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A GRAPHICAL TOOL FOR EXTREME VALUE COPULA SELECTION BASED ON THE PICKANDS DEPENDENCE FUNCTION

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Abstract: We present a graphical tool that was primarily proposed by Michiels et al. [18] and later modified by Durante et al. [4]. We also improve this method to select the better fit of the given data among some extreme value copulas based on the Pickands dependence function. We conduct a Monte Carlo simulation study to investigate its performance. Also, the graphical method is illustrated by a real data example.

Key words: Copula, Pickands function, Extreme value copula.

1. Introduction

A copula is a joint distribution of the random variables U and V, each of which is marginally uniformly distributed as $U(0,1)$. Sklar's [20] theorem states that for any bivariate random variables X,Y with a cumulative distribution function (CDF)

$$H(x,y) = P(X \leq x, Y \leq y)$$

and the marginal CDF $F(x) = P(X \leq x)$ and $G(y) = P(Y \leq y)$ then there exist a copula such as:

$$H(x,y) = C(F(x),G(y)) = C(u,v),$$

where $u = F(x)$ and $v = g(y)$. From the modelling perspective, Sklar's Theorem allows us to separate the modelling of the marginal distributions $F(x), G(y)$ from the dependence structure, which is expressed in C.

One of the most important fields of statistics is the extreme value (EV) theory. The estimation of the events outside the range of data should be estimated by the EV distributions such as the daily maximum air temperature, and annual maximum sea levels. The EV distribution is the limiting distribution for the minimum or the maximum of random observations. Pickands [19] states that the pair (X,Y) has an EV dependence if and only if its copula C can be expressed for all $u,v \in (0,1)$

$$C(u,v) = \exp\Big(\log(uv)A(\frac{\log(v)}{\log(uv)})\Big),$$

where $A(.)$ is the Pickands dependence function defined on $[0,1] \to [1/2,1]$. The Pickand's dependence function has some properties as follows:

- $A(0) = A(1) = 1$.
- A is the convex function.
- $\max(1-t,t) \leq A(t) \leq 1$ for all $t \in [0,1]$.

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The non-parametric estimation of the Pickands dependence is an important issue when dealing with extreme events. Let $\{X_i, Y_i\}_{i=1}^n$ be a n random observation from the random variables X and Y with the joint distribution function $H(x, y)$, copula $C(u, v)$ and the marginal distribution functions $F(x)$ and $G(y)$. Also, let $U_i = F(x_i)$ and $V_i = G(y_i)$ then put $S_i = -\log(U_i)$ and $T_i = -\log(V_i)$. For every $t \in [0, 1]$

$$\xi_i(t) = \min(\frac{S_i}{1-t}, \frac{T_i}{t}).$$

Pickands [19] introduced the non-parametric Pickands dependence function estimator as follows:

$$\hat{A}_P = \Big(\frac{1}{n}\sum_{i=1}^{n}\xi_i(t)\Big)^{-1}.$$

This estimator does not satisfy the conditions of the Pickands dependence function $A(.)$. Capéraá et al. [3] proposed an estimator called the CFG as follows:

$$\hat{A}_{CFG} = \exp\Big(-\gamma - \frac{1}{n}\sum_{i=1}^{n}\xi_i(t)\Big)^{-1},$$

where γ is Euler's constant that is $\gamma = -\int_0^{\inf}\log(x)\exp(x)dx$. In practice, marginals are rarely known. Thus, F and G should be estimated by their empirical counterparts $\hat{F}_n$ and $\hat{G}_n$ (Genest et al. [9]). In this paper, we use the corrected estimator $\hat{A}_{CFG}$ that is studied in Gudendorf et al. [10].

In the past few years, a certain number of papers have emerged which use Bernstein polynomials for the modelling of the extremal dependence, i.e. (Marcon et al. [16]; Guillotte et al. [11]; Marcon et al. [17]), to name a few. Also, Ahmadabadi et al. [1] investigated a new nonparametric approach using the Bernstein copula approximation. They used the Kernel regression method in order to derive an intrinsic estimator satisfying all the properties of the Pickands dependence function. See (Vettori et al. [21]) for a review.

The selection of EV copulas is an important issue when dealing with extreme situations. For this reason, many authors developed a tool for EV copulas selection. Michiels et al. [18] introduced a graphical tool for copula selection, based on the principal coordinate analysis. The main idea of this paper is that calculating the distance between the empirical copula and the parametric families of copulas then the calculated distances are visualized in 2D space via principal coordinate analysis. Also, Durante et al. [4] proposed the graphical tool in order to detect which families of copulas are closer to the empirical copula in tail dependence behavior.

In this study, we present a graphical tool that was firstly proposed by Michiels et al. [18] and later modified by Durante et al. [4]. We also improve this method to select the better fit of the given data among some extreme value copulas based on the Pickands dependence function. The EV copulas exhibit a similar upper tail dependence structure in terms of the tail concentration function. Thus, the tail concentration function proposed in Durante et al. [4] may fail to detect the tail dependence structure for the extreme value copulas for the same dependence level. In Figure 1, tail concentration functions of five EV copulas with the same Kendall's tau ($\tau = 0.5$) are presented. From this figure, it is hard to distinguish tail concentration function visually for the same dependence level among all EV copulas. For this reason, we prefer using the Pickands function in order to select the best suited extremely distributed random variables in the graphical method proposed in Durante et al. [4]. The extreme value copula is characterized by the Pickands dependence function; therefore, it can be useful in determining the best-fitted model for the bivariate extreme events. Although the test statistic proposed by Genest et al. [8] are consistent and effective tools for distinguishing between the symmetric and asymmetric extreme value copulas, the processing time is drawn out when dealing with big data because the test procedure involves the bootstrap method for estimating the p-values of the test statistic. For all these reasons, the graphical method based on the Pickands function can be used for determining the best-fitted EV copula for underlying data.

