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Oscillatory Criteria of Nonlinear Higher Order Ψ-Hilfer Fractional Differential Equations

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Abstract

In this paper, we study the forced oscillatory theory for higher order fractional differential equations with damping term via Ψ-Hilfer fractional derivative. We get sufficient conditions which ensure the oscillation of all solutions and give an illustrative example for our results. The Ψ-Hilfer fractional derivative according to the choice of the Ψ function is a generalization of the different fractional derivatives defined earlier. The results obtained in this paper are a generalization of the known results in the literature, and present new results for some fractional derivatives.

1. Introduction

Arbitrary order differential and integration notions are notions that combine and generalize integer order derivatives and n-fold integrals. Fractional differential theory is a very good tool that can be used to describe the inherited properties of various items and operations. This is an important advantage for fractional derivatives compared to integer order derivatives. This advantage of fractional derivatives is used in mathematical modeling of the mechanical and electrical properties of objects, in many other fields such as fluid theory, electrical circuits, electro-analytical chemistry [1]-[6]. Many definitions of fractional derivatives and integrals have been made, for more details, we recommend the monographs [7]-[10]. In recent years, the behavior of solutions of fractional differential equations has been an attractive area for researchers. Especially, the oscillation behavior of solutions has been studied by many researchers [11]-[19]. We also refer the reader to the papers [20], [21] for the oscillation of dynamic equations on time scales and to the papers [22], [23] for the oscillation of functional differential equations.
In [14] the authors considered the oscillatory criteria of nonlinear fractional differential equations by taking fractional initial value problem

$$\begin{aligned} D_a^{\mu} x(t) + f_1(t,x) &= v(t) + f_2(t,x), \quad t > a,\ 0 < \mu \leq 1 \\ \lim_{t \to a^+} J_a^{1-\mu} x(t) &= b, \end{aligned}$$

where D_a^{μ} shows μ order Riemann-Liouville fractional derivative, $J_a^{1-\mu}$ is $1-\mu$ order Riemann-Liouville fractional integral. Recently, in [24] Vivek et al. studied the oscillatory theory for Ψ-Hilfer fractional type fractional differential equations

$$\begin{aligned} {}^{H}\mathbb{D}_{a^+}^{\mu,\nu;\Psi} x(t) + f_1(t,x) &= \omega(t) + f_2(t,x), \qquad 0 < \mu < 1,\ 0 \leq \nu \leq 1 \\ I_{a^+}^{1-\eta;\Psi} x(t) &= b_1 \end{aligned}$$

where ${}^{H}\mathbb{D}_{a^+}^{\mu,\nu;\Psi}$ denotes Ψ-Hilfer fractional derivative and $I_{a^+}^{1-\eta;\Psi}$ is the Ψ-Riemann-Liouville fractional integral with $\eta = \mu + \nu(1-\mu)$.

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In [18] the authors examined oscillation of the solutions of forced fractional differential equations with damping term via the Riemann-Liouville fractional derivative

$$\begin{aligned} \left(D_{0^+}^{1+\mu}y\right)(t)+p(t)\left(D_{0^+}^{\mu}y\right)(t)+q(t)f(y(t)) &= g(t), \quad t>0 \\ \left(I_{0^+}^{1-\mu}y\right)(0^+) &= b, \end{aligned}$$

where $\mu\in(0,1)$.

In this paper, inspired by the above articles, we studied the oscillation properties of forced fractional differential equations with damping term

$$\begin{aligned} D\left({}^H\mathbb{D}_{a^+}^{\mu,\nu;\Psi}y(x)\right)+p(x){}^H\mathbb{D}_{a^+}^{\mu,\nu;\Psi}y(x)+q(x)f(y(x)) &= g(x), \quad x>0 \qquad (1.1)\\ \left(\frac{1}{\Psi'(x)}\frac{d}{dx}\right)^{m-i}I_{a^+}^{1-\eta;\Psi}y(x)|_a &= y_i,\ i=1,2,\dots,m \end{aligned}$$

where $m-1<\mu<m$, $0\le\nu\le1$ is a constant and $\eta=\mu+\nu(m-\mu)$, ${}^H\mathbb{D}_{a^+}^{\mu,\nu;\Psi}y(x)$ is the Ψ-Hilfer fractional differential operator of order μ type ν of $y(x)$. Throughout this paper, we assume that

