b}{2})\right|+(1-t)^{\frac{1}{p}}|f'(a)|\right)dt\\
&+\tfrac{1}{p(b-a)}\int_0^1\Big[b^2(t^{\frac{2}{p}-1}+t^{\frac{1}{p}-1})+|b|\left|\tfrac{a+b}{2}\right|(t^{\frac{1}{p}-1}(1-t)^{\frac{1}{p}}+2(1-t)^{\frac{1}{p}-1}+t^{\frac{1}{p}}(1-t)^{\frac{1}{p}-1})+\left|\tfrac{a+b}{2}\right|^2(1-t)^{\frac{2}{p}-1}\Big]\\
&\times\left(t^{\frac{1}{p}}|f'(b)|+(1-t)^{\frac{1}{p}}\left|f'(\tfrac{a+b}{2})\right|\right)dt\\
&\leq \tfrac{5}{12(b-a)}\left(2a^2|f'(a)|+(a+b)^2\left|f'(\tfrac{a+b}{2})\right|+2b^2|f'(b)|\right)\\
&+\tfrac{1}{4p(b-a)}\left(|a|(|a|+|b|)|f'(a)|+2(a^2+b^2)\left|f'(\tfrac{a+b}{2})\right|+|b|(|a|+|b|)|f'(b)|\right)B(\tfrac{1}{p},\tfrac{1}{p})\\
&+\tfrac{1}{12p(b-a)}\left((7|a|+|b|)(|a|+|b|)|f'(a)|+2(5a^2+6|a||b|+5b^2)\left|f'(\tfrac{a+b}{2})\right|+(|a|+7|b|)(|a|+|b|)|f'(b)|\right)B(\tfrac{1}{p},\tfrac{2}{p}).
\end{aligned}$$

□

Theorem 2.11. *Let $f\in D[a,b]$ such that $|f'|^s\in L[a,b]$ and p-convex function on $\mathbb{R}$. Then,*

$$\begin{aligned}
\left|f(\tfrac{a+b}{2})-\tfrac{1}{b-a}\int_a^b f(x)dx\right| &\leq \tfrac{1}{p(b-a)}\left(\tfrac{p}{p+1}\right)^{\frac{1}{s}}\left(2|a|+\left|\tfrac{a+b}{2}\right|\right)\left(|a|+\left|\tfrac{a+b}{2}\right|\right)\left(\left|f'(\tfrac{a+b}{2})\right|^s+|f'(a)|^s\right)^{\frac{1}{s}}\\
&+\tfrac{1}{p(b-a)}\left(\tfrac{p}{p+1}\right)^{\frac{1}{s}}\left(2|b|+\left|\tfrac{a+b}{2}\right|\right)\left(|b|+\left|\tfrac{a+b}{2}\right|\right)\left(\left|f'(\tfrac{a+b}{2})\right|^s+|f'(b)|^s\right)^{\frac{1}{s}}.
\end{aligned}$$

Proof. From Lemma 2.7, Hölder inequality, triangle inequality and the p-convexity of $|f'|^s$, we have