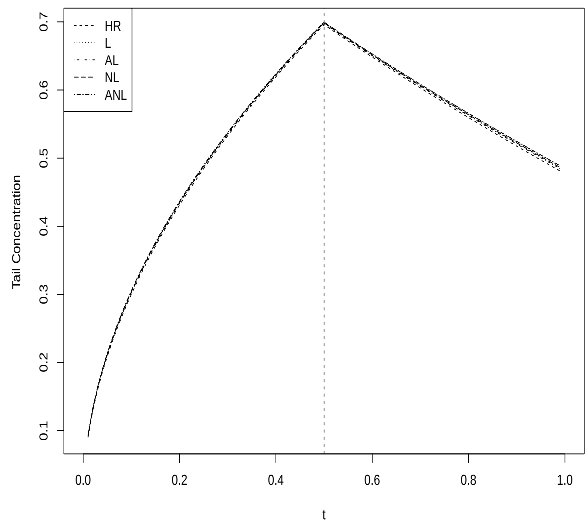


FIGURE 1. Tail concentration function for EV copulas with $\tau = 0.5$

The remainder of the study is organized as follows. In section 2, some EV copulas with Pickands dependence functions are introduced. In section 3, a graphical method to select EV copulas is presented. Some advantages of the proposed methods are discussed. Also, we performed a graphical method to show how accurately it works for a simulated data set from the EV copulas. In section 4, we apply the proposed method to the Danube dataset. Finally, the conclusion is given in the last section.

2. Some parametric extreme value copulas

Constructing a Pickands dependence function is one of the popular methods to obtain an EV copula. In this section, five EV copulas are introduced. Logistic (L) or Gumbel-Hougaard copula dating back to Gumbel [12] and Hougaard [13] can be considered as one of the oldest bivariate extreme value models. The logistic copula is the only copula that is at the same time as the extreme value and Archimedean copula. The Pickands dependence function of the Logistic copula with the dependence parameter θ given by:

$$A_L(t) = (t^\theta + (1-t)^\theta)^{\frac{1}{\theta}}.$$

The bivariate Asymmetric Logistic (AL) copulas Pickands dependence function with the dependence parameter $1 \leq \theta < \infty$ and asymmetry parameters α, β is given by

$$A_{AL}(t) = (1-\alpha)(1-t) + (1-\beta)t + \left((\alpha t)^\theta + (\beta(1-t))^\theta\right)^{\frac{1}{\theta}},$$

where $0 \leq \alpha,\ \beta \leq 1$. the Asymmetric Logistic copula adds further exibility to the Logistic copula. Note that by taking $\alpha = \beta = 1$, we can obtain the Logistic model, and by allowing $\alpha = \beta$, the Asymmetric Logistic copula is symmetric. The complete dependence is obtained. The complete dependence is obtained when $\alpha = \beta = 1$ and $\theta \to 0$. And, also the independence is obtained when $\theta = 1$ and $\alpha = 0$ or $\beta = 0$.

The bivariate Pickands function of the Negative Logistic (NL) model dating back to Galambos [7] is given by

$$A_{NL}(t) = 1 - (t^{-\theta} + (1-t)^{-\theta})^{-\frac{1}{\theta}},$$

where $\theta \in [0, \infty)$. Independence is obtained as $\theta = 0$ and complete dependence is obtained when $\theta \to \infty$.

The bivariate Asymmetric Negative Logistic copula dating back to Joe [15] is an extension of the Negative Logistic copula. The Joe copula has two parameters α and β which allow the model to be asymmetric. Pickands dependence function of Asymmetric Negative Logistic copula is given by

$$A_{ANL}(t) = 1 - ((\alpha t)^{-\theta} + (\beta(1-t))^{-\theta})^{-\frac{1}{\theta}},$$

where $0 \leq \alpha,\ \beta \leq 1$ and $\theta \in (0, \infty)$. Note that if $\alpha = \beta = 1$, we obatain the Negative Logistic copula. If $\alpha = \beta$, then the Asymmetric Negative Logistic copula is symmetric. Independence is obtained as $\alpha = \beta = 0$ or $\theta \to 0$ and complete dependence is obtained when $\alpha = \beta = 1$ and $\theta \to \infty$.

The Pickands dependence function of the bivariate Húsler-Reiss copula with parameter $\theta > 0$ is

$$A_{HR}(t) = (1-t)\phi(Z_{1-t}) + t\phi(Z_t),$$

where $\phi(.)$ is the standard normal distribution function and $Z_t = \frac{1}{\theta} + \frac{\theta}{2}\log(\frac{t}{1-t})$. Independence is obtained as $\theta \to 0$ and complete dependence is obtained when $\theta \to \infty$. For more details, see Húsler [14].

For the basics of the multivariate extreme value distributions and their Pickands dependence function see Dutfoy et al. [5] and Breachmann [2].

