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Research Article

Congruences with q-generalized Catalan numbers and q-harmonic numbers

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Abstract

In this paper, we give some congruences related to $q-$generalized Catalan numbers, $q-$harmonic numbers and alternating $q-$harmonic numbers, using combinatorial identities and some known congruences.

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1. Introduction

The Catalan numbers play an important role in combinatorics, number theory and linear algebra in [7–9, 12, 14]. In [18], Shapiro gave the generalized Catalan numbers $B_{n,k}$ as follows:

$$B_{n,k} = \frac{k}{n}\binom{2n}{n-k} = \binom{2n-1}{n-k} - \binom{2n-1}{n-k-1}, \; 0 \le k \le n.$$

These numbers $B_{n,k}$ are the entries of the Catalan triangles and satisfy the recurrence relation

$$B_{n,k} = B_{n-1,k-1} + 2B_{n-1,k} + B_{n-1,k+1}, \quad k \ge 2,$$

with the initial conditions $B_{n,0} = 0 = B_{n,n+m}$, $m \ge 1$. They have several applications by authors [4, 12, 18]. Note that for $k = 1$, $B_{n,1}$ are the well known Catalan numbers

$$C_n = \frac{1}{n+1}\binom{2n}{n}, \quad n \geqslant 1.$$

The harmonic numbers are given by

$$H_0 = 0 \text{ and } H_n = \sum_{k=1}^{n} \frac{1}{k}, \quad n \in \mathbb{N}^* = \mathbb{N} - \{0\}.$$

In [21], Wolstenholme discovered that for any prime number $p \ge 5$,

$$H_{p-1} \equiv 0 \pmod{p^2}.$$

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The $q-$harmonic numbers and $q-$alternating harmonic numbers are given by

$$H_n(q)=\sum_{k=1}^{n}\frac{1}{[k]_q},\quad \widetilde{H}_n(q)=\sum_{k=1}^{n}\frac{q^k}{[k]_q}\quad\text{and}\quad I_n(q)=\sum_{k=1}^{n}\frac{(-1)^k}{[k]_q},$$

where $[0]_q=1$ and $[k]_q=(1-q^k)/(1-q)=1+q+q^2+...+q^{k-1}$.

It is seen that for $0<k<p$,

$$\frac{1}{[p-k]_q}\equiv-\frac{q^k}{[k]_q}\pmod{[p]_q}. \tag{1.1}$$

The $q-$Pochhammer symbol is given by

$$(x;q)_0=1\text{ and }(x;q)_n=\prod_{k=0}^{n-1}\left(1-xq^k\right).$$

For any $m,n\in\mathbb{N}$, the $q-$binomial coefficients are defined by

$$\begin{bmatrix}n\\m\end{bmatrix}_q=\frac{(q;q)_n}{(q;q)_m(q;q)_{n-m}},$$

if $n\geq m$, and if $n<m$, then $\begin{bmatrix}n\\m\end{bmatrix}_q=0$. It is clear that

$$\lim_{q\to1}\begin{bmatrix}n\\m\end{bmatrix}_q=\binom{n}{m},$$

where $\binom{n}{m}$ is the usual binomial coefficient. The $q-$binomial coefficients satisfy the recurrence relation

$$\begin{bmatrix}n+1\\m\end{bmatrix}_q=q^m\begin{bmatrix}n\\m\end{bmatrix}_q+\begin{bmatrix}n\\m-1\end{bmatrix}_q.$$

In [17], Pan and Cao defined the $q-$Fermat quotient by

$$Q_p(m,q)=\frac{(q^m;q^m)_{p-1}/(q;q)_{p-1}-1}{[p]_q},$$

where m is nonnegative integer such that $p\nmid m$. There are many generalizations of the Catalan number [2,7], one of which is $q-$analogue of the Catalan number. In [6], Fürlinger and Hofbauer defined this number by

$$C_n(q)=\frac{1}{[n+1]_q}\begin{bmatrix}2n\\n\end{bmatrix}_q,$$

and Guo and Zeng [8] defined $q-$generalized Catalan numbers as follows:

$$B_{n,k}(q)=\frac{[k]_q}{[n]_q}\begin{bmatrix}2n\\n-k\end{bmatrix}_q,\quad 1\leq k\leq n.$$

In [3,19], the authors showed that for an odd prime p,

$$H_{p-1}(q)\equiv\frac{p-1}{2}(1-q)+\frac{p^2-1}{24}(1-q)^2\ [p]_q\pmod{[p]_q^2}, \tag{1.2}$$

and for any prime $p\geq5$

$$\widetilde{H}_{p-1}(q)\equiv\frac{p-1}{2}(q-1)+\frac{p^2-1}{24}(1-q)^2\ [p]_q\pmod{[p]_q^2}. \tag{1.3}$$

In [20], Tauraso gave that for positive integer α and $k=0,1,2,...,p-1$,

$$\begin{bmatrix}\alpha p-1\\k\end{bmatrix}_q\equiv(-1)^k q^{-\binom{k+1}{2}}\left(1-\alpha[p]_q H_k(q)\right)\pmod{[p]_q^2}.$$