(A) $p(x)\in C(\mathbb{R}^+,\mathbb{R})$, $q(x)\in C(\mathbb{R}^+,\mathbb{R}^+)$, $f(x)\in C(\mathbb{R},\mathbb{R})$ and $f(y)/y>0$ for all $y\neq0$, $g(x)\in C(\mathbb{R}^+,\mathbb{R})$.

Definition 1.1 ([25]). *A solution $y(x)$ of problem (1.1) is said to be oscillatory if it has arbitrarily large zeros for $x\ge x_0$ there exists a sequence of zeros $\{x_n\}$ of y such that $\lim_{n\to\infty}x_n=\infty$. Otherwise, y is said to be non-oscillatory.*

2. Preliminaries

In this section, we mention some basic definitions and theorems which will be used in the study.

Definition 2.1 ([8]). *Let f be a function defined on $[a,b]$, $(-\infty<a<b<\infty)$. μ-th left-sided and right-sided Riemann-Liouville fractional integrals of f are given by*

$$I_{a^+}^{\mu}f(x):=\frac{1}{\Gamma(\mu)}\int_a^x(x-t)^{\mu-1}f(t)dt, \quad x>a,\ \mu>0$$

and

$$I_{b^-}^{\mu}f(x):=\frac{1}{\Gamma(\mu)}\int_x^b(t-x)^{\mu-1}f(t)dt, \quad x<b,\ \mu>0$$

respectively.

Definition 2.2 ([8]). *Assume $\lceil\mu\rceil=m$, $m\in\mathbb{N}_0$ and $f(x)\in C^m(a,b)$. Left-sided and right-sided Riemann-Liouville fractional derivatives of f of order μ, are defined respectively by*

$$\begin{aligned} D_{a^+}^{\mu}f(x) &= \left(\frac{d}{dx}\right)^m I_{a^+}^{m-\mu}f(x)\\ &= \frac{1}{\Gamma(m-\mu)}\left(\frac{d}{dx}\right)^m\int_a^x(x-t)^{m-\mu-1}f(t)dt \end{aligned}$$

and

$$\begin{aligned} D_{b^-}^{\mu}f(x) &= (-1)^m\left(\frac{d}{dx}\right)^m I_{b^-}^{m-\mu}f(x)\\ &= \frac{(-1)^m}{\Gamma(m-\mu)}\left(\frac{d}{dx}\right)^m\int_x^b(t-x)^{m-\mu-1}f(t)dt. \end{aligned}$$

In [26], Hilfer generalized the Riemann-Liouville fractional derivative operator by introducing a right-sided fractional derivative operator.

Definition 2.3. *Let $\lceil\mu\rceil=m$, $m\in\mathbb{N}_0$, $\nu\in[0,1]$ and $f(x)\in C^n(a,b)$. The left-sided and right sided Hilfer fractional derivatives of f of order μ and type ν are given by*

$$D_{a^+}^{\mu,\nu}f(x)=I_{a^+}^{\eta-\mu}\left(\frac{d}{dx}\right)^m I_{a^+}^{(1-\nu)(m-\mu)}f(x)$$

and

$$D_{b^-}^{\mu,\nu}f(x)=I_{b^-}^{\eta-\mu}\left(-\frac{d}{dx}\right)^m I_{b^-}^{(1-\nu)(m-\mu)}f(x)$$

where $\eta=\mu+\nu(m-\mu)$.

Due to a large number of definitions, the next definition is a significant approach because of the kernel has an arbitrary function Ψ.