3. Graphical tool to select extreme value copula

In this section, we present a graphical tool that can help in the selection of the appropriate EV copula for underlying data set. Let $(X_i, Y_i)_{i=1}^n$ be a random sample from the EV copulas and $(U_i, V_i)_{i=1}^n$ be associated with the pseudo-observations. Consider a set of m EV copula's Pickands dependence function $A_1(.), \ldots, A_m(.)$ which belong to a different EV copula. A dissimilarity between the empirical estimate of the Pickands function $A_n(.)$ and the parametric Pickands function $A_i(.)$ for $i = 1, \ldots, m$ can be defined by

$$d(A_n, A_i) = \int_0^1 |A_n(t) - A_i(t)|^2 dt,\ i = 1, \ldots, m. \tag{3.1}$$

Similarly, the dissimilarity between the i-th and the j-th Pickands function is computed as

$$d(A_i, A_j) = \int_0^1 |A_i(t) - A_j(t)|^2 dt,\ 1 < i \neq j < m. \tag{3.2}$$

Let us give the procedure of graphical tool for selection of appropriate extreme value copula for the given data set. The procedure can be provided by following:

- For $i = 1, \ldots, m$ estimate dependence parameter(s) of a Pickands dependence function $A_i(.)$ from the family of the EV copula.
- For $i = 1, \ldots, m$ compute the dissimilarity between $A_i(.)$ and the corrected empirical estimate $\Delta_{(emp,i)} = d(\hat{A}_{CFG}, A_i)$ by using Eq. (3.1).
- For the m EV copulas Pickands function $A_1, \ldots, A_m$ compute mutual dissimilarities between $\Delta_{(i,j)} = d(A_i, A_j)$ by using Eq. (3.2).

- Symmetric square matrix of the dimension $m+1$, $D=\sigma_{(i,j)}$ can be defined as the following:

$$\sigma_{(1,j)} = \Delta_{(emp,j+1)},\ j=2,\dots,m+1,$$
$$\sigma_{(i,j)} = \Delta_{(i-1,j-1)},\ i,j=2,\dots,m+1,\ i<j,$$
$$\sigma_{(i,i)} = 0,\ i=1,\dots,m+1.$$

- Using the dissimilarity matrix D, a non-metric multidimensional scaling (NMDS) technique can be performed.

Dissimilarity matrix D contains L^2- type distances which contain the information about the relation among the $A_n(.)$ (empirical Pickands function), $A_1(.),\dots,A_m(.)$. In order to obtain a two-dimensional representation through the ranking of distances between A_n and $A_1,\dots,A_m$, a a non-metric multidimensional scaling (MDS) technique can be performed on D. Finally, the m points $p_i=(x_i,y_i)$ corresponding to Pickands function A_i and $p_{emp}=(x_{emp},y_{emp})$ corresponding to the empirical Pickands dependence function estimation A_n can be visualized in a two dimensional graph.

For Figures 1-5, we apply the NMDS method based on the Pickands function for each generated data sets from EV copulas. The procedure provides a graphical representation of the empirical Pickands function and the five fitted EV copulas (L: Logistic, AL: Asymmetric logistic, NL: Negative logistic, ANL: Asymmetric negative logistic, and HR: Husler-Reiss) in two dimensions for a stress a value lower than 100th of a percent of 0.05. As can be seen from Figures 1-5, the charts are often useful to determine the true data generating process except for asymmetric EV copulas.

Now, in order to assess the performance of graphical method for EV copulas, we conduct simulation study. Let the five points $p_i=(x_i,y_i)$, $i=1,\dots,5$ be corresponding to Pickands dependence function $A_i(.)$ of five EV copulas and $p_{emp}=(x_{emp},y_{emp})$ be corresponding to empirical estimation of Pickands dependence function, which are obtained by NMDS method in a 2D graph. We may define an Euclidean distances d_i^2 from the points p_i, $i=1,\dots,5$ to p_{emp} given by following:

$$d_i^2 = (x_{i;1}-x_{emp,1})^2 + (x_{i;2}-x_{emp,2})^2,\ \ i=1,\dots,5.$$

Thus, the point p_i, corresponding to Pickands function A_i, with smallest distance d_i^2 is the best choice for given data among all possible five EV copulas. By repeating this process K times for the randomly generated EV copula then we can measure the performance of the graphical method. Let $(X_{i,k},Y_{i,k})_{i=1,\dots,n}^{k=1,\dots,K}$ be K Monte Carlo samples of size n from EV copula. Also, $P_{i,k}=(x_{i,k},y_{i,k})_{i=1,\dots,5}^{k=1,\dots,K}$ and $P_{emp,k}=(x_{emp,k},y_{emp,k})^{k=1,\dots,K}$ be the points obtained by NMDS method in a 2D graph. The simulation procedure goes as follows. We can define Euclidean distances in 2D space for K Monte Carlo samples from EV copula as following:

$$d_{i,k}^2 = (x_{i,k;1}-x_{emp,k;1})^2 + (x_{i,k;2}-x_{emp,k;2})^2,\ \ i=1,\dots,5,\ \ k=1,\dots,K.$$

We can calculate the ranks of $d_{i,k}^2$ associated to index i for all K Monte Carlo samples given by $r_{i,k}$. Hence, the smallest rank of $r_{i,k}$, $k=1,\dots K$ indicates that the Pickands dependence function $A_i(.)$ is as close as to empirical Pickands dependence function $A_n(.)$ than other Pickands dependence function for the Monte Carlo samples of $k=1,\dots,K$ in 2D graph. For the overall performance, we define the mean of ranks $r_{i,k}$ as $\overline{r}_i=\sum_{k=1}^{K} r_{i,k}/K,\ \ i=1,\dots,5$ for all EV copulas.

