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Fixed Point Results for Zamfirescu Mappings in *A*-metric Spaces

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Abstract. In the present paper, we extend the Zamfirescu results ([9]) to A-metric spaces. Firstly, we define the notion of Zamfirescu mapping in A-metric spaces. After, we also obtain a fixed point theorem for such mappings. The established results carry some well-known results from the literature (see [2, 3, 5, 9]) to A-metric spaces. An appropriate example is also given.

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1. Introduction and Preliminaries

Zamfirescu's fixed point theorem is one of the most important extensions of Banach contraction principle. In 1972, Zamfirescu [9] obtained the following a very interesting fixed point theorem.

Theorem 1.1. *Let (U, d) be a complete metric space and let $T : U \to U$ be a mapping for which there exists the real numbers a, b and c satisfying $a \in (0, 1)$; $b, c \in \left(0, \frac{1}{2}\right)$ such that for each pair $u, v \in U$, at least one of the following conditions holds:*
(Z_1) $d(Tu, Tv) \leq ad(u, v)$;
(Z_2) $d(Tu, Tv) \leq b\,[d(u, Tu) + d(v, Tv)]$;
(Z_3) $d(Tu, Tv) \leq c\,[d(u, Tv) + d(v, Tu)]$.
Then T has a unique fixed point u^* and the Picard iteration $\{u_n\}$ defined by $u_{n+1} = Tu_n$ converges to u^* for any arbitrary fixed $u_0 \in U$.

An operator T satisfying the contractive conditions (Z_1), (Z_2) and (Z_3) in the above theorem is called Zamfirescu mapping. Zamfirescu's theorem is a generalization of Banach's, Kannan's and Chatterjea's fixed point theorems (see [2, 3, 5]). Many researchers studied to obtain new classes of contraction mappings in different metric spaces. Some of them are D^*-metric space (see [8]), G-metric space (see [6]), S-metric space (see [7]), A-metric space (see [1]), etc., as a generalization of the usual metric space. These generalizations helped the development of fixed point theory.

In 2006, Mustafa and Sims [6] introduced the notion of G-metric space. After, Sedghi et al. [7] defined the concept of D^*-metric and S-metric spaces. Also, they proved some fixed point theorems in such spaces. Every G-metric space is a D^*-metric space and every D^*-metric space is an S-metric space. That is, S-metric space is a generalization of the G-metric space and the D^*-metric space.

In 2015, Abbas et al. [1] introduced the concept of an A-metric space as follows:

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Definition 1.2. Let U be nonempty set. Suppose a mapping $A : U^t \to \mathbb{R}$ satisfies the following conditions:
(A_1) $A(u_1, u_2, ..., u_{t-1}, u_t) \geq 0$,
(A_2) $A(u_1, u_2, ..., u_{t-1}, u_t) = 0$ if and only if $u_1 = u_2 = ... = u_{t-1} = u_t$,
(A_3) $A(u_1, u_2, ..., u_{t-1}, u_t) \leq A(u_1, u_1, ..., (u_1)_{t-1}, v) + A(u_2, u_2, ..., (u_2)_{t-1}, v) + ... + A(u_{t-1}, u_{t-1}, ..., (u_{t-1})_{t-1}, v) + A(u_t, u_t, ..., (u_t)_{t-1}, v)$,
for any $u_i, v \in U$, $(i = 1, 2, ..., t)$. Then, (U, A) is said to be an A-metric space.

It is clear that the an A-metric space for $t = 2$ reduces to ordinary metric d and an A-metric space for $t = 3$ reduces to S-metric spaces. So, an A-metric space is a generalization of the G-metric space, the D^*-metric space and the S-metric space.