$$\begin{aligned}
\left|f(\tfrac{a+b}{2})-\tfrac{1}{b-a}\int_a^b f(x)dx\right| &\leq \tfrac{1}{p(b-a)}\left(\int_0^1 |g_1(t)h_1(t)|^{\frac{s}{s-1}}dt\right)^{\frac{s-1}{s}}\left(\int_0^1\left|f'(t^{\frac{1}{p}}\tfrac{a+b}{2}+(1-t)^{\frac{1}{p}}a)\right|^s dt\right)^{\frac{1}{s}}\\
&+\tfrac{1}{p(b-a)}\left(\int_0^1 |g_2(t)h_2(t)|^{\frac{s}{s-1}}dt\right)^{\frac{s-1}{s}}\left(\int_0^1\left|f'(t^{\frac{1}{p}}b+(1-t)^{\frac{1}{p}}\tfrac{a+b}{2})\right|^s dt\right)^{\frac{1}{s}}\\
&\leq \tfrac{1}{p(b-a)}\left(\int_0^1\left|\left(t^{\frac{1}{p}}\tfrac{a+b}{2}+((1-t)^{\frac{1}{p}}-1)a\right)\left(t^{\frac{1}{p}-1}\tfrac{a+b}{2}-(1-t)^{\frac{1}{p}-1}a\right)\right|^{\frac{s}{s-1}}dt\right)^{\frac{s-1}{s}}\left(\int_0^1\left|f'(t^{\frac{1}{p}}\tfrac{a+b}{2}+(1-t)^{\frac{1}{p}}a)\right|^s dt\right)^{\frac{1}{s}}\\
&+\tfrac{1}{p(b-a)}\left(\int_0^1\left|\left(b(t^{\frac{1}{p}}-1)+(1-t)^{\frac{1}{p}}\tfrac{a+b}{2}\right)\left(t^{\frac{1}{p}-1}b-(1-t)^{\frac{1}{p}-1}\tfrac{a+b}{2}\right)\right|^{\frac{s}{s-1}}dt\right)^{\frac{s-1}{s}}\left(\int_0^1\left|f'(t^{\frac{1}{p}}b+(1-t)^{\frac{1}{p}}\tfrac{a+b}{2})\right|^s dt\right)^{\frac{1}{s}}\\
&\leq \tfrac{1}{p(b-a)}\left(\int_0^1\left(\left|\tfrac{a+b}{2}\right|+2|a|\right)^{\frac{s}{s-1}}\left(\left|\tfrac{a+b}{2}\right|+|a|\right)^{\frac{s}{s-1}}dt\right)^{\frac{s-1}{s}}\left(\int_0^1\left(t^{\frac{1}{p}}\left|f'(\tfrac{a+b}{2})\right|^s+(1-t)^{\frac{1}{p}}|f'(a)|^s\right)dt\right)^{\frac{1}{s}}\\
&+\tfrac{1}{p(b-a)}\left(\int_0^1\left(2|b|+\left|\tfrac{a+b}{2}\right|\right)^{\frac{s}{s-1}}\left(|b|+\left|\tfrac{a+b}{2}\right|\right)^{\frac{s}{s-1}}dt\right)^{\frac{s-1}{s}}\left(\int_0^1\left(t^{\frac{1}{p}}|f'(b)|^s+(1-t)^{\frac{1}{p}}\left|f'(\tfrac{a+b}{2})\right|^s\right)dt\right)^{\frac{1}{s}}\\
&\leq \tfrac{1}{p(b-a)}\left(\tfrac{p}{p+1}\right)^{\frac{1}{s}}\left(2|a|+\left|\tfrac{a+b}{2}\right|\right)\left(|a|+\left|\tfrac{a+b}{2}\right|\right)\left(\left|f'(\tfrac{a+b}{2})\right|^s+|f'(a)|^s\right)^{\frac{1}{s}}\\
&+\tfrac{1}{p(b-a)}\left(\tfrac{p}{p+1}\right)^{\frac{1}{s}}\left(2|b|+\left|\tfrac{a+b}{2}\right|\right)\left(|b|+\left|\tfrac{a+b}{2}\right|\right)\left(\left|f'(\tfrac{a+b}{2})\right|^s+|f'(b)|^s\right)^{\frac{1}{s}}.
\end{aligned}$$

□

Additionally, we will use the following lemma to obtain a similar result to the right side of the Hermite-Hadamard inequality for p-convex functions given in Theorem 1.3.