Definition 2.4 ([8])**.** *Let f be a function defined on (a,b), $(-\infty\leq a<b\leq\infty)$ and $\mu>0$ and also assume $\Psi(x)$ is a positive monotone and increasing function on $(a,b]$, $\Psi'(x)$ is continuous on (a,b). The left and right-sided fractional integrals of f with respect to Ψ of order μ are given by*

$$I_{a^+}^{\mu;\Psi}f(x)=\frac{1}{\Gamma(\mu)}\int_a^x(\Psi(x)-\Psi(t))^{\mu-1}f(t)\Psi'(t)dt$$

and

$$I_{b^-}^{\mu;\Psi}f(x)=\frac{1}{\Gamma(\mu)}\int_x^b(\Psi(t)-\Psi(x))^{\mu-1}f(t)\Psi'(t)dt.$$

Lemma 2.5 ([8])**.** *Assume $\mu>0$ and $\nu>0$. Then,*

$$I_{a^+}^{\mu;\Psi}I_{a^+}^{\nu;\Psi}f(x)=I_{a^+}^{\mu+\nu;\Psi}f(x)$$

and

$$I_{b^-}^{\mu;\Psi}I_{b^-}^{\nu;\Psi}f(x)=I_{b^-}^{\mu+\nu;\Psi}f(x)$$

semigroup property hold.

Definition 2.6 ([8])**.** *Assume f is a function defined on $[a,b]$, $\Psi(x)\neq 0$ and $\lceil\mu\rceil=m$, $m\in\mathbb{N}$. The right and left-sided Riemann-Liouville derivatives of f with respect to another function Ψ of order μ are given respectivly by*

$$\begin{aligned}D_{a^+}^{\mu;\Psi}f(x)&=\left(\frac{1}{\Psi'(x)}\frac{d}{dx}\right)^m I_{a^+}^{m-\mu;\Psi}f(x)\\&=\frac{1}{\Gamma(m-\mu)}\left(\frac{1}{\Psi'(x)}\frac{d}{dx}\right)^m\int_a^x(\Psi(x)-\Psi(t))^{m-\mu-1}\Psi'(t)f(t)dt\end{aligned}$$

and

$$\begin{aligned}D_{b^-}^{\mu;\Psi}f(x)&=\left(-\frac{1}{\Psi'(x)}\frac{d}{dx}\right)^m I_{b^-}^{m-\mu;\Psi}f(x)\\&=\frac{1}{\Gamma(m-\mu)}\left(-\frac{1}{\Psi'(x)}\frac{d}{dx}\right)^m\int_x^b(\Psi(t)-\Psi(x))^{m-\mu-1}\Psi'(t)f(t)dt.\end{aligned}$$

In [27], Sousa and Oliveria presented a new fractional derivative which unifies Hilfer fractional derivative and Riemann-Lioville derivative with respect to another function.

Definition 2.7. *Assume $\lceil\mu\rceil=m$, $m\in\mathbb{N}$ and $\nu\in[0,1]$. Also let $f\in C^n([a,b],\mathbb{R})$, $-\infty\leq a<b\leq\infty$, Ψ be an increasing function on $[a,b]$ and $\Psi'(x)\neq 0$, for all $x\in[a,b]$. The right and left-sided $\Psi-$Hilfer fractional derivatives of f of order μ and type ν, are given by*

$$^H\mathbb{D}_{a^+}^{\mu,\nu;\Psi}f(x)=I_{a^+}^{\nu(m-\mu);\Psi}\left(\frac{1}{\Psi'(x)}\frac{d}{dx}\right)^m I_{a^+}^{(1-\nu)(m-\mu);\Psi}f(x)$$

and

$$^H\mathbb{D}_{b^-}^{\mu,\nu;\Psi}f(x)=I_{b^-}^{\nu(m-\mu);\Psi}\left(-\frac{1}{\Psi'(x)}\frac{d}{dx}\right)^m I_{b^-}^{(1-\nu)(m-\mu);\Psi}f(x).$$

Remark 2.8 ([27])**.** *The Ψ-Hilfer fractional derivative can be given as following for $\eta=\mu+\nu(m-\mu)$*

$$^H\mathbb{D}_{a^+}^{\mu,\nu;\Psi}f(x)=I_{a^+}^{\eta-\mu;\Psi}D_{a^+}^{\eta;\Psi}f(x)$$

and

$$^H\mathbb{D}_{b^-}^{\mu,\nu;\Psi}f(x)=I_{b^-}^{\eta-\mu;\Psi}(-1)^m D_{b^-}^{\eta;\Psi}f(x).$$