Let us consider the bivariate random data from the EV copulas. We simulate the bivariate 1000 Monte sample of sizes 250 and 500 from the Logistic, Asymmetric logistic, Negative logistic, Asymmetric negative logistic, and Húsler-Reiss EV copula models by using the following combinations:

TABLE 1. Mean of the ranks for different EV copulas with $n = 250$

True Copula	$\overline{r}_L$	$\overline{r}_{AL}$	$\overline{r}_{HR}$	$\overline{r}_{NL}$	$\overline{r}_{ANL}$
$L(\theta = 0.1)$	**1.4713**	2.9713	1.7459	3.8504	4.9610
$L(\theta = 0.9)$	**2.4090**	3.4545	2.8363	2.5727	3.7272
$AL(\theta = 0.1, \alpha = 0.2, \beta = 0.8)$	2.19375	4.5687	2.1375	**1.6750**	4.4250
$AL(\theta = 0.1, \alpha = 0.8, \beta = 0.2)$	2.2083	4.5416	2.5000	**1.5833**	4.1666
$AL(\theta = 0.1, \alpha = 0.5, \beta = 0.5)$	2.870	**2.118**	4.538	3.223	2.251
$AL(\theta = 0.9, \alpha = 0.2, \beta = 0.8)$	**2.73**	2.99	2.92	2.98	3.38
$AL(\theta = 0.9, \alpha = 0.8, \beta = 0.2)$	2.9629	3.0740	**2.6913**	2.9135	3.3580
$AL(\theta = 0.9, \alpha = 0.5, \beta = 0.5)$	2.4375	**2.3125**	3.1250	3.1875	3.9375
$NL(\theta = 10)$	3.8619	2.9079	2.0083	**1.2887**	4.9330
$NL(\theta = 1)$	2.4814	4.0370	2.2592	**1.9629**	4.2592
$ANL(\theta = 1, \alpha = 0.2, \beta = 0.8)$	**2.2777**	3.5222	2.6222	2.3888	4.1888
$ANL(\theta = 1, \alpha = 0.8, \beta = 0.2)$	2.42	3.52	2.46	**2.30**	4.30
$ANL(\theta = 1, \alpha = 0.5, \beta = 0.5)$	**1.8401**	3.3848	3.5555	2.4986	3.7208
$ANL(\theta = 10, \alpha = 0.2, \beta = 0.8)$	2.1645	4.6195	2.1413	**1.7146**	4.3598
$ANL(\theta = 10, \alpha = 0.8, \beta = 0.2)$	2.181	4.614	2.156	**1.703**	4.346
$ANL(\theta = 10, \alpha = 0.5, \beta = 0.5)$	2.341	2.909	4.529	3.175	**2.046**
$HR(\theta = 0.1)$	3.1612	3.2096	**2.6935**	2.7580	3.1774
$HR(\theta = 0.9)$	2.5116	4.2558	**2.1395**	2.2558	3.8372

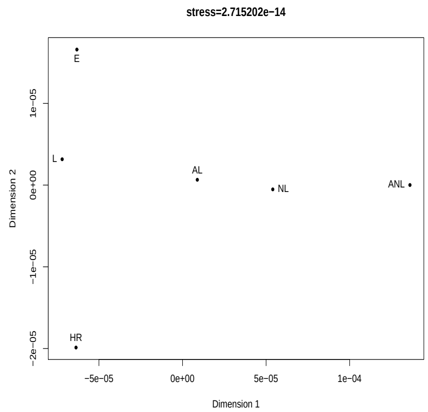
1. Logistic copula with dependence parameters $\theta = 0.1$ (Strong dependence), $\theta = 0.9$ (Mild dependence)
2. Asymmetric logistic copula
   (a) $\theta = 0.1, \alpha = 0.2, \beta = 0.8$ (Strong dependence and asymmetric Pickands function with $\alpha < \beta$)
   (b) $\theta = 0.1, \alpha = 0.8, \beta = 0.2$ (Strong dependence and asymmetric Pickands function with $\alpha > \beta$)
   (c) $\theta = 0.1, \alpha = 0.5, \beta = 0.5$ (Strong dependence and asymmetric Pickands function with $\alpha = \beta$)
   (d) $\theta = 0.9, \alpha = 0.2, \beta = 0.8$ (Mild dependence and asymmetric Pickands function with $\alpha < \beta$)
   (e) $\theta = 0.9, \alpha = 0.8, \beta = 0.2$ (Mild dependence and asymmetric Pickands function with $\alpha > \beta$)
   (f) $\theta = 0.9, \alpha = 0.5, \beta = 0.5$ (Mild dependence and asymmetric Pickands function with $\alpha = \beta$)
3. Negative logistic copula with the dependence parameters $\theta = 10$ (Strong dependence), $\theta = 1$ (Mild dependence)
4. Asymmetric negative logistic copula
   (a) $\theta = 10, \alpha = 0.2, \beta = 0.8$ (Strong dependence and asymmetric Pickands function with $\alpha < \beta$)
   (b) $\theta = 10, \alpha = 0.8, \beta = 0.2$ (Strong dependence and asymmetric Pickands function with $\alpha > \beta$)
   (c) $\theta = 10, \alpha = 0.5, \beta = 0.5$ (Strong dependence and asymmetric Pickands function with $\alpha = \beta$)
   (d) $\theta = 1, \alpha = 0.2, \beta = 0.8$ (Mild dependence and asymmetric Pickands function with $\alpha < \beta$)
   (e) $\theta = 1, \alpha = 0.8, \beta = 0.2$ (Mild dependence and asymmetric Pickands function with $\alpha > \beta$)
   (f) $\theta = 1, \alpha = 0.5, \beta = 0.5$ (Mild dependence and asymmetric Pickands function with $\alpha > \beta$)