Example 1.3 ([1]). Let $U = \mathbb{R}$. Define a function $A : U^t \to \mathbb{R}$ by

$$\begin{aligned} A(u_1, u_2, ..., u_{t-1}, u_t) &= |u_1 - u_2| + |u_1 - u_3| + ... + |u_1 - u_t| \\ &\quad + |u_2 - u_3| + |u_2 - u_4| + ... + |u_2 - u_t| \\ &\quad \vdots \\ &\quad + |u_{t-2} - u_{t-1}| + |u_{t-2} - u_t| + |u_{t-1} - u_t| \\ &= \sum_{i=1}^{t} \sum_{i<j} \left| u_i - u_j \right|. \end{aligned}$$

Then (U, A) is an A-metric space.

In 2017, Fernandez et al. [4] introduced the generalized Lipschitz mapping, Chatterjea's and Kannan's mappings in an A-cone metric space over Banach algebra. Also, they proved some fixed point theorems for the above mappings in complete A-cone metric space (U, A) over Banach algebra.

Next, we state the following useful lemmas and definition.

Lemma 1.4 ([1]). *Let (U, A) be an A-metric space. Then $A(u, u, \ldots, u, v) = A(v, v, \ldots, v, u)$ for all $u, v \in U$.*

Lemma 1.5 ([1]). *Let (U, A) be an A-metric space. Then for all $u, v, z \in U$ we have $A(u, u, \ldots, u, z) \leq (t-1) A(u, u, \ldots, u, v) + A(z, z, \ldots, z, v)$ and $A(u, u, \ldots, u, z) \leq (t-1) A(u, u, \ldots, u, v) + A(v, v, \ldots, v, z)$.*

Definition 1.6 ([1]). Let (U, A) be an A-metric space.
(i) A sequence $\{u_n\}$ in U is said to converge to a point $u \in U$ if $A(u_n, u_n, \ldots, u_n, u) \to 0$ as $n \to \infty$,
(ii) A sequence $\{u_n\}$ in U is called a Cauchy sequence if $A(u_n, u_n, \ldots, u_n, u_m) \to 0$ as $n, m \to \infty$,
(iii) The A-metric space (U, A) is said to be complete if every Cauchy sequence in U is convergent.

2. Main Results

In this section, following the ideas of Zamfirescu [9] we first introduce the notion of Zamfirescu mappings in A-metric space as follows:

Definition 2.1. Let (U, A) be an A-metric space and let $T : U \to U$ be a mapping. T is called a A-Zamfirescu mapping (AZ mapping), if and only if, there are real numbers, $0 \leq a < 1$, $0 \leq b, c < \frac{1}{t}$ such that for all $u, v \in U$, at least one of the next conditions is true:

$$(AZ_1)\ A(Tu, Tu, \ldots, Tu, Tv) \leq aA(u, u, \ldots, u, v),$$
$$(AZ_2)\ A(Tu, Tu, \ldots, Tu, Tv) \leq b\left[A(Tu, Tu, \ldots, Tu, u) + A(Tv, Tv, \ldots, Tv, v)\right],$$
$$(AZ_3)\ A(Tu, Tu, \ldots, Tu, Tv) \leq c\left[A(Tu, Tu, \ldots, Tu, v) + A(Tv, Tv, \ldots, Tv, u)\right].$$

It is clear that if we take $t = 2$ in the Definition 2.1, we obtain the definition of Zamfirescu [9] in ordinary metric space. Before giving the our main result in A-metric space, we need the following significant lemma.

Lemma 2.2. *Let (U, A) be an A-metric space and let $T : U \to U$ be a mapping. If T is a AZ mapping, then there is $0 \leq \delta < 1$ such that*

$$A(Tu, Tu, \ldots, Tu, Tv) \leq \delta A(u, u, \ldots, u, v) + t\delta A(Tu, Tu, \ldots, Tu, u)$$

and

$$A(Tu, Tu, \ldots, Tu, Tv) \leq \delta A(u, u, \ldots, u, v) + t\delta A(Tv, Tv, \ldots, Tv, u)$$

for all $u, v \in U$.