It is clearly seen that

$$\begin{bmatrix} \alpha p-1 \\ k \end{bmatrix}_q \equiv (-1)^k q^{\alpha pk-\binom{k+1}{2}} \left(1-\alpha [p]_q \widetilde{H}_k(q)\right) \pmod{[p]_q^2}. \tag{1.4}$$

In [16], Pan established that for any odd prime p,

$$\begin{aligned} &2\sum_{k=1}^{(p-1)/2} \frac{1}{[2k]_q} + 2Q_p(2,q) - Q_p^2(2,q)[p]_q \\ &\equiv \left(Q_p(2,q)(1-q) + \frac{p^2-1}{8}(1-q)^2\right)[p]_q \pmod{[p]_q^2} \end{aligned}$$

and for any prime p,

$$q^{kp} \equiv 1 - k(1-q)[p]_q + \binom{k}{2}(1-q)^2 [p]_q^2 \pmod{[p]_q^3}. \tag{1.5}$$

In [10], He obtained that for any prime $p \geq 5$,

$$\begin{aligned} I_{p-1}(q) \equiv\ & -2Q_p(2,q) - \frac{(p-1)(1-q)}{2} \\ & + [p]_q \left(Q_p^2(2,q) + Q_p(2,q)(1-q) + \frac{(p^2-1)}{12}(1-q)^2\right) \pmod{[p]_q^2}. \end{aligned} \tag{1.6}$$

In [9], Gutiérrez et al. gave some identities involving well-known Catalan numbers. For example, for $n \geq 1$,

$$\sum_{k=1}^{n} kB_{n,k}^2 = \frac{n(n+1)}{2} C_n C_{n-1}.$$

In [14], Miana and Romero showed the following identity that for $1 \leq m \leq n$,

$$\sum_{k=1}^{m} B_{n,k} B_{n,n+k-m} (n+2k-m)^3 = \binom{2n}{n}\binom{2(n-1)}{m-1}\left(n^2+4n-2nm+m^2\right).$$

In [11], He and Wang established several $q-$congruences involving Catalan numbers. Some of these extend the results of Z.-W. Sun. For example, for an odd prime p,

$$\sum_{k=0}^{p-1} \frac{C_k^2(q)}{(-q;q)_k^4} q^{4k+2} \equiv -\frac{3(1+q)^2}{4} \pmod{[p]_q}.$$

In [15], Ömür and Koparal gave some congruences involving the numbers $B_{p,k-d}$. For example, for $1 \leq d \leq p-1$,

$$\sum_{k=1}^{p-1} (-1)^k B_{p,k} B_{p,k-d} \equiv 4(-1)^d \left(1 + \left(1+(-1)^d\right)\frac{p}{d} - 2pH_d\right) \pmod{p^2},$$

and in [13], they proved some congruences involving the generalized Catalan numbers and harmonic numbers modulo p^2. For example, for a prime $p > 3$ and $2 \leq d \leq p-1$,

$$\begin{aligned} \sum_{k=1}^{p-1} \frac{B_{p,k}B_{p,k-d}}{k} \equiv\ & 4(-1)^d \Big\{H_d + H_{d-1} + \frac{2}{p} + p\Big(4H_d^2 + 4H_{d-1}^2 \\ & -2H_{d,2} - 2H_{d-1,2} - H_{p+d}^2 - H_{p+d-1}^2\Big)\Big\} \pmod{p^2}, \end{aligned}$$

where $H_{n,m}$ is harmonic number of order m.

In [5], Elkhiri et al. gave some congruences with the generalized Catalan numbers and harmonic numbers modulo p^2. For example, for $1 < d < p-1$ and prime number $p > 3$,

$$\begin{aligned}\sum_{k=d+1}^{p-1} B_{p,k}B_{p,k-d}H_k \equiv\ & 4(-1)^d \left(d+7p+8dp-3+\frac{1+2d}{p}+\frac{3p}{2d^2}+\frac{4p-2}{d}\right. \\ & +p\left(6d+\frac{7}{2}\right)H_d^2+\left(3d-10p-8dp+1-\frac{2p}{d}\right)H_d \\ & \left.-p(2d+1)H_{p+d-1}^2-p\left(2d+\frac{3}{2}\right)H_{d-1,2}\right) \pmod{p^2}.\end{aligned}$$

In [1], Abel's partial summation formula asserts that for every pair of families $(a_k)_{k=1}^n$ and $(b_k)_{k=1}^n$ of complex numbers, there is the relation

$$\sum_{k=1}^{n} a_k b_k = \sum_{k=1}^{n-1}\left[(a_k-a_{k+1})\left(\sum_{j=1}^{k} b_j\right)\right]+a_n\left(\sum_{j=1}^{n} b_j\right). \tag{1.7}$$

2. Some congruences

In this section, we will start with some lemmas and then derive our results about congruences.