Lemma 2.12. *Let $p \in (0,1]$ and $f \in D[a,b]$. If $f' \in L[a,b]$, then the following equality holds:*

$$bf(b)-af(a)-\int_a^b f(x)dx=\frac{1}{p}\int_0^1\left[b^2t^{\frac{2}{p}-1}+\frac{a^2}{t-1}(1-t)^{\frac{2}{p}}+abt^{\frac{1}{p}-1}(1-t)^{\frac{1}{p}}+ab\frac{t^{\frac{1}{p}}}{t-1}(1-t)^{\frac{1}{p}}\right]f'(t^{\frac{1}{p}}b+(1-t)^{\frac{1}{p}}a)dt.$$

Proof. If we apply the partial integration formula and change the variable as $x=t^{\frac{1}{p}}b+(1-t)^{\frac{1}{p}}a$, then we get the desired equality.

$$\frac{1}{p}\int_0^1\left[b^2t^{\frac{2}{p}-1}-a^2(1-t)^{\frac{2}{p}-1}+abt^{\frac{1}{p}-1}(1-t)^{\frac{1}{p}}-abt^{\frac{1}{p}}(1-t)^{\frac{1}{p}-1}\right]f'(t^{\frac{1}{p}}b+(1-t)^{\frac{1}{p}}a)dt$$

$$=\int_0^1\left(t^{\frac{1}{p}}b+(1-t)^{\frac{1}{p}}a\right)f'(t^{\frac{1}{p}}b+(1-t)^{\frac{1}{p}}a)\frac{1}{p}(t^{\frac{1}{p}-1}b-(1-t)^{\frac{1}{p}-1}a)dt$$

$$=\left[(t^{\frac{1}{p}}b+(1-t)^{\frac{1}{p}}a)f(t^{\frac{1}{p}}b+(1-t)^{\frac{1}{p}}a)\right]_0^1-\int_0^1 f(t^{\frac{1}{p}}b+(1-t)^{\frac{1}{p}}a)\frac{1}{p}(t^{\frac{1}{p}-1}b-(1-t)^{\frac{1}{p}-1}a)dt$$

$$=bf(b)-af(a)-\int_a^b f(x)dx.$$

□

Theorem 2.13. *Let $f \in D[a,b]$. If $|f'|$ is p-convex on $\mathbb{R}$, then the following inequality holds:*

$$\left|bf(b)-af(a)-\int_a^b f(x)dx\right|\leq\frac{1}{p+1}(|a|+|b|)^2\left(\left|f'(a)\right|+\left|f'(b)\right|\right).$$

Proof. Using Lemma 2.12 above and convexity of $|f'|$, we have

$$\left|(bf(b)-af(a))-\int_a^b f(x)dx\right|=\frac{1}{p}\left|\int_0^1\left[b^2t^{\frac{2}{p}-1}-a^2(1-t)^{\frac{2}{p}-1}+abt^{\frac{1}{p}-1}(1-t)^{\frac{1}{p}}-abt^{\frac{1}{p}}(1-t)^{\frac{1}{p}-1}\right]f'(t^{\frac{1}{p}}b+(1-t)^{\frac{1}{p}}a)dt\right|$$

$$\leq\frac{1}{p}\int_0^1\left|b^2t^{\frac{2}{p}-1}-a^2(1-t)^{\frac{2}{p}-1}+abt^{\frac{1}{p}-1}(1-t)^{\frac{1}{p}}-abt^{\frac{1}{p}}(1-t)^{\frac{1}{p}-1}\right|\left|f'(t^{\frac{1}{p}}b+(1-t)^{\frac{1}{p}}a)\right|dt$$

$$\leq\frac{1}{p}\int_0^1\left(\left|b^2t^{\frac{2}{p}-1}\right|+\left|a^2(1-t)^{\frac{2}{p}-1}\right|+\left|abt^{\frac{1}{p}-1}(1-t)^{\frac{1}{p}}\right|+\left|abt^{\frac{1}{p}}(1-t)^{\frac{1}{p}-1}\right|\right)\left(t^{\frac{1}{p}}|f'(b)|+(1-t)^{\frac{1}{p}}|f'(a)|\right)dt$$

$$\leq\frac{1}{p}\int_0^1(|a|+|b|)^2\left(t^{\frac{1}{p}}|f'(b)|+(1-t)^{\frac{1}{p}}|f'(a)|\right)dt$$