Proof. Let's assume that (AZ_2) is hold. From Lemma 1.5, we have

$$\begin{aligned} A(Tu,Tu,\ldots,Tu,Tv) &\leq b\,[A(Tu,Tu,\ldots,Tu,u)+A(Tv,Tv,\ldots,Tv,v)] \\ &\leq b\left[\begin{array}{c} A(Tu,Tu,\ldots,Tu,u)+(t-1)\,A(Tv,Tv,\ldots,Tv,Tu) \\ +A(Tu,Tu,\ldots,Tu,v) \end{array}\right] \\ &\leq b\,[A(Tu,Tu,\ldots,Tu,u)+(t-1)\,A(Tv,Tv,\ldots,Tv,Tu) \\ &\quad +(t-1)\,A(Tu,Tu,\ldots,Tu,u)+A(u,u,\ldots,u,v)]\,. \end{aligned}$$

Thus,

$$(1-b\,(t-1))\,A(Tu,Tu,\ldots,Tu,Tv)\leq tbA(Tu,Tu,\ldots,Tu,u)+bA(u,u,\ldots,u,v).$$

From the fact that $0\leq b<\frac{1}{t}$, we get

$$A(Tu,Tu,\ldots,Tu,Tv)\leq\frac{b}{1-b\,(t-1)}A(u,u,\ldots,u,v)+\frac{tb}{1-b\,(t-1)}A(Tu,Tu,\ldots,Tu,u).$$

Assume that (AZ_3) is hold. From Lemmas 1.4 and 1.5, similarly we get

$$\begin{aligned} A(Tu,Tu,\ldots,Tu,Tv) &\leq c\,[A(Tu,Tu,\ldots,Tu,v)+A(Tv,Tv,\ldots,Tv,u)] \\ &\leq c\left[\begin{array}{c} A(Tu,Tu,\ldots,Tu,v)+(t-1)\,A(Tv,Tv,\ldots,Tv,Tu) \\ +A(Tu,Tu,\ldots,Tu,u) \end{array}\right] \\ &\leq c\,[A(Tu,Tu,\ldots,Tu,v)+(t-1)\,A(Tv,Tv,\ldots,Tv,Tu) \\ &\quad +(t-1)\,A(Tu,Tu,\ldots,Tu,v)+A(v,v,\ldots,v,u)]\,. \end{aligned}$$

Thus,

$$(1-c\,(t-1))\,A(Tu,Tu,\ldots,Tu,Tv)\leq tcA(Tu,Tu,\ldots,Tu,v)+cA(u,u,\ldots,u,v).$$

From the fact that $0\leq c<\frac{1}{t}$, we get

$$A(Tu,Tu,\ldots,Tu,Tv)\leq\frac{c}{1-c\,(t-1)}A(u,u,\ldots,u,v)+\frac{tc}{1-c\,(t-1)}A(Tu,Tu,\ldots,Tu,v).$$

Therefore, denoting by

$$\delta=max\left\{a,\frac{b}{1-b\,(t-1)},\frac{c}{1-c\,(t-1)}\right\},$$

we have that $0\leq\delta<1$. Thus, the following inequalities hold:

$$A(Tu,Tu,\ldots,Tu,Tv)\leq\delta A(u,u,\ldots,u,v)+t\delta A(Tu,Tu,\ldots,Tu,u) \tag{2.1}$$

and

$$A(Tu,Tu,\ldots,Tu,Tv)\leq\delta A(u,u,\ldots,u,v)+t\delta A(Tv,Tv,\ldots,Tv,u) \tag{2.2}$$

for all $u,v\in U$. □

Theorem 2.3. *Let (U,A) be a complete A-metric space and let $T:U\to U$ be an AZ mapping. Then U has a unique fixed point and Picard iteration process $\{u_n\}$ defined by $u_{n+1}=Tu_n$ converges to a fixed point of T.*

Proof. Let $u_0\in U$ be arbitrary and $\{u_n\}$ be the Picard iteration as $u_{n+1}=Tu_n$.