TABLE 2. Mean of the ranks for different EV copulas with $n=500$

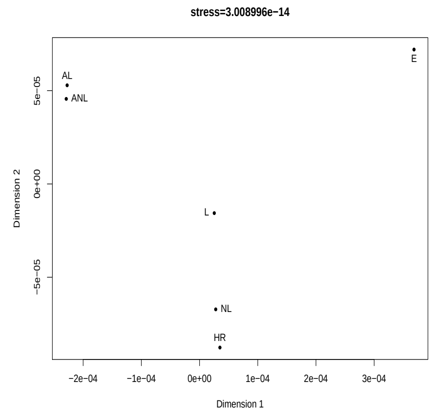
True Copula	$\overline{r}_L$	$\overline{r}_{AL}$	$\overline{r}_{HR}$	$\overline{r}_{NL}$	$\overline{r}_{ANL}$
$L(\theta=0.1)$	**1.360**	2.784	1.973	3.934	4.949
$L(\theta=0.9)$	**2.0458**	4.0917	2.8990	2.2385	3.7247
$AL(\theta=0.1,\alpha=0.2,\beta=0.8)$	2.2248	4.5574	2.2129	**1.5741**	4.4306
$AL(\theta=0.1,\alpha=0.8,\beta=0.2)$	2.2299	4.5328	2.1934	**1.5839**	4.4598
$AL(\theta=0.1,\alpha=0.5,\beta=0.5)$	2.509	**1.771**	4.800	3.567	2.353
$AL(\theta=0.9,\alpha=0.2,\beta=0.8)$	**2.4117**	3.3176	3.1529	2.7058	3.4117
$AL(\theta=0.9,\alpha=0.8,\beta=0.2)$	**2.5022**	3.3452	3.1748	2.6905	3.2869
$AL(\theta=0.9,\alpha=0.5,\beta=0.5)$	3.6692	**2.4307**	2.9000	2.6230	3.3769
$NL(\theta=10)$	3.9509	2.6666	2.3039	**1.0980**	4.9803
$NL(\theta=1)$	2.3617	4.0265	2.6648	**1.8510**	4.0957
$ANL(\theta=1,\alpha=0.2,\beta=0.8)$	**1.820**	4.454	2.508	1.857	4.361
$ANL(\theta=1,\alpha=0.8,\beta=0.2)$	1.9	4.3	2.4	**1.7**	4.7
$ANL(\theta=1,\alpha=0.5,\beta=0.5)$	**1.6170**	3.4361	4.0212	2.6276	3.2978
$ANL(\theta=10,\alpha=0.2,\beta=0.8)$	2.1415	4.5752	4.5752	**1.6106**	4.4247
$ANL(\theta=10,\alpha=0.8,\beta=0.2)$	2.2788	4.5576	2.1538	**1.5769**	4.4326
$ANL(\theta=10,\alpha=0.5,\beta=0.5)$	2.462	2.475	4.797	3.544	**1.722**
$HR(\theta=0.1)$	3.2142	3.0357	**2.9285**	3.0000	2.8214
$HR(\theta=0.9)$	2.6736	4.621	**1.7568**	1.9847	3.8736

5. Húsler-Reiss copula with dependence parameters $\theta=0.9$ (Strong dependence), $\theta=0.1$ (Mild dependence)

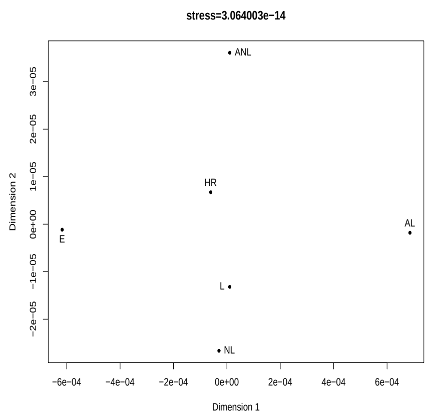
Tables 1-2 represent $\overline{r}_L$, $\overline{r}_{AL}$, $\overline{r}_{HR}$, $\overline{r}_{NL}$ and $\overline{r}_{ANL}$ which are obtained from 1000 Monte Carlo samples with 250 and 500 sizes, respectively. From these tables, if the True EV copula possesses to symmetric dependence structure, graphical method performs well. As an example, when the data is generated from Logistic copula with $\theta=0.1$ and $n=250$, smallest value of $\overline{r}_i$, $i=1,\dots,5$ is $\overline{r}_L=1.4713$ among five EV copulas. This means that the points which correspond to Pickands dependence function of Logistic copula ($A_L(.)$) in 2D graph is the closest to points correspond to A_n in 2D graph. Also, we can conclude from the Table 1-2, if the true EV copula possesses to asymmetric dependence structure, the graphical method does not perform well except for EV copula with equal asymmetry parameters. Also, mean of the ranks is decreased when the sample of size is increased.