$$=\frac{1}{p+1}(|a|+|b|)^2(|f'(a)|+|f'(b)|).$$

□

Theorem 2.14. *Let $f \in D[a,b]$, $s \in (1,\infty)$ such that $\frac{1}{s}<p$ and $|f'|^s \in L[a,b]$. If $|f'|^s$ is p-convex on $\mathbb{R}$, then the following inequality holds:*

$$\left|bf(b)-af(a)-\int_a^b f(x)dx\right|\leq\frac{1}{p}\left(\frac{p}{p+1}\right)^{\frac{1}{s}}(|a|+|b|)^2\left(|f'(b)|^s+|f'(a)|^s\right)^{\frac{1}{s}}.$$

Proof. From Lemma 2.12, Hölder inequality, triangle inequality and the p-convexity of $|f'|^s$ we can write the following:

$$\left|bf(b)-af(a)-\int_a^b f(x)dx\right| = \frac{1}{p}\left|\int_0^1 \left[b^2t^{\frac{2}{p}-1}+\frac{a^2}{t-1}(1-t)^{\frac{2}{p}}+abt^{\frac{1}{p}-1}(1-t)^{\frac{1}{p}}+ab\frac{t^{\frac{1}{p}}}{t-1}(1-t)^{\frac{1}{p}}\right]f'(t^{\frac{1}{p}}b+(1-t)^{\frac{1}{p}}a)dt\right|$$

$$\le \frac{1}{p}\left(\int_0^1\left|b^2t^{\frac{2}{p}-1}-\frac{a^2}{1-t}(1-t)^{\frac{2}{p}}+abt^{\frac{1}{p}-1}(1-t)^{\frac{1}{p}}-ab\frac{t^{\frac{1}{p}}}{1-t}(1-t)^{\frac{1}{p}}\right|^{\frac{s}{s-1}}dt\right)^{\frac{s-1}{s}}\left(\int_0^1\left|f'(t^{\frac{1}{p}}b+(1-t)^{\frac{1}{p}}a)\right|^s dt\right)^{\frac{1}{s}}$$

$$\le \frac{1}{p}\left(\int_0^1\left[b^2t^{\frac{2}{p}-1}+a^2(1-t)^{\frac{2}{p}-1}+|ab|t^{\frac{1}{p}-1}(1-t)^{\frac{1}{p}}+|ab|t^{\frac{1}{p}}(1-t)^{\frac{1}{p}-1}\right]^{\frac{s}{s-1}}dt\right)^{\frac{s-1}{s}}\left(\int_0^1(t^{\frac{1}{p}}|f'(b)|^s+(1-t)^{\frac{1}{p}}|f'(a)|^s)dt\right)^{\frac{1}{s}}$$

$$\le \frac{1}{p}\left(\int_0^1\left[b^2+2|ab|+a^2\right]^{\frac{s}{s-1}}dt\right)^{\frac{s-1}{s}}\left(\int_0^1(t^{\frac{1}{p}}|f'(b)|^s+(1-t)^{\frac{1}{p}}|f'(a)|^s)dt\right)^{\frac{1}{s}}$$

$$= \frac{1}{p}\left(\frac{p}{p+1}\right)^{\frac{1}{s}}(|a|+|b|)^2(|f'(b)|^s+|f'(a)|^s)^{\frac{1}{s}}.$$

□

2.2. Applications

By using p-convexity of the function and the derived inequalities, some bounds and inequalities involving Beta functions can be obtained. To do this we use the function $f(x)=x^2$. Let us show that it is p-convex function on any p-convex set of real number:

$$\begin{aligned} f(t^{\frac{1}{p}}x+(1-t)^{\frac{1}{p}}y) &= (t^{\frac{1}{p}}x+(1-t)^{\frac{1}{p}}y)^2 \\ &= (t^{\frac{1}{p}})^2x^2+2t^{\frac{1}{p}}(1-t)^{\frac{1}{p}}xy+((1-t)^{\frac{1}{p}})^2y^2 \\ &\le (t^{\frac{1}{p}})^2x^2+t^{\frac{1}{p}}(1-t)^{\frac{1}{p}}(x^2+y^2)+((1-t)^{\frac{1}{p}})^2y^2 \\ &= t^{\frac{1}{p}}x^2(t^{\frac{1}{p}}+(1-t)^{\frac{1}{p}})+(1-t)^{\frac{1}{p}}y^2(t^{\frac{1}{p}}+(1-t)^{\frac{1}{p}}) \\ &\le t^{\frac{1}{p}}x^2(t+1-t)+(1-t)^{\frac{1}{p}}y^2(t+1-t) \\ &= t^{\frac{1}{p}}x^2+(1-t)^{\frac{1}{p}}y^2 \\ &= t^{\frac{1}{p}}f(x)+(1-t)^{\frac{1}{p}}f(y). \end{aligned}$$

Moreover by making use of some theorems in Section 2.1, we suggest an upper bound for error in numerical integration of p-convex functions via composite trapezoid rule.
Using Theorem 2.3 and Theorem 2.10, we obtain two similar results involving Beta functions.

Proposition 2.15. *Let $a,b\in\mathbb{R}$ with $a<b$ and $\alpha>1$. Then*

$$4ab(3a^2+2ab+3b^2)\alpha\mathrm{B}(\alpha,2\alpha)+3ab(a+b)^2\alpha\mathrm{B}(\alpha,\alpha)+(5ab^3+a^3b+8a^4+6b^4)\ge 0.$$

Proof. Applying Theorem 2.3 for $f(x)=\frac{x^3}{3}$, whose derivative is p-convex function, then making substitution $\alpha=\frac{1}{p}$ with $0<p<1$, we have the desired inequality. □

Proposition 2.16. *Let $a,b\in\mathbb{R}$ and $\alpha>1$. Then*

$$47a^4+16a^3b+30a^2b^2+24ab^3+43b^4\ge\alpha(a+b)^2\left[6\left(2ab-3(a^2+b^2)\right)\mathrm{B}(\alpha,\alpha)+2\left(6ab-19(a^2+b^2)\right)\mathrm{B}(\alpha,2\alpha)\right].$$

Proof. Applying the same ideas in proof of Proposition 2.15 to Theorem 2.10 yield to the desired inequality. □

Making some algebraic manipulations in both propostion above, we can get an inequality with respect to one variable. Considerig these propositions for positive numbers a,b with $a<b$, dividing both side by b^4, taking $t=\frac{a}{b}$ $(0<t<1)$, multiplying both side of inequality with $(1-t)^{\alpha}$ $(\alpha>1)$, then integrating both side with respect to t on $[0,1]$, we have the following corollaries corresponding to Proposition 2.15 and Proposition 2.16, respectively:

Corollary 2.17. *Let $\alpha>1$. Then*

$$4\alpha\left(\frac{7}{\alpha+1}-\frac{14}{\alpha+2}+\frac{9}{\alpha+3}-\frac{2}{\alpha+4}\right)\mathrm{B}(\alpha,2\alpha)\quad+3\alpha\left(-\frac{4}{\alpha+1}+\frac{8}{\alpha+2}-\frac{5}{\alpha+3}+\frac{1}{\alpha+4}\right)\mathrm{B}(\alpha,\alpha)$$
$$\le\left(\frac{20}{\alpha+1}-\frac{40}{\alpha+2}+\frac{51}{\alpha+3}-\frac{33}{\alpha+4}+\frac{8}{\alpha+5}\right).$$