Theorem 2.9 ([27]). *Let $f \in C^m[a,b]$, $\lceil \mu \rceil = m$ and $\nu \in [0,1]$. Then*

$$I_{a^+}^{\mu;\Psi}\, {}^H\mathbb{D}_{a^+}^{\mu,\nu;\Psi} f(x) = f(x) - \sum_{i=1}^{m} \frac{(\Psi(x)-\Psi(a))^{\eta-i}}{\Gamma(\eta-i+1)} \left(\frac{1}{\Psi'(x)}\frac{d}{dx}\right)^{m-i} I_{a^+}^{(1-\nu)(m-\mu);\Psi} f(a) \tag{2.1}$$

and

$$I_{b^-}^{\mu;\Psi}\, {}^H\mathbb{D}_{b^-}^{\mu,\nu;\Psi} f(x) = f(x) - \sum_{i=1}^{m} \frac{(-1)^i (\Psi(b)-\Psi(x))^{\eta-i}}{\Gamma(\eta-i+1)} \left(\frac{1}{\Psi'(x)}\frac{d}{dx}\right)^{m-i} I_{b^-}^{(1-\nu)(m-\mu);\Psi} f(b).$$

3. Main results

Theorem 3.1. *Assume (A) and the following conditions meet*

$$\liminf_{x\to+\infty} (\Psi(x))^{1-\eta} I_{T^+}^{\mu;\Psi} \left(\frac{M}{V(x)} + \frac{1}{V(x)}\int_{x_0}^{x} g(t)V(t)dt\right) = -\infty, \tag{3.1}$$

$$\limsup_{x\to+\infty} (\Psi(x))^{1-\eta} I_{T^+}^{\mu;\Psi} \left(\frac{M}{V(x)} + \frac{1}{V(x)}\int_{x_0}^{x} g(t)V(t)dt\right) = \infty, \tag{3.2}$$

where $M \in \mathbb{R}$ is a constant and $V(t) = \exp\int_{x_0}^{t} p(\xi)d\xi$. Then each solution of (1.1) oscillates for every sufficiently large T.

Proof. To obtain contradiction, assume that $y(x)$ is a non-oscillatory solution of (1.1). We can suppose that there exist $T>0$, $x_0 > x$ without losing any generality, such that $y(x) > 0$ for all $x \geq x_0$. According to (1.1) and *(A)*,

$$\begin{aligned}
\left[{}^H\mathbb{D}_{a^+}^{\mu,\nu;\Psi} y(x)\; V(x)\right]' &= D\left[{}^H\mathbb{D}_{a^+}^{\mu,\nu;\Psi} y(x)\right] V(x) + {}^H\mathbb{D}_{a^+}^{\mu,\nu;\Psi} y(x)\; p(x)V(x) \\
&= -q(x)f(y(x))V(x) + g(x)V(x) \\
&< g(x)V(x).
\end{aligned}$$

Integrating the inequality from x_0 to x, we get

$${}^H\mathbb{D}_{a^+}^{\mu,\nu;\Psi} y(x)\; V(x) < {}^H\mathbb{D}_{a^+}^{\mu,\nu;\Psi} y(x_0)\; V(x_0) + \int_{x_0}^{x} g(t)V(t)dt = M + \int_{x_0}^{x} g(t)V(t)dt,$$

where $M = {}^H\mathbb{D}_{a^+}^{\mu,\nu;\Psi} y(x_0)\; V(x_0)$. From (2.1) we can obtain

$$y(x) < \sum_{i=1}^{m} \frac{(\Psi(x)-\Psi(a))^{\eta-i}}{\Gamma(\eta-i+1)} y_i + I_{a^+}^{\mu;\Psi}\left(\frac{M}{V(x)} + \frac{1}{V(x)}\int_{x_0}^{x} g(t)V(t)dt\right). \tag{3.3}$$