Lemma 2.1. *For any prime p, let n and d be integer numbers such that $0 \le d < n < p$. We have*

$$\begin{aligned}&\frac{[n]_q[n-d]_q}{[p-n]_q[p-n+d]_q} \\ \equiv\ & q^{2n-d}\left(1+[p]_q\left(\frac{q^n}{[n]_q}+\frac{q^{n-d}}{[n-d]_q}+2(1-q)\right)\right) \pmod{[p]_q^2}.\end{aligned}$$

Proof. By using $[p+n]_q = [p]_q + q^p[n]_q$ and $[p-n]_q = [p]_q - q^{p-n}[n]_q$ for $0 < n < p$, we have

$$\begin{aligned}\frac{1}{[p-n]_q} &= \frac{[p+n]_q}{[p-n]_q[p+n]_q} = \frac{[p]_q+q^p[n]_q}{\left([p]_q+q^p[n]_q\right)\left([p]_q-q^{p-n}[n]_q\right)} \\ &= \frac{[p]_q+q^p[n]_q}{[p]_q^2-[n]_q\left([p]_q(q^{p-n}-q^p)+q^{2p-n}[n]_q\right)} \\ &\equiv \frac{[p]_q+q^p[n]_q}{[n]_q\left([p]_q(q^p-q^{p-n})-q^{2p-n}[n]_q\right)} \pmod{[p]_q^2},\end{aligned} \tag{2.1}$$

and for $0 < n-d < p$, we replace in (2.1) n by $n-d$ to obtain,

$$\frac{1}{[p-n+d]_q} \equiv \frac{[p]_q+q^p[n-d]_q}{[n-d]_q\left([p]_q(q^p-q^{p-n+d})-q^{2p-n+d}[n-d]_q\right)} \pmod{[p]_q^2}. \tag{2.2}$$

Thus, combining (2.1) and (2.2), we write

$$\begin{aligned}&\frac{[n]_q[n-d]_q}{[p-n]_q[p-n+d]_q} \\ \equiv\ & \frac{\left([p]_q+q^p[n]_q\right)\left([p]_q+q^p[n-d]_q\right)}{q^{2p}\left([p]_q(1-q^{-n})-q^{p-n}[n]_q\right)\left([p]_q(1-q^{-n+d})-q^{p-n+d}[n-d]_q\right)} \\ \equiv\ & \frac{1}{q^{2p-2n+d}}+[p]_q\frac{q^{p+d}[n-d]_q+\left(q^p-q^{p-n+d}+q^{d-n}\right)[n]_q}{q^{4p-3n+2d}[n]_q[n-d]_q} \pmod{[p]_q^2}.\end{aligned}$$

By (1.5), we complete the proof of lemma. □

Lemma 2.2. *For $n \in \mathbb{N}^*$, we have*

$$\sum_{k=1}^{n} \frac{q^{2k}}{[k]_q} = \widetilde{H}_n(q) - q(1-q)[n]_q, \tag{2.3}$$

$$\sum_{k=1}^{n} \frac{q^{3k}}{[k]_q} = \widetilde{H}_n(q) + (1-q)\left(2 - [n+1]_q \frac{q(1+q^n)+2}{1+q}\right), \tag{2.4}$$

$$\sum_{k=1}^{n} q^{-k}\widetilde{H}_k(q) = \frac{q}{q-1}\left(H_n(q) - q^{-(n+1)}\widetilde{H}_n(q)\right), \tag{2.5}$$

$$\sum_{k=1}^{n} q^{-2k}\widetilde{H}_k(q) = \frac{q^2}{1+q}\left(q^{-2n-2}[2n+2]_q \widetilde{H}_n(q) - q^{-n}[n]_q - n\right). \tag{2.6}$$

Proof. By exchanging the sums and some elementary operations, the proof is clearly obtained. □

Lemma 2.3. *For $n \in \mathbb{N}^*$, we have*

$$\sum_{k=1}^{n} \frac{(-q)^k}{[k]_q} = I_n(q) + (1-q)\frac{(-1)^{n+1}+1}{2} \tag{2.7}$$

and

$$\sum_{k=1}^{n} \frac{(-1)^k}{[k]_q} q^{2k} = I_n(q) - (1-q)\left(\frac{(-1)^n - 3}{2} + [n+1]_{-q}\right). \tag{2.8}$$

Proof. From (1.7), we have

$$\begin{aligned}
&\sum_{k=1}^{n} \frac{(-1)^k}{[k]_q} q^{2k} \\
= \;& \sum_{k=1}^{n-1}\left(q^{2k} - q^{2k+2}\right)\sum_{i=1}^{k}\frac{(-1)^i}{[i]_q} + q^{2n}I_n(q) \\
= \;& (1-q^2)\sum_{k=1}^{n-1} q^{2k}\sum_{i=1}^{k}\frac{(-1)^i}{[i]_q} + q^{2n}I_n(q) \\
= \;& (1-q^2)\sum_{i=1}^{n-1}\frac{(-1)^i}{[i]_q}\sum_{k=i}^{n-1} q^{2k} + q^{2n}I_n(q) \\
= \;& \left(1-q^{2n}\right)I_{n-1}(q) - (1-q)\sum_{k=1}^{n-1}(-1)^k\left(1+q^k\right) + q^{2n}I_n(q) \\
= \;& \left(1-q^{2n}\right)I_{n-1}(q) - (1-q)\left(\frac{(-1)^{n+1}-1}{2} + \frac{1-(-q)^n}{1+q} - 1\right) + q^{2n}I_n(q) \\
= \;& I_n(q) - (1-q)\left((-1)^n + (-q)^n + \frac{(-1)^{n+1}-1}{2} + \frac{1-(-q)^n}{1+q} - 1\right) \\
= \;& I_n(q) - (1-q)\left(\frac{(-1)^n - 3}{2} + \frac{1+(-1)^n q^{n+1}}{1+q}\right),
\end{aligned}$$