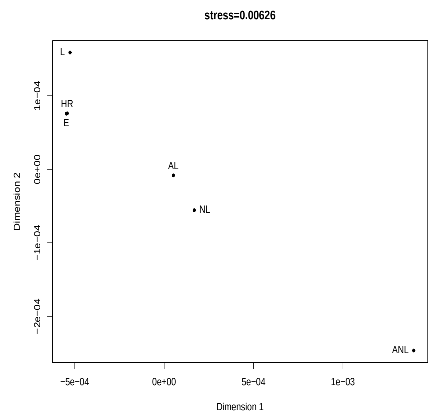
(a) Logistic Copula with $\theta = 0.1$



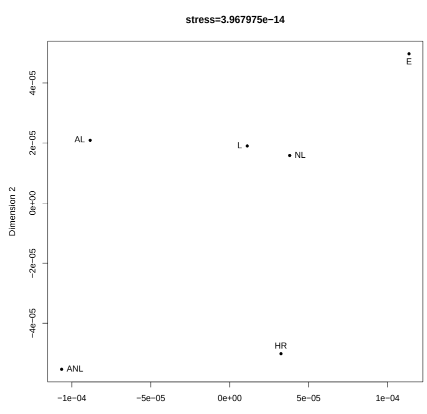
(b) Logistic Copula with $\theta = 0.9$



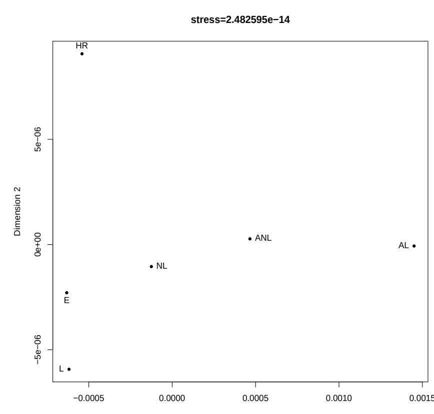
(c) Husler-Reiss Copula with $\theta = 0.1$



(d) Husler-Reiss Copula with $\theta = 0.9$

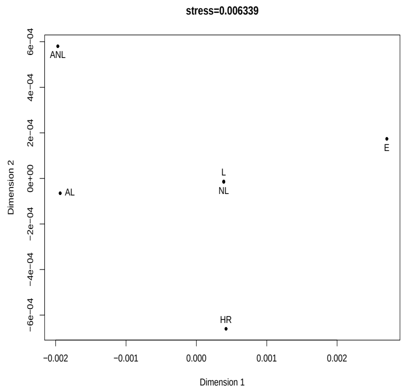


(e) Negative logistic Copula with $\theta = 1$

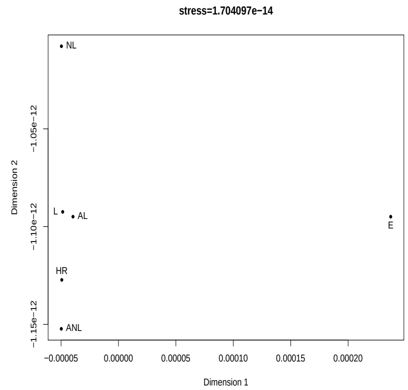


(f) Negative logistic Copula with $\theta = 10$

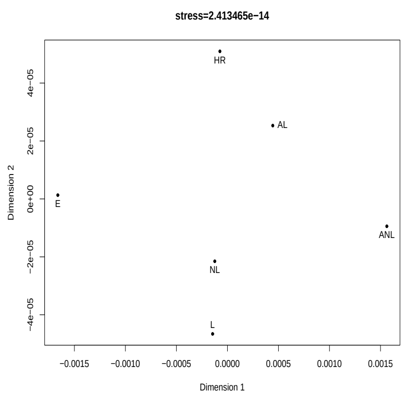
FIGURE 2. Graphical representation based on the two dimensional NMDS with data generated by Logistic, Husler-Reiss and Negative logistic copula



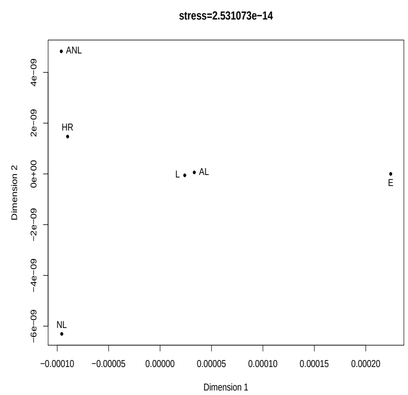
(a) Asymmetric logistic Copula with $\theta = 0.1$, $\alpha = 0.2$, $\beta = 0.8$



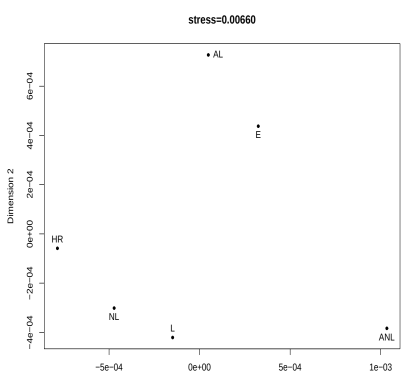
(b) Asymmetric logistic Copula with $\theta = 0.9$, $\alpha = 0.2$, $\beta = 0.8$



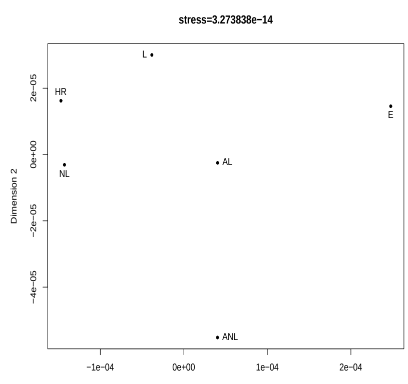
(c) Asymmetric logistic Copula with $\theta = 0.1$, $\alpha = 0.8$, $\beta = 0.2$