Multiplying the both sides of inequality (3.3) with $(\Psi(x))^{1-\eta}$ we obtain

$$\begin{aligned}
(\Psi(x))^{1-\eta} y(x) &\leq (\Psi(x))^{1-\eta} \sum_{i=1}^{m} \frac{(\Psi(x)-\Psi(a))^{\eta-i}}{\Gamma(\eta-i+1)} y_i \\
&\quad + (\Psi(x))^{1-\eta} I_{a^+}^{\mu;\Psi}\left(\frac{M}{V(x)} + \frac{1}{V(x)}\int_{x_0}^{x} g(t)V(t)dt\right) \\
&\leq (\Psi(x))^{1-\eta} \sum_{i=1}^{m} \frac{(\Psi(x)-\Psi(a))^{\eta-i}}{\Gamma(\eta-i+1)} y_i \\
&\quad + (\Psi(x))^{1-\eta} \frac{1}{\Gamma(\mu)} \int_a^T (\Psi(x)-\Psi(\tau))^{\mu-1}\Psi'(\tau)\left(\frac{M}{V(\tau)} + \frac{1}{V(\tau)}\int_{x_0}^{\tau} g(t)V(t)dt\right) d\tau \\
&\quad + (\Psi(x))^{1-\eta} I_{T^+}^{\mu;\Psi}\left(\frac{M}{V(x)} + \frac{1}{V(x)}\int_{x_0}^{x} g(t)V(t)dt\right),\ x \geq T.
\end{aligned}$$

Define

$$\Phi(x) = (\Psi(x))^{1-\eta} \sum_{i=1}^{m} \frac{(\Psi(x)-\Psi(a))^{\eta-i}}{\Gamma(\eta-i+1)} y_i$$

and

$$\Psi(x,T)=(\Psi(x))^{1-\eta}\frac{1}{\Gamma(\mu)}\int_a^T(\Psi(x)-\Psi(\tau))^{\mu-1}\Psi'(\tau)\left(\frac{M}{V(\tau)}+\frac{1}{V(\tau)}\int_{x_0}^{\tau}g(t)V(t)dt\right)d\tau.$$

Then we get,

$$0<(\Psi(x))^{1-\eta}y(x)\leq\Phi(x)+\Psi(x,T)+(\Psi(x))^{1-\eta}I_{T^+}^{\mu;\Psi}\left(\frac{M}{V(x)}+\frac{1}{V(x)}\int_{x_0}^{x}g(t)V(t)dt\right),\ x\geq T. \tag{3.4}$$

We take two cases as follows.

Case(1): Assume $0<\mu\leq 1$ and so on $0<\eta\leq 1$. Then $m=1$ and $|\Phi(x)|=\left|y_1(\Psi(x))^{1-\eta}\frac{(\Psi(x)-\Psi(a))^{\eta-1}}{\Gamma(\eta)}\right|$. For $x>T_1>T$, we get

$$|\Phi(x)|=\left|y_1\frac{1}{\Gamma(\eta)}\left(\frac{\Psi(x)-\Psi(a)}{\Psi(x)}\right)^{\eta-1}\right|\leq\frac{|y_1|}{\Gamma(\eta)}\left(\frac{\Psi(T_1)-\Psi(a)}{\Psi(T_1)}\right)^{\eta-1}:=c_1(T_1).$$