If we take $u=u_n$ and $v=u_{n-1}$ at the inequality (2.2), we obtain that

$$\begin{aligned} A\left(u_{n+1},u_{n+1},\ldots,u_{n+1},u_n\right) &= A\left(Tu_n,Tu_n,\ldots,Tu_n,Tu_{n-1}\right) \\ &\leq \delta A\left(u_n,u_n,\ldots,u_n,u_{n-1}\right)+t\delta A\left(u_n,u_n,\ldots,u_n,u_n\right). \end{aligned}$$

From above the inequality, we get

$$\begin{aligned} A\left(u_{n+1},u_{n+1},\ldots,u_{n+1},u_n\right) &\leq \delta A\left(u_n,u_n,\ldots,u_n,u_{n-1}\right) \\ &\leq \delta^2 A\left(u_{n-1},u_{n-1},\ldots,u_{n-1},u_{n-2}\right) \\ &\vdots \\ &\leq \delta^n A\left(u_1,u_1,\ldots,u_1,u_0\right) \\ &= \delta^n A\left(u_0,u_0,\ldots,u_0,u_1\right). \end{aligned}$$

Let $m > n$. Using Lemma 1.5 and the above inequality, we get

$$\begin{aligned}
A(u_n, u_n, \ldots, u_n, u_m) &\leq (t-1)\,\delta A(u_n, u_n, \ldots, u_n, u_{n+1}) + A(u_{n+1}, u_{n+1}, \ldots, u_{n+1}, u_m) \\
&\leq (t-1)\,\delta A(u_n, u_n, \ldots, u_n, u_{n+1}) + (t-1)\,A(u_{n+1}, u_{n+1}, \ldots, u_{n+1}, u_{n+2}) \\
&\quad + A(u_{n+2}, u_{n+2}, \ldots, u_{n+2}, u_m) \\
&\leq (t-1)\,\delta A(u_n, u_n, \ldots, u_n, u_{n+1}) + (t-1)\,A(u_{n+1}, u_{n+1}, \ldots, u_{n+1}, u_{n+2}) \\
&\quad + \cdots + (t-1)\,A(u_{m-2}, u_{m-2}, \ldots, u_{m-2}, u_{m-1}) \\
&\quad + A(u_{m-1}, u_{m-1}, \ldots, u_{m-1}, u_m) \\
&= (t-1)\left[\delta^n A(u_0, u_0, \ldots, u_0, u_1) + \delta^{n+1} A(u_0, u_0, \ldots, u_0, u_1)\right. \\
&\quad \left. + \cdots + \delta^{m-2} A(u_0, u_0, \ldots, u_0, u_1)\right] + \delta^{m-1} A(u_0, u_0, \ldots, u_0, u_1) \\
&= (t-1)\,\delta^n A(u_0, u_0, \ldots, u_0, u_1)\left[1 + \delta + \delta^2 + \cdots + \delta^{m-n-2}\right] \\
&\quad + \delta^{m-1} A(u_0, u_0, \ldots, u_0, u_1) \\
&\leq (t-1)\,\delta^n A(u_0, u_0, \ldots, u_0, u_1)\,\frac{\delta^m}{1-\delta} + \delta^{m-1} A(u_0, u_0, \ldots, u_0, u_1) \\
&= \left[(t-1)\,\frac{\delta^{m+n}}{1-\delta} + \delta^{m-1}\right] A(u_0, u_0, \ldots, u_0, u_1)\,.
\end{aligned}$$

We know that $0 \leq \delta < 1$ from Lemma 2.2. Suppose that $A(u_0, u_0, \ldots, u_0, u_1) > 0$. If we take limit as $m, n \to \infty$ in above inequality we get

$$\lim_{n,m\to\infty} A(u_n, u_n, \ldots, u_n, u_m) = 0.$$

Therefore $\{u_n\}$ is a Cauchy sequence in U. Also, assume that $A(u_0, u_0, \ldots, u_0, u_1) = 0$, then $A(u_n, u_n, \ldots, u_n, u_m) = 0$ for all $m > n$ and $\{u_n\}$ is a Cauchy sequence in U. Since (U, A) is a complete metric space, $u_n \to u^* \in U$ as $n \to \infty$.