as claimed. Similarly, the other equality is obtained. □

Corollary 2.4. *Let p be an odd prime. For $0 < d \leq p-2$,*

$$I_{p-d-1}(q) \equiv -2Q_p(2,q) - I_d(q) - (1-q)\frac{p-(-1)^d}{2} \pmod{[p]_q}, \tag{2.9}$$

$$\sum_{k=d+1}^{p-1}(-1)^k\frac{q^{2k}}{[k]_q}\equiv -2Q_p(2,q)-I_d(q) - (1-q)\left(\frac{p-(-1)^d}{2}+\frac{1+(-q)^{d+1}}{1+q}\right) \pmod{[p]_q},$$

and

$$\sum_{k=1}^{p-d-1}(-1)^k\frac{q^{2k}}{[k]_q} \equiv -2Q_p(2,q)-I_d(q)-(1-q)\left(\frac{p-3}{2}+\frac{1+(-q)^{-d}}{1+q}\right) \pmod{[p]_q}.$$

Proof. By (1.1), we have

$$I_{p-d-1}(q)=I_{p-1}(q)+\sum_{k=1}^{d}\frac{(-1)^k}{[p-k]_q}\equiv I_{p-1}(q)-\sum_{i=1}^{d}\frac{(-q)^k}{[k]_q} \pmod{[p]_q}.$$

(1.6) and (2.7) yield congruence

$$\begin{aligned} I_{p-d-1}(q) &\equiv -2Q_p(2,q)-I_d(q)-\frac{(p-1)(1-q)}{2}-(1-q)\frac{(-1)^{d+1}+1}{2} \\ &= -2Q_p(2,q)-I_d(q)-(1-q)\frac{p-(-1)^d}{2} \pmod{[p]_q}, \end{aligned}$$

as claimed. Similarly, using (1.6) and (2.8), the other congruences are obtained. □

Lemma 2.5. *Let $p\geqslant 5$ be any prime. For $0<d\leq p-3$,*

$$\sum_{k=d+1}^{p-2}(-q)^k\,\widetilde{H}_{p-k-1}(q)\equiv\frac{1}{1+q}\Big((-q)^{d+1}H_d(q)-2Q_p(2,q)-I_d(q) - \frac{(1-q)}{2}\left(p-(-1)^d+(p-1)(-q)^{d+1}\right)\Big) \pmod{[p]_q},$$

and

$$\sum_{k=d+1}^{p-1}(-q)^k\,\widetilde{H}_{p-k+d-1}(q)\equiv -\frac{(-q)^d}{q+1}\Big(2Q_p(2,q)+I_d(q)-(-q)^{-d}\,\widetilde{H}_d(q) +\frac{(q-1)^2(p-1)}{2}\Big) \pmod{[p]_q}.$$

Proof. Consider that

$$\begin{aligned} \sum_{k=d+1}^{p-2}(-q)^k\widetilde{H}_{p-k-1}(q) &= q^{p-1}\sum_{k=1}^{p-d-2}(-q)^{-k}\widetilde{H}_k(q) \\ &= q^{p-1}\sum_{i=1}^{p-d-2}\frac{q^i}{[i]_q}\sum_{k=i}^{p-d-2}(-q)^{-k} \\ &= q^{p-1}\sum_{i=1}^{p-d-2}\frac{q^i}{[i]_q}\left(\sum_{k=0}^{p-d-2}(-q)^{-k}-\sum_{k=0}^{i-1}(-q)^{-k}\right) \\ &= \frac{q^p}{1+q}\sum_{i=1}^{p-d-2}\frac{q^i}{[i]_q}\left((-q)^{-i}-(-q)^{-p+d+1}\right) \\ &= \frac{1}{1+q}\left((-q)^{d+1}\,\widetilde{H}_{p-d-2}(q)+q^pI_{p-d-2}(q)\right), \end{aligned}$$

and by (1.5),

$$\sum_{k=d+1}^{p-2}(-q)^k\widetilde{H}_{p-k-1}(q)\equiv\frac{1}{1+q}\left((-q)^{d+1}\widetilde{H}_{p-d-2}(q)+I_{p-d-2}(q)\right)\pmod{[p]_q}.$$

From congruence $\widetilde{H}_{p-d-2}(q)\equiv\widetilde{H}_{p-1}(q)+H_{d+1}(q)\pmod{[p]_q}$ and (1.1), we have

$$\begin{aligned}&\sum_{k=d+1}^{p-2}(-q)^k\,\widetilde{H}_{p-k-1}(q)\\ \equiv\;&\frac{1}{1+q}\left((-q)^{d+1}\left(\widetilde{H}_{p-1}(q)+H_{d+1}(q)\right)+I_{p-d-2}(q)\right)\\ \equiv\;&\frac{1}{1+q}\left((-q)^{d+1}\left(\widetilde{H}_{p-1}(q)+H_{d}(q)\right)+I_{p-d-1}(q)\right)\pmod{[p]_q},\end{aligned}$$

and by (1.3),

$$\begin{aligned}&\sum_{k=d+1}^{p-2}(-q)^k\,\widetilde{H}_{p-k-1}(q)\\ \equiv\;&\frac{1}{1+q}\left((-q)^{d+1}\left(\frac{p-1}{2}(q-1)+H_{d}(q)\right)+I_{p-d-1}(q)\right)\pmod{[p]_q}.\end{aligned}$$