(d) Asymmetric logistic Copula with $\theta = 0.9$, $\alpha = 0.8$, $\beta = 0.2$

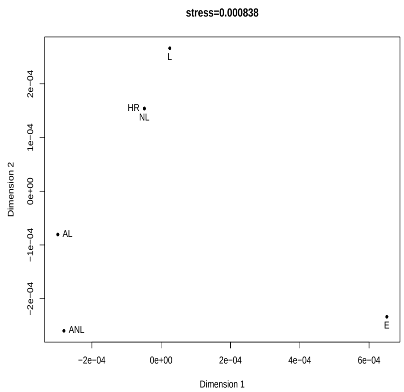


(e) Asymmetric logistic Copula with $\theta = 0.1$, $\alpha = 0.5$, $\beta = 0.5$

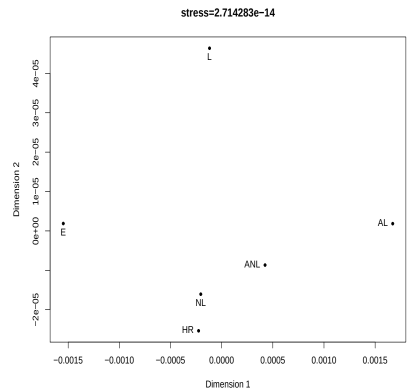


(f) Asymmetric logistic Copula with $\theta = 0.9$, $\alpha = 0.5$, $\beta = 0.5$

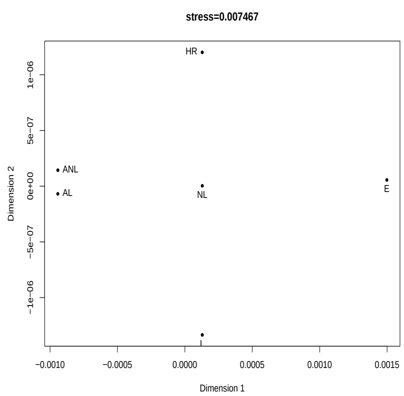
FIGURE 3. Graphical representation based on the two dimensional NMDS with the data generated by the asymmetric logistic copula



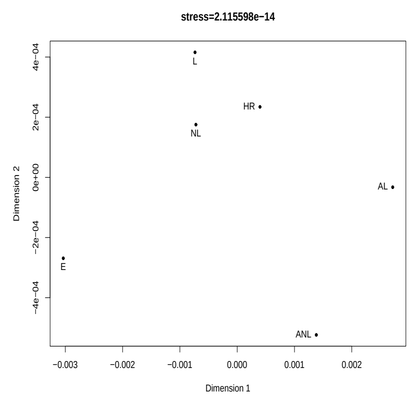
(a) Asymmetric negative logistic Copula with $\theta = 1$, $\alpha = 0.2$, $\beta = 0.8$



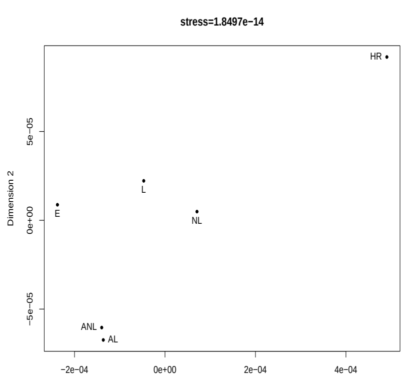
(b) Asymmetric negative logistic Copula with $\theta = 10$, $\alpha = 0.2$, $\beta = 0.8$



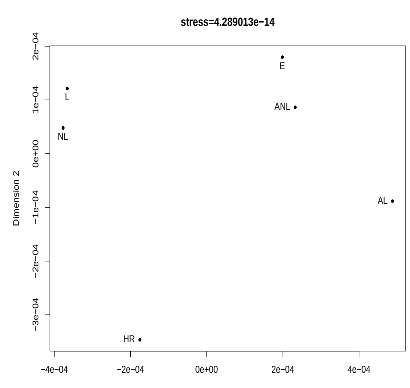
(c) Asymmetric negative logistic Copula with $\theta = 1$, $\alpha = 0.8$, $\beta = 0.2$



(d) Asymmetric negative logistic Copula with $\theta = 10$, $\alpha = 0.8$, $\beta = 0.2$



(e) Asymmetric negative logistic Copula with $\theta = 1$, $\alpha = 0.5$, $\beta = 0.5$



(f) Asymmetric negative logistic Copula with $\theta = 10$, $\alpha = 0.5$, $\beta = 0.5$

FIGURE 4. Graphical representation based on the two dimensional NMDS with the data generated by the asymmetric negative logistic copula

4. Real data example

To demonstrate the graphical method for the selection of the best-fitted EV copulas, in this section, we fit the EV copulas to the Danube data set which is available in the R package copula. According to this package, the Danube dataset contains ranks of base of observations from the Global River Discharge project of the Oak Ridge National Laboratory Distributed Active Archive Centre (ORNL DAAC), a NASA data centre. The measurements are the monthly average of rate for two stations situated at Scharding (Austria) on the Inn River and Nagymaros (Hungary) on the Danube.

TABLE 3. Estimation of dependence and asymmetry parameters for five EV copulas

EV Copula	CvM	α	β	θ
L	8.233616×10^{-5}	—	—	0.4872
AL	0.000333	0.8582	0.9992	0.4445
NL	9.257569×10^{-5}	—	—	1.3332
ANL	0.001962	0.9158	0.9995	1.0034
HR	0.000152	—	—	1.7981

The scatter plot of the pseudo-observations of the Danube data set is displayed in Figure 5. From Figure 5, symmetrical dependence structures are observed. Also, the Danube data set has a heavy right tail dependence structure.