Furthermore we have

$$\begin{aligned}
|\Psi(x,T)| &= \left|(\Psi(x))^{1-\eta}\frac{1}{\Gamma(\mu)}\int_a^T(\Psi(x)-\Psi(\tau))^{\mu-1}\Psi'(\tau)\left(\frac{M}{V(\tau)}+\frac{1}{V(\tau)}\int_{x_0}^{\tau}g(t)V(t)dt\right)d\tau\right|\\
&\leq \frac{1}{\Gamma(\mu)}\int_a^T\left|\frac{(\Psi(x)-\Psi(\tau))^{\mu-1}}{(\Psi(x))^{\eta-1}}\Psi'(\tau)\left(\frac{M}{V(\tau)}+\frac{1}{V(\tau)}\int_{x_0}^{\tau}g(t)V(t)dt\right)\right|d\tau\\
&\leq \frac{1}{\Gamma(\mu)}\int_a^T\left(\frac{\Psi(x)-\Psi(\tau)}{(\Psi(x))^{1-\nu}}\right)^{\mu-1}\Psi'(\tau)\left|\frac{M}{V(\tau)}+\frac{1}{V(\tau)}\int_{x_0}^{\tau}g(t)V(t)dt\right|d\tau\\
&\leq \frac{1}{\Gamma(\mu)}\int_a^T\left(\frac{\Psi(T_1)-\Psi(\tau)}{(\Psi(T_1))^{1-\nu}}\right)^{\mu-1}\Psi'(\tau)\left|\frac{M}{V(\tau)}+\frac{1}{V(\tau)}\int_{x_0}^{\tau}g(t)V(t)dt\right|d\tau\\
&:= c_2(T,T_1).
\end{aligned}$$

Using inequality (3.4), we obtain

$$0<(\Psi(x))^{1-\eta}y(x)\leq c_1(T_1)+c_2(T,T_1)+(\Psi(x))^{1-\eta}I_{T^+}^{\mu;\Psi}\left(\frac{M}{V(x)}+\frac{1}{V(x)}\int_{x_0}^{x}g(t)V(t)dt\right),$$

and then

$$(\Psi(x))^{1-\eta}I_{T^+}^{\mu;\Psi}\left(\frac{M}{V(x)}+\frac{1}{V(x)}\int_{x_0}^{x}g(t)V(t)dt\right)\geq-[c_1(T_1)+c_2(T,T_1)]. \tag{3.5}$$

Taking limit of (3.5) as $x\to\infty$ we get

$$\liminf_{x\to\infty}(\Psi(x))^{1-\eta}I_{T^+}^{\mu;\Psi}\left(\frac{M}{V(x)}+\frac{1}{V(x)}\int_{x_0}^{x}g(t)V(t)dt\right)\geq-[c_1(T_1)+c_2(T,T_1)]>-\infty$$

which contradicts the condition (3.1).

Case(2): Assume $\mu>1$. Then $m\geq 2$ and $m-1\leq\eta\leq m$. For $x\geq T_2$ we get

$$\begin{aligned}
|\Phi(x)| &= \left|(\Psi(x))^{1-\eta}\sum_{i=1}^{m}\frac{(\Psi(x)-\Psi(a))^{\eta-i}}{\Gamma(\eta-i+1)}y_i\right|\\
&\leq \left(\frac{\Psi(x)-\Psi(a)}{\Psi(x)}\right)^{\eta-1}\sum_{i=1}^{m}\frac{(\Psi(x)-\Psi(a))^{1-i}}{\Gamma(\eta-i+1)}|y_i|\\
&\leq \sum_{i=1}^{m}\frac{(\Psi(x)-\Psi(a))^{1-i}}{\Gamma(\eta-i+1)}|y_i|\\
&\leq \sum_{i=1}^{m}\frac{(\Psi(T_2)-\Psi(a))^{1-i}}{\Gamma(\eta-i+1)}|y_i|:=c_3(T_2).
\end{aligned}$$

Since $\eta=\mu+\nu(m-\mu)\geq\mu$, $\frac{(\Psi(x)-\Psi(a))^{\mu-1}}{(\Psi(x))^{\eta-1}}\leq 1$ for $m-1<\mu<m$. Then we have

$$\begin{aligned}|\Psi(x,T)| &= \left|(\Psi(x))^{1-\eta}\frac{1}{\Gamma(\mu)}\int_a^T(\Psi(x)-\Psi(\tau))^{\mu-1}\Psi'(\tau)\left(\frac{M}{V(\tau)}+\frac{1}{V(\tau)}\int_{x_0}^{\tau}g(t)V(t)dt\right)d\tau\right|\\ &\leq \frac{1}{\Gamma(\mu)}\int_a^T\frac{(\Psi(x)-\Psi(\tau))^{\mu-1}}{(\Psi(x))^{\eta-1}}\Psi'(\tau)\left|\frac{M}{V(\tau)}+\frac{1}{V(\tau)}\int_{x_0}^{\tau}g(t)V(t)dt\right|d\tau\\ &\leq \frac{1}{\Gamma(\mu)}\int_a^T\Psi'(\tau)\left|\frac{M}{V(\tau)}+\frac{1}{V(\tau)}\int_{x_0}^{\tau}g(t)V(t)dt\right|d\tau\\ &:= c_4(T).\end{aligned}$$