(2.9) yields that

$$\begin{aligned}\sum_{k=d+1}^{p-2}(-q)^k\,\widetilde{H}_{p-k-1}(q)\equiv&\frac{(-q)^{d+1}}{1+q}H_d(q)+\frac{(-q)^{d+1}}{1+q}\frac{p-1}{2}(q-1)\\ &+\frac{1}{1+q}\left(-2Q_p(2,q)-I_d(q)-\frac{(1-q)\left(p-(-1)^d\right)}{2}\right)\pmod{[p]_q}.\end{aligned}$$

Thus, the proof of this congruence is finish. Similarly, with help of the equality

$$\sum_{k=d+1}^{p-1}(-q)^k\,\widetilde{H}_{p-k+d-1}(q)=q^{p+d-1}(-1)^d\sum_{k=d}^{p-2}(-q)^{-k}\,\widetilde{H}_k(q),$$

the proof of other congruence is obtained. □

Lemma 2.6. *Let $p\geqslant 5$ be any prime. For $0<d\leq p-3$,*

$$\begin{aligned}&\sum_{k=d+1}^{p-1}q^k\,[k]_q\,\widetilde{H}_{p-k+d-1}(q)\\ \equiv\;&\frac{q^d}{1-q}\left(\frac{1}{q-1}\left(\frac{q^{1-d}+q^d}{q+1}\widetilde{H}_d(q)-H_d(d)-q^d\frac{2q-1-q^{-d}}{q+1}\right)\right.\\ &\left.-\frac{p-1}{2}\left(q\left(1-q^d\right)+1+q^d\right)-\frac{q^d}{q+1}(-p+d)\right)\pmod{[p]_q},\end{aligned}$$

and

$$\begin{aligned}\sum_{k=d+1}^{p-1}q^k\,[k]_q\,\widetilde{H}_{p-k-1}(q)\equiv&\frac{1}{1-q}\left(\frac{1}{q-1}\left(\widetilde{H}_d(q)-\frac{1+q^{d+1}\left(1-q^{d+1}+q\right)}{q+1}H_d(q)\right)\right.\\ &\left.-\frac{1}{q+1}\left(\frac{p-1}{2}\left(1+q^{d+1}\right)\left(q\,(1-q)\,[d]_q+2\right)+q\,[d]_q-p+d+2\right)\right)\pmod{[p]_q}.\end{aligned}$$

Proof. Using (1.2), (1.3), (1.5) and (2.6), the proof is similar to the proof of Lemma 2.5. □

Lemma 2.7. *Let p be an odd prime. For $0 < d \le p-2$,*

$$\sum_{k=d+1}^{p-1} q^{2k}\frac{[k]_q}{[k-d]_q} \equiv q^{2d}[d]_q\left(H_d(q)-\frac{p-1+q\,(2-q\,(p-5))}{2\,[2]_q}\right)+q^d\left([d]_q-\frac{[d+2]_q}{[2]_q}\right) \pmod{[p]_q}, \tag{2.10}$$

and

$$\sum_{k=d+1}^{p-1} q^k[k]_q \equiv -\frac{1}{[2]_q}\left([p]_q+q\,[d]_q\,[d+1]_q\right) \pmod{[p]_q^2}. \tag{2.11}$$

Proof. Observed that

$$\begin{aligned}&\sum_{k=d+1}^{p-1} q^{2k}\frac{[k]_q}{[k-d]_q}\\ =\;& q^{2d}\sum_{k=1}^{p-d-1} q^{2k}\frac{[k+d]_q}{[k]_q} = q^{2d}\sum_{k=1}^{p-d-1} q^{2k}\frac{[k]_q+q^k[d]_q}{[k]_q}\\ =\;& q^{2d}\sum_{k=1}^{p-d-1} q^{2k}\left(1+[d]_q\frac{q^k}{[k]_q}\right) = q^{2d}\left(\sum_{k=1}^{p-d-1} q^{2k}+[d]_q\sum_{k=1}^{p-d-1}\frac{q^{3k}}{[k]_q}\right).\end{aligned}$$

Using (2.4) and the congruence $\widetilde{H}_{p-d-1}(q) \equiv \widetilde{H}_{p-1}(q)+H_d(q) \pmod{[p]_q}$, we get

$$\begin{aligned}\sum_{k=d+1}^{p-1} q^{2k}\frac{[k]_q}{[k-d]_q} \equiv\;& q^{2d}\Bigg(\frac{q^2-q^{2p-2d}}{1-q^2}+[d]_q\Bigg(\widetilde{H}_{p-1}(q)+H_d(q)\\ &+(1-q)\left(2-[p-d]_q\frac{q\left(1+q^{p-d-1}\right)+2}{q+1}\right)\Bigg)\Bigg) \pmod{[p]_q}.\end{aligned}$$