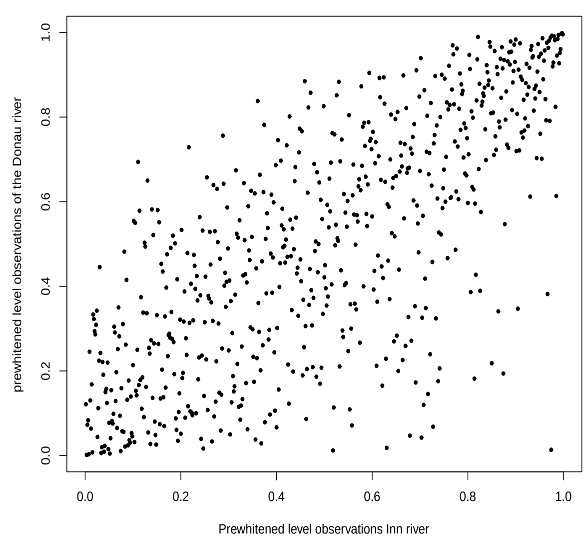
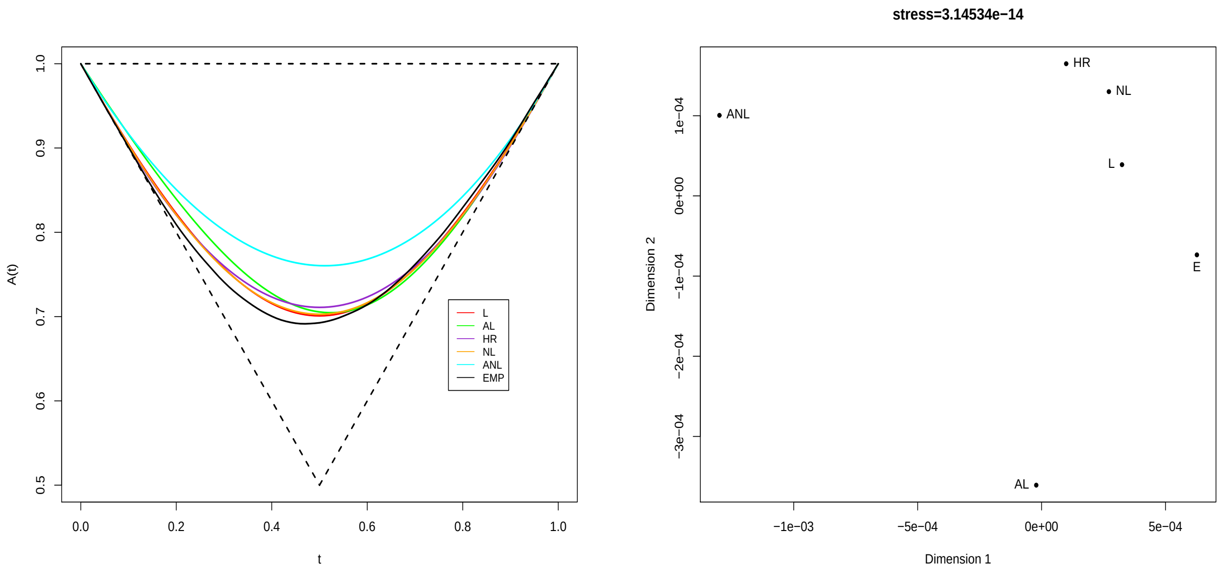


FIGURE 5. Scatter plot of danube dataset



(a) Parametric and Non-parametric estimation of the Pickands function for the danube data

(b) Graphical representation based on two dimensional NMDS for danube dataset

FIGURE 6. Fiiting results for danube dataset

In Figure 6(a), the parametric and non-parametric estimation of the Pickands dependence functions is displayed. From this figure, we can observe that the Logistic copula's Pickands dependence the function is much closer to the empirical Pickands dependence function for the Danube dataset. Also Table 1 represents the CvM distances between $A_n, A_L, \ldots, A_{HR}$, and estimation of dependence parameters for five EV copulas. On the other hand, Figure 6(b) displays the two-dimensional representation of the EV copula test spaces with the Danube dataset based the on NMDS method. When Figure 6(b) is examined; it can be concluded that the Logistic copulas are the most appropriate EV copulas for the Danube data set.

5. Conclusions

In this study, we proposed a graphical method based on NMDS to select the best-fitted EV copulas for underlying data. Also, we discussed some advantages of the proposed methods. If practitioners are interested in modelling for extreme situations which consist of a big data size, the graphical method can be useful to select the EV copulas. We performed the graphical method to see how accurately it works for the simulated data set from EV copulas. From the simulation study, when the dependence structure is symmetric, the procedure is useful to identify the true EV copula which is data generated. On the other hand when the data has asymmetric dependence the structure of the graphical procedure fails except for with the Asymmetric EV copula with equal asymmetry parameters. This problem can be overcome by using the Bernstein polynomial based on the Pickands dependence function estimator in the procedure of the graphical method. The main advantage of Bernstein polynomials is their flexibility against data that has a complex structure. So, Bernstein polynomials can take on an extremely wider range of shapes than simple estimators. Also, to demonstrate the graphical method for the selection of the best-fitted EV copulas, we fitted the EV copulas to the real data set. We have shown that the graphical procedure can lead to acceptable results.

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