Using inequality (3.4), we conclude that

$$(\Psi(x))^{1-\eta}I_{T^+}^{\mu;\Psi}\left(\frac{M}{V(x)}+\frac{1}{V(x)}\int_{x_0}^{x}g(t)V(t)dt\right)\geq-[c_3(T_2)+c_4(T)],\tag{3.6}$$

hence taking limit of (3.6) as $x\to\infty$ we obtain

$$\liminf_{x\to\infty}(\Psi(x))^{1-\eta}I_{T^+}^{\mu;\Psi}\left(\frac{M}{V(x)}+\frac{1}{V(x)}\int_{x_0}^{x}g(t)V(t)dt\right)\geq-[c_3(T_2)+c_4(T)]>-\infty,$$

which contradicts (3.1).

Consequently, we conclude that $y(x)$ is an oscillatory solution of (1.1). If $y(x)$ is eventually negative, a contradiction can be obtained with (3.2) similarly. □

Choosing a special Ψ function and specific μ and ν real numbers, the Ψ-Hilfer fractional derivative turn into 22 different fractional derivative which is defined before. Sousa and Oliveira remarked on all of these 22 different situations in [27].

Remark 3.2. *If we take limit $\nu\to 0$ and $\Psi(x)=x^{\rho}$, then we have the following fractional derivative*

$${}^{H}\mathbb{D}_{a^+}^{\mu,\nu;\Psi}y(x)={}^{H}\mathbb{D}_{a^+}^{\mu,0;x^{\rho}}y(x)=\left(\frac{1}{x^{\rho-1}}\frac{d}{dx}\right)^{m}I_{a^+}^{m-\mu;x^{\rho}}y(x)$$

which is defined by Katugampola in [28].

Remark 3.3. *If we take limit $\nu\to 1$ and $\Psi(x)=x^{\rho}$, then we have Caputo type Katugampola fractional derivative which is defined in [29] as follows*

$${}^{H}\mathbb{D}_{a^+}^{\mu,1;x^{\rho}}y(x)=I_{a^+}^{(m-\mu);x^{\rho}}\left(\frac{1}{x^{\rho-1}}\frac{d}{dx}\right)^{m}y(x).$$

Remark 3.4. *If we take limit $\nu\to 0$ and $\Psi(x)=\ln x$, then we have Hadamard fractional derivative*

$${}^{H}\mathbb{D}_{a^+}^{\mu,0;\ln x}y(x)=\left(x\frac{d}{dx}\right)^{m}I_{a^+}^{m-\mu;\ln x}y(x).$$

Example 3.5. *Consider the initial value problem*

$$\begin{aligned}D\left({}^{H}\mathbb{D}_{a^+}^{\frac{3}{2},0;\ln x}y(x)\right)-\frac{1}{x}\,{}^{H}\mathbb{D}_{a^+}^{\frac{3}{2},0;\ln x}y(x)+e^{x^2}y^3e^y &= x\sin(\ln x)\\ I_{a^+}^{\frac{3}{2};\ln x}y(t) &= b.\end{aligned}\tag{3.7}$$