By the congruence $[p-k]_q \equiv -q^{-k}[k]_q \pmod{[p]_q}$ and (1.5), we have

$$\begin{aligned}\sum_{k=d+1}^{p-1} q^{2k}\frac{[k]_q}{[k-d]_q} \equiv\;& q^{2d}\Bigg(-q^{-2d}\frac{[2d+2]_q}{1+q}+[d]_q\Bigg(\widetilde{H}_{p-1}(q)+H_d(q)\\ &+(1-q)\left(2+q^{-d}[d]_q\frac{q\left(1+q^{-d-1}\right)+2}{1+q}\right)\Bigg)\Bigg) \pmod{[p]_q}.\end{aligned}$$

By (1.3), the proof is clearly given. Similarly, the proof of other congruence is given. □

Theorem 2.8. *Let $p \geqslant 5$ be any prime. For $0 < d \le p-3$,*

$$\begin{aligned}\sum_{k=d+1}^{p-1} q^{k(2p-d+k)}B_{p,k}(q)B_{p,k-d}(q) \equiv\;& 4(-1)^d q^{-\binom{d+1}{2}}\Bigg(-[d+1]_q\\ &+[p]_q\Bigg((p-1)\frac{(q+1)(1+q^d)}{2}-(p+d)\,q^{d+1}+d+3p-1\\ &+q^dH_d\frac{3q+1}{q-1}-\widetilde{H}_d\frac{q+3}{q-1}\Bigg)\Bigg) \pmod{[p]_q^2}.\end{aligned}$$

Proof. Observed that

$$\sum_{k=d+1}^{p-1} q^{k(2p-d+k)}B_{p,k}(q)\,B_{p,k-d}(q)$$

$$= \sum_{k=d+1}^{p-1} q^{k(2p-d+k)} \frac{[k]_q [k-d]_q}{[p]_q^2} \begin{bmatrix} 2p \\ p-k \end{bmatrix}_q \begin{bmatrix} 2p \\ p-k+d \end{bmatrix}_q$$

and for $0 \le d \le p-4$

$$\sum_{k=d+1}^{p-1} q^{k(2p-d+k)} B_{p,k}(q) B_{p,k-d}(q)$$
$$= \frac{[2p]_q^2}{[p]_q^2} \sum_{k=d+1}^{p-1} q^{k(2p-d+k)} \frac{[k]_q [k-d]_q}{[p-k]_q [p-k+d]_q} \begin{bmatrix} 2p-1 \\ p-k-1 \end{bmatrix}_q \begin{bmatrix} 2p-1 \\ p-k+d-1 \end{bmatrix}_q .$$

By equality $[2p]_q = (1+q^p)[p]_q$, equals that for $0 \le d \le p-4$

$$(1+q^p)^2 \sum_{k=d+1}^{p-1} q^{k(2p-d+k)} \frac{[k]_q [k-d]_q}{[p-k]_q [p-k+d]_q} \begin{bmatrix} 2p-1 \\ p-k-1 \end{bmatrix}_q \begin{bmatrix} 2p-1 \\ p-k+d-1 \end{bmatrix}_q .$$

Then using the congruence (1.4) and Lemma 2.1, we show that for $1 \le d \le p-3$

$$\sum_{k=d+1}^{p-1} q^{k(2p-d+k)} B_{p,k}(q) B_{p,k-d}(q)$$
$$\equiv (-1)^d (1+q^p)^2 q^{p(3p+d-5)-\binom{d+1}{2}} \left(\sum_{k=d+1}^{p-1} q^k \right.$$
$$-[2p]_q \left(\sum_{k=d+1}^{p-1} q^k \widetilde{H}_{p-k-1}(q) + \sum_{k=d+1}^{p-1} q^k \widetilde{H}_{p-k+d-1}(q) \right)$$
$$\left. +[p]_q \left(\sum_{k=d+1}^{p-1} \frac{q^{2k}}{[k]_q} + q^{-d} \sum_{k=d+1}^{p-1} \frac{q^{2k}}{[k-d]_q} \right) \right) \pmod{[p]_q^2},$$

and by some combinatorial operations,

$$\equiv (-1)^d (1+q^p)^2 q^{p(3p+d-5)-\binom{d+1}{2}} \left\{ \sum_{k=d+1}^{p-1} q^k \right.$$
$$- [2p]_q q^{p-1} \left(\sum_{k=1}^{p-d-2} q^{-k} \widetilde{H}_k(q) + q^d \sum_{k=d}^{p-2} q^{-k} \widetilde{H}_k(q) \right)$$
$$\left. + [p]_q \left(\sum_{k=d+1}^{p-1} \frac{q^{2k}}{[k]_q} + q^d \sum_{k=1}^{p-d-1} \frac{q^{2k}}{[k]_q} \right) \right\} \pmod{[p]_q^2}.$$