We show that u^* is a fixed point of T. From (2.1), we have

$$\begin{aligned}
A(Tu^*, Tu^*, \ldots, Tu^*, u^*) &\leq (t-1)\,A(Tu^*, Tu^*, \ldots, Tu^*, Tu_n) + A(Tu_n, Tu_n, \ldots, Tu_n, u^*) \\
&\leq (t-1)\left[\delta A(u^*, u^*, \ldots, u^*, u_n) + t\delta A(u^*, u^*, \ldots, u^*, Tu_n)\right] \\
&\quad + A(u_{n+1}, u_{n+1}, \ldots, u_{n+1}, u^*) \\
&= (t-1)\left[\delta A(u^*, u^*, \ldots, u^*, u_n) + t\delta A(u^*, u^*, \ldots, u^*, u_{n+1})\right] \\
&\quad + A(u_{n+1}, u_{n+1}, \ldots, u_{n+1}, u^*)\,.
\end{aligned}$$

If we take limit for $n \to \infty$ in above inequality, we obtain that $Tu^* = u^*$. That is, u^* is a fixed point of the mapping T. Now, we show that the uniqueness of fixed point of T. Assume that u^* and v^* are fixed point of T. That is, $Tu^* = u^*$ and $Tv^* = v^*$. From (2.1), we have

$$\begin{aligned}
A(u^*, u^*, \ldots, u^*, v^*) &= A(Tu^*, Tu^*, \ldots, Tu^*, Tv^*) \\
&\leq \delta A(u^*, u^*, \ldots, u^*, v^*) + t\delta A(Tu^*, Tu^*, \ldots, Tu^*, u^*)\,.
\end{aligned}$$

Thus,

$$A(u^*, u^*, \ldots, u^*, v^*) \leq \delta A(u^*, u^*, \ldots, u^*, v^*)\,.$$

This implies that $A(u^*, u^*, \ldots, u^*, v^*) = 0 \Longrightarrow u^* = v^*$ and hence, T has a unique fixed point in U. □

Remark 2.4. Putting $t = 2$ in Theorem 2.3, we obtain the Theorem 1.1. Hence, Theorem 2.3 is a generalization of Theorem 1.1 of Zamfirescu [9] in A-metric space.

Example 2.5. Let $U = \mathbb{R}$. Define a function $A : U^t \to [0, \infty)$ by

$$A(u_1, u_2, ..., u_{t-1}, u_t) = \sum_{i=1}^{t} \sum_{i<j} \left|u_i - u_j\right|$$

for all $u_i \in U$, $i = 1, 2, ..., t$. Then (U, A) is complete A-metric space.

If we define $T : U \to U$ by $Tu = \frac{2u}{7}$, then T satisfy the conditions of Theorem 2.3. For all $u_i \in U,\ i = 1, 2, ..., t,$

$$\begin{aligned}
A(Tu_1, Tu_2, ..., Tu_{t-1}, Tu_t) &= A(\frac{2u_1}{7}, \frac{2u_2}{7}, ..., \frac{2u_{t-1}}{7}, \frac{2u_t}{7}) \\
&= \frac{2}{7}|u_1 - u_2| + \frac{2}{7}|u_1 - u_3| + ... + \frac{2}{7}|u_1 - u_t| \\
&\quad + \frac{2}{7}|u_2 - u_3| + \frac{2}{7}|u_2 - u_4| + ... + \frac{2}{7}|u_2 - u_t| \\
&\quad \vdots \\
&\quad + \frac{2}{7}|u_{t-2} - u_{t-1}| + \frac{2}{7}|u_{t-2} - u_t| + \frac{2}{7}|u_{t-1} - u_t| \\
&= \frac{2}{7}\sum_{i=1}^{t}\sum_{i<j}\left|u_i - u_j\right| \\
&= \frac{2}{7}A(u_1, u_2, ..., u_{t-1}, u_t)
\end{aligned}$$

where $\frac{2}{7} \in [0, 1)$. This implies that T is a AZ mapping. And $u = 0$ is the unique fixed point of T in U as asserted by Theorem 2.3.

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this article.

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