Corollary 2.18. *For* $\alpha > 1$,

$$6\left(-\tfrac{16}{\alpha+1}-\tfrac{32}{\alpha+2}+\tfrac{32}{\alpha+3}+\tfrac{16}{\alpha+4}-\tfrac{3}{\alpha+5}\right)\alpha\mathrm{B}(\alpha,\alpha) \quad -2\left(\tfrac{128}{\alpha+1}-\tfrac{256}{\alpha+2}+\tfrac{236}{\alpha+3}-\tfrac{108}{\alpha+4}+\tfrac{19}{\alpha+5}\right)\alpha\mathrm{B}(\alpha,2\alpha)$$
$$\leq\left(\tfrac{160}{\alpha+1}-\tfrac{320}{\alpha+2}+\tfrac{360}{\alpha+3}-\tfrac{204}{\alpha+4}+\tfrac{47}{\alpha+5}\right).$$

Using the inequalities obtained via Hölder inequality, we can have the following generalized inequalities with respect to s.

Proposition 2.19. *Let* $a,b\in(0,\infty)$ *with* $a<b$ *and* $0<p,\ \alpha<1$. *Then,*

$$\left|\frac{a^{2\alpha+1}+b^{2\alpha+1}}{2}-\frac{b^{2\alpha+2}-a^{2\alpha+2}}{2(\alpha+1)(b-a)}\right|\leq\frac{3}{2p}\left(\frac{p}{p+1}\right)^{\alpha}\frac{(a+b)^2\left(a^2+b^2\right)^{\alpha}}{(b-a)}.$$

Proof. In Theorem 2.5, let $f(x)=\frac{s}{s+2}x^{\frac{2}{s}+1}$ on $[0,\infty)$ and $a<b$. Then $|f'(x)|^s$ is p-convex. We have

$$\left|\frac{s}{2(s+2)}\left(a^{\frac{2}{s}+1}+b^{\frac{2}{s}+1}\right)-\frac{1}{b-a}\frac{s}{s+2}\frac{s}{2(s+1)}\left(b^{\frac{2}{s}+2}-a^{\frac{2}{s}+2}\right)\right|\leq\frac{3}{2p(b-a)}\left(\frac{p}{p+1}\right)^{\frac{1}{s}}(a+b)^2\left(a^2+b^2\right)^{\frac{1}{s}}. \quad (2.7)$$

The substitution $\alpha=\frac{1}{s}$ and algebraical manipulations yield to desired inequality. ☐

Some algebraic manipulations in proposition above yield to the inequality involving a hypergeometric function.

Proposition 2.20. *For* $s>1$ *and* $0<p<1$,

$$\frac{1}{4}\frac{-9s^3+34s+16s^2+12}{(3s+2)(s+2)(s+1)}\leq\frac{3}{2p}\left(\frac{p}{p+1}\right)^{\frac{1}{s}}\frac{1}{s+1}\left(\frac{2}{(3s+2)}\left((s+1)^2\cdot{}_2F_1\left(\frac{1}{2},-\frac{1}{s};\frac{3}{2};-1\right)+2^{\frac{1}{s}}s(4s+3)\right)-s\right)$$

where ${}_2F_1$ *is hypergeometric function, i.e.*

$${}_2F_1(\alpha,\beta;\gamma;z)=\frac{1}{B(\beta,\gamma-\beta)}\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-tz)^{-\alpha}dt,\ \ (\gamma>\beta>0).$$

Proof. It is clear that the expression inside the absolute value in (2.7) is less then or equal to right side. Multiplying this inequality with $b-a$, dividing both side by $b^{\frac{2}{s}+2}$, taking $t=\frac{a}{b}$ and integrating both side with respect to t on $[0,1]$, we have desired result. ☐

Proposition 2.21. *Let* $0<p,\alpha<1$ *and* $a,b\in(0,\infty)$ *with* $a<b$. *Then*

$$\left|\left(\tfrac{a+b}{2}\right)^{2\alpha+1}-\tfrac{1}{1+\alpha}\left(\tfrac{b^{2\alpha+2}-a^{2\alpha+2}}{b-a}\right)\right|$$
$$\leq\tfrac{1}{2^{2\alpha+1}}\tfrac{2\alpha+1}{p(b-a)}\left(\tfrac{p}{p+1}\right)^{\alpha}\left(\left(2ab+5a^2+b^2\right)^{\alpha}(3a+b)(5a+b)+(a+3b)(a+5b)\left(2ab+a^2+5b^2\right)^{\alpha}\right).$$