By (1.5), (2.3) and (2.5), we write that for $1 \le d \le p-3$,

$$\sum_{k=d+1}^{p-1} q^{k(2p-d+k)} B_{p,k}(q) B_{p,k-d}(q)$$
$$\equiv (-1)^d (1+q^p)^2 q^{p(3p+d-5)-\binom{d+1}{2}} \left(-[d+1]_q \right.$$
$$+ [p]_q \left(3 - 2q^{d+1} + \widetilde{H}_{p-1}(q) \left(1 + q^d \frac{2q}{q-1} \right) \right.$$
$$- \widetilde{H}_d(q) \left(\frac{q+1}{q-1} \right) + q^d \widetilde{H}_{p-d-1}(q) \left(\frac{3q-1}{q-1} \right)$$
$$\left. \left. - \frac{2}{q-1} \left(H_{p-d-1}(q) + q^d \left(H_{p-1}(q) - H_d(q) \right) \right) \right) \right) \pmod{[p]_q^2}.$$

Since congruence $\widetilde{H}_{p-d-1}(q) \equiv \widetilde{H}_{p-1}(q) + H_d(q) \pmod{[p]_q}$, (1.2), (1.3) and (1.5), the proof is complete. □

Theorem 2.9. *Let $p \geqslant 5$ be any prime. For $0 < d \leq p-3$,*

$$\sum_{k=d+1}^{p-1} (-1)^k q^{k(2p-d+k)} B_{p,k}(q) B_{p,k-d}(q) \equiv 4(-1)^d \frac{(1-q)}{1+q} q^{-\binom{d+1}{2}} \left(\frac{1+(-q)^{d+1}}{1-q} \right.$$
$$+ [p]_q \left(\left((-q)^d + 1 \right) (2Q_p(2,q) + I_d(q)) - \frac{2}{1-q} \left((-q)^{d+1} H_d(q) + \widetilde{H}_d(q) \right) \right.$$
$$-\frac{1}{2} \left(2(d-1) + p(5+q) - (-q)^d ((2d+1)q + p(q+1) - 1) \right.$$
$$\left.\left.\left. - (q-1)(-1)^d \right) \right) \right) \pmod{[p]_q^2}.$$

Proof. Observe that

$$\sum_{k=d+1}^{p-1} (-1)^k q^{k(2p-d+k)} B_{p,k}(q) B_{p,k-d}(q)$$
$$= \sum_{k=d+1}^{p-1} (-1)^k q^{k(2p-d+k)} \frac{[k]_q [k-d]_q}{[p]_q^2} \begin{bmatrix} 2p \\ p-k \end{bmatrix}_q \begin{bmatrix} 2p \\ p-k+d \end{bmatrix}_q$$
$$= \frac{[2p]_q^2}{[p]_q^2} \sum_{k=d+1}^{p-1} (-1)^k q^{k(2p-d+k)} \frac{[k]_q [k-d]_q}{[p-k]_q [p-k+d]_q} \begin{bmatrix} 2p-1 \\ p-k-1 \end{bmatrix}_q \begin{bmatrix} 2p-1 \\ p-k+d-1 \end{bmatrix}_q,$$

and by the equality $[2p]_q = (1+q^p)[p]_q$, equals

$$(1+q^p)^2 \sum_{k=d+1}^{p-1} (-1)^k q^{k(2p-d+k)} \frac{[k]_q [k-d]_q}{[p-k]_q [p-k+d]_q}] \begin{bmatrix} 2p-1 \\ p-k-1 \end{bmatrix}_q \begin{bmatrix} 2p-1 \\ p-k+d-1 \end{bmatrix}_q.$$

Then using (1.4) and Lemma 2.1, we show that for $1 \leq d \leq p-3$,

$$\sum_{k=d+1}^{p-1} (-1)^k q^{k(2p-d+k)} B_{p,k}(q) B_{p,k-d}(q)$$
$$\equiv (-1)^d (1+q^p)^2 q^{p(3p-5+d)-\binom{d+1}{2}} \left(\sum_{k=d+1}^{p-1} (-q)^k \right.$$
$$+ [p]_q \left(\sum_{k=d+1}^{p-1} (-1)^k \frac{q^{2k}}{[k]_q} + (-q)^d \sum_{k=1}^{p-1-d} (-1)^k \frac{q^{2k}}{[k]_q} \right)$$
$$\left. - [2p]_q \left(\sum_{k=d+1}^{p-1} (-q)^k \widetilde{H}_{p-k-1}(q) + \sum_{k=d+1}^{p-1} (-q)^k \widetilde{H}_{p-k+d-1}(q) \right) \right) \pmod{[p]_q^2}.$$

By Corollary 2.4 and the congruence $[2p]_q \equiv [p]_q \pmod{[p]_q^2}$, we have

$$\sum_{k=d+1}^{p-1} (-1)^k q^{k(2p-d+k)} B_{p,k}(q) B_{p,k-d}(q)$$
$$\equiv (-1)^d (1+q^p)^2 q^{p(3p-5+d)-\binom{d+1}{2}} \left(\sum_{k=d+1}^{p-1} (-q)^k \right.$$
$$+ [p]_q \left((-2Q_p(2,q) - I_d(q)) \left(1 + (-q)^d \right) \right.$$

$$-(1-q)\left(\frac{p-(-1)^d}{2}+(-q)^d\frac{p-3}{2}+\frac{2+(-q)^d(1-q)}{q+1}\right.$$
$$\left.\left.\left.-2\left(\sum_{k=d+1}^{p-1}(-q)^k\widetilde{H}_{p-k-1}(q)+\sum_{k=d+1}^{p-1}(-q)^k\widetilde{H}_{p-k+d-1}(q)\right)\right)\right)\right) \pmod{[p]_q^2}.$$