Proof. In Theorem 2.11, let $f(x)=\frac{s}{s+2}x^{\frac{2}{s}+1}$ on $[0,\infty)$ and $a<b$. Then $|f'(x)|^s$ is p-convex. We have

$$\left|\tfrac{s}{2(s+2)}\left(\tfrac{a+b}{2}\right)^{\frac{2}{s}+1}-\tfrac{s}{s+2}\tfrac{s}{2(s+1)}\left(\tfrac{b^{\frac{2}{s}+2}-a^{\frac{2}{s}+2}}{b-a}\right)\right|\leq\tfrac{1}{p(b-a)}\left(\tfrac{p}{p+1}\right)^{\frac{1}{s}}\left(2a+\tfrac{a+b}{2}\right)\left(a+\tfrac{a+b}{2}\right)\left(\left(\tfrac{a+b}{2}\right)^2+a^2\right)^{\frac{1}{s}}$$
$$+\tfrac{1}{p(b-a)}\left(\tfrac{p}{p+1}\right)^{\frac{1}{s}}\left(2b+\tfrac{a+b}{2}\right)\left(b+\tfrac{a+b}{2}\right)\left(\left(\tfrac{a+b}{2}\right)^2+b^2\right)^{\frac{1}{s}}.$$

The substitution $\alpha=\frac{1}{s}$ and algebraical manipulations yield to desired inequality. ☐

Proposition 2.22. *Let* $0<\alpha<p<1$ *and* $a,b\in(0,\infty)$ *with* $a<b$. *Then*

$$\frac{b^{2\alpha+2}-a^{2\alpha+2}}{2(\alpha+1)}\leq\frac{1}{p}\left(\frac{p}{p+1}\right)^{\alpha}(a+b)^2\left(b^2+a^2\right)^{\alpha}.$$

Proof. In Theorem 2.14, let $f(x)=\frac{s}{s+2}x^{\frac{2}{s}+1}$ on $[0,\infty)$ and $a<b$. Then $|f'(x)|^s$ is p-convex. We have

$$\left|\frac{s}{(s+2)}\left(b^{\frac{2}{s}+2}-a^{\frac{2}{s}+2}\right)-\frac{s}{s+2}\frac{s}{2(s+1)}\left(b^{\frac{2}{s}+2}-a^{\frac{2}{s}+2}\right)\right|\leq\frac{1}{p}\left(\frac{p}{p+1}\right)^{\frac{1}{s}}(a+b)^2\left(b^2+a^2\right)^{\frac{1}{s}}.$$

The substitution $\alpha=\frac{1}{s}$ and algebraical manipulations yield to desired inequality. ☐

When the same idea in the proof of Proposition 2.20 is applied to the inequality in Propositon 2.22, we have the following result involving a hypergeometric function.

Corollary 2.23. *For* $s>1$ *and* $p\in(0,1]$ *with* $p>\frac{1}{s}$,

$$\frac{s+1}{s}\leq\frac{1}{p}\left(\frac{p}{p+1}\right)^{\frac{1}{s}}\left(2(\frac{s+1}{s})^2\,{}_2F_1(\frac{1}{2},-\frac{1}{s};\frac{3}{2};-1)+2^{\frac{1}{s}+3}+2(3\,2^{\frac{1}{s}}-1)\frac{1}{s}-3\right).$$

Moreover by making use of some theorems in main results, we can find an upper bound for error in numerical integration of p-convex functions via composite trapezoid rule.
Let f be an integrable function on $[a,b]$ and P be a partition of the interval $[a,b]$, i.e. $P: a=x_0<x_1<\cdots<x_{n-1}<x_n=b$ and $\Delta x_i=x_i-x_{i-1}$. Then

$$\int_a^b f(x)dx=\sum_{k=0}^{n-1}\frac{f(x_k)+f(x_{k+1})}{2}\Delta x_k+E(f,P) \tag{2.8}$$

where $E(f,P)$ is called the error of integral with respect to P. There are some ways to estimate an upper bound for $E(f,P)$. For p-convex functions, we suggest the following proposition:

Proposition 2.24. *Let* $f:\mathbb{R}\to\mathbb{R}$ *be differentiable function and* $|f'|\in L[a,b]$ *and p-convex function on* $\mathbb{R}$. *Suppose that P is a partition of* $[a,b]$. *Then,*

$$|E(f,P)|\leq\frac{3}{2(p+1)}\sum_{k=0}^{n-1}(|x_k|+|x_{k+1}|)^2\left(|f'(x_k)|+|f'(x_{k+1})|\right).$$

Proof. Applying Theorem 2.2 on $[x_k,x_{k+1}]$, we have

$$\left|\frac{f(x_k)+f(x_{k+1})}{2}-\frac{1}{x_{k+1}-x_k}\int_{x_k}^{x_{k+1}}f(x)dx\right|\leq\frac{3}{2(p+1)(x_{k+1}-x_k)}(|x_k|+|x_{k+1}|)^2\left(|f'(x_k)|+|f'(x_{k+1})|\right). \tag{2.9}$$

Then using (2.8) and (2.9), we get the desired result as follows:

$$\begin{aligned}|E(f,P)| &=\left|\sum_{k=0}^{n-1}\frac{f(x_k)+f(x_{k+1})}{2}\Delta x_k-\int_a^b f(x)dx\right|\\ &=\left|\sum_{k=0}^{n-1}\left(\frac{f(x_k)+f(x_{k+1})}{2}\Delta x_k-\int_{x_k}^{x_{k+1}}f(x)dx\right)\right|\\ &\leq\sum_{k=0}^{n-1}\left|\frac{f(x_k)+f(x_{k+1})}{2}\Delta x_k-\int_{x_k}^{x_{k+1}}f(x)dx\right|\\ &=\sum_{k=0}^{n-1}\Delta x_k\left|\frac{f(x_k)+f(x_{k+1})}{2}-\frac{1}{x_{k+1}-x_k}\int_{x_k}^{x_{k+1}}f(x)dx\right|.\end{aligned}$$

□

Proposition 2.25. *Let* $f\in D[a,b]$ *such that* $|f'|^s\in L[a,b]$ *and p-convex function on* $\mathbb{R}$. *Suppose that P is a partition of* $[a,b]$. *Then*

$$|E(f,P)|\leq\frac{3}{2p}\left(\frac{p}{p+1}\right)^{\frac{1}{s}}\sum_{k=0}^{n-1}(|x_k|+|x_{k+1}|)^2\left(|f'(x_k)|^s+|f'(x_{k+1})|^s\right)^{\frac{1}{s}}.$$

Proof. Applying Theorem 2.5 in a similar way to proof of the proposition. □

3. Conclusion

In this article, some upper boundaries related to Hermite-Hadamard type inequalities for the functions of real numbers whose derivatives are p-convex are obtained and by means of these results some interesting applications are given. Basically, setting three integral equalities containing the derivative of a function, we present new inequalities involving p-convex functions. Then, these are extended to the powers of the derivative of the function via the Hölder inequality. For the applications section, it has been shown that $f(x)=x^2$ is p-convex and through this, the inequalities related to Beta and Hypergeometric functions are obtained. In addition, an upper bound has been obtained for the errors in numerical integration via the composite trapezoid rule of the functions whose derivative and some powers of derivative are p-convex. This study is based on the fact that p-convex functions are defined on real numbers and some applications are obtained via only few examples of functions. In the future, more interesting inequalities regarding special functions can be obtained through different examples of p-convex functions. The introduction of p-convex functions and their properties for n dimensional case are given in [22]. By making use of that study, the existence of similar results can be investigated for multiple integrals.

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