By Lemma 2.5, we can rewrite

$$\sum_{k=d+1}^{p-1}(-1)^k q^{k(2p-d+k)}B_{p,k}(q)\,B_{p,k-d}(q)$$
$$\equiv (-1)^d(1+q^p)^2 q^{p(3p-5+d)-\binom{d+1}{2}}\left(\frac{1+(-q)^{d+1}}{q-1}\right.$$
$$+[p]_q\left(\frac{2}{1-q}\left(\widetilde{H}_d(q)+(-q)^{d+1}H_d(q)\right)-(2Q_p(2,q)+I_d(q))\left(1+(-q)^d\right)\right.$$
$$\left.\left.+\frac{1}{2}\left((-q)^d(1-p-9q+5pq)+\left((-1)^d-p\right)(1-q)+6\right)\right)\right) \pmod{[p]_q^2}.$$

By (1.5), we have the proof of the congruence. □

Theorem 2.10. *Let $p\geqslant 5$ be any prime. For $0<d\leq p-3$,*

$$\sum_{k=d+1}^{p-1} q^{3\binom{k}{2}-k(d-3p-1)}B_{p,k}^2(q)B_{p,k-d}(q)$$
$$\equiv 8(-1)^{d+1}q^{9\binom{p}{2}-\binom{d}{2}+d(p-1)}\left(\frac{1+(-q)^{d+1}}{q+1}\right.$$
$$+[p]_q(1-q)\left(\frac{1}{1+q}\left(\frac{(-q)^d}{2}((p-1)(1-7q)-q-4)-\frac{3}{2}\right.\right.$$
$$\left.-3(-1)^d+\frac{1+qp}{q}+\left(2+(-q)^d\right)(I_d(q)+2Q_p(2,q))\right)$$
$$\left.\left.-\frac{1}{q}\left(1-2q(-1)^d\left(1+q^d\right)\right)-\frac{2}{1-q^2}\left(\widetilde{H}_d(q)+2(-q)^{d+1}H_d(q)\right)\right)\right) \pmod{[p]_q^2}.$$

Proof. By (1.4), (1.5), Lemma 2.1, Corollary 2.4 and Lemma 2.5, the proof is similar to the proof of Theorem 2.8. □

Theorem 2.11. *Let $p\geqslant 5$ be any prime. For $0<d\leq p-3$,*

$$\sum_{k=d+1}^{p-1}(-1)^k q^{3\binom{k}{2}-k(d-3p-1)}B_{k,d}^2(q)B_{p,k-d}(q)\equiv 8(-1)^d q^{-\binom{d+1}{2}}\left([d+1]_q\right.$$
$$-[p]_q\left(q^d\left(\left(\frac{1}{2}(p-q-1)-q(d+p)\right)+\frac{5q+1}{q-1}H_d(q)\right)\right.$$
$$\left.\left.-2\frac{q+2}{q-1}\widetilde{H}_d(q)+\frac{11}{2}p+q(p-1)+d-3\right)\right) \pmod{[p]_q^2}.$$

Proof. By Lemma 2.1, (1.4), (1.5), (2.3) and (2.5), the proof is similar to the proof of Theorem 2.8. □

Theorem 2.12. *Let $p\geqslant 5$ be any prime. For $0<d\leq p-3$,*

$$\sum_{k=d+1}^{p-1} q^{k(k+2p-d)}[k]_q\,B_{p,k}(q)\,B_{p,k-d}(q)\equiv\frac{4(-1)^d}{[2]_q}q^{-\binom{d+1}{2}}\left(-q\,[d+1]_q\,[d]_q\right.$$

$$+[p]_q\left(\frac{q^d}{1-q}\left(\frac{p+1-q^3(p-1)+q(1-q)(2p-4)}{1-q}+q^2\left(p+q^d\right)\right)\right.$$
$$+2\frac{q^{2d}}{1-q}\left(\frac{1-q^2}{2}(p-1)+\frac{2q-1}{q-1}-p+d\right)$$
$$+\frac{2d-p(1-q)}{1-q}+[d+1]_q\left(\left(1+q^{d+1}\right)(p-1)+2\frac{q}{1-q}\right)$$
$$+[d]_q\left(-q^d\frac{p-1+q(2+q(5p-3+2d))}{2}+q(3p-3+d)+1\right)$$
$$-\frac{q^d\left(q^d(1-3q^2)+1+3q^2+4q\right)+2}{(1-q)^2}H_d(q)\Bigg)$$
$$\left.-2\left([d]_q\frac{1+q^d}{1-q}-\frac{2(1+q)}{(1-q)^2}\right)\widetilde{H}_d(q)\right)\Bigg)\pmod{[p]_q^2}.$$

Proof. By Lemma 2.1, (1.4), (1.5), (2.6), (2.10) and (2.11), the proof is clearly given. □

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