Here $\mu=3/2$, $\nu=0$, $\Psi=\ln x$, $p(x)=-1/x$, $q(x)=e^{x^2}$, $f(y)=y^3e^y$, $g(x)=x\sin(\ln x)$ and $V(x)=\exp\int_{x_0}^{x}p(t)dt=x_0/x$. Then

$$\begin{aligned}\int_{x_0}^{x}g(t)V(t)dt &= \int_{x_0}^{x}t\sin(\ln t)\frac{x_0}{t}dt\\ &= x_0\int_{\ln x_0}^{\ln x}e^{\xi}\sin\xi\,d\xi\\ &= \frac{x_0}{2}\left[x\left(\sin(\ln x)-\cos(\ln x)\right)+x_0\left(\cos(\ln x_0)-\sin(\ln x_0)\right)\right]\\ &= \frac{x_0}{2}\left[\frac{2x}{\sqrt{2}}\sin\left(\ln x-\frac{\pi}{4}\right)+x_0\left(\cos(\ln x_0)-\sin(\ln x_0)\right)\right].\end{aligned}$$

Set $x_0 = 1$. *Then, we can obtain*

$$\begin{aligned}
I_{a^+}^{\frac{3}{2};\ln x}\left(\frac{M}{V(x)}+\frac{1}{V(x)}\int_{x_0}^{x} g(t)V(t)dt\right) &= \frac{1}{\Gamma(3/2)}\int_a^x (\ln x-\ln t)^{1/2}\left(\frac{M}{V(t)}+\frac{1}{V(t)}\left[\frac{t}{\sqrt{2}}\sin\left(\ln t-\frac{\pi}{4}\right)+\frac{1}{2}\right]\right)\frac{dt}{t}\\
&= \frac{2}{\sqrt{\pi}}\int_a^x (\ln x-\ln t)^{1/2}\left(\left(M+\frac{1}{2}\right)t+\frac{t^2}{\sqrt{2}}\sin\left(\ln t-\frac{\pi}{4}\right)\right)\frac{dt}{t}.
\end{aligned}$$

Set $\ln x - \ln t = \xi^2$. *Then the above integral can be written as the form:*

$$\begin{aligned}
&\frac{2}{\sqrt{\pi}}\int_a^x (\ln x-\ln t)^{1/2}\left(\left(M+\frac{1}{2}\right)t+\frac{t^2}{\sqrt{2}}\sin\left(\ln t-\frac{\pi}{4}\right)\right)\frac{dt}{t}\\
&= \frac{2}{\sqrt{\pi}}\int_{\sqrt{\ln\frac{x}{a}}}^{0}\xi\left(\left(M+\frac{1}{2}\right)xe^{-\xi^2}+\frac{x^2e^{-2\xi^2}}{\sqrt{2}}\sin\left(\ln x-\xi^2-\frac{\pi}{4}\right)\right)(-2\xi)d\xi\\
&= \frac{2(2M+1)x}{\sqrt{\pi}}\int_0^{\sqrt{\ln\frac{x}{a}}}\xi^2e^{-\xi^2}d\xi+\frac{2\sqrt{2}x^2}{\sqrt{\pi}}\int_0^{\sqrt{\ln\frac{x}{a}}}\xi^2e^{-2\xi^2}\sin\left(\ln x-\xi^2-\frac{\pi}{4}\right)d\xi\\
&= \frac{2(2M+1)x}{\sqrt{\pi}}\int_0^{\sqrt{\ln\frac{x}{a}}}\xi^2e^{-\xi^2}d\xi+\frac{2\sqrt{2}x^2}{\sqrt{\pi}}\sin\left(\ln x-\frac{\pi}{4}\right)\int_0^{\sqrt{\ln\frac{x}{a}}}\xi^2e^{-2\xi^2}\cos(\xi^2)d\xi\\
&\quad -\frac{2\sqrt{2}x^2}{\sqrt{\pi}}\cos\left(\ln x-\frac{\pi}{4}\right)\int_0^{\sqrt{\ln\frac{x}{a}}}\xi^2e^{-2\xi^2}\sin(\xi^2)d\xi.
\end{aligned}$$

Letting $x\to+\infty$, *because of* $|\xi^2e^{-2\xi^2}\cos(\xi^2)|\le\xi^2e^{-2\xi^2}$, $|\xi^2e^{-2\xi^2}\sin(\xi^2)|\le\xi^2e^{-2\xi^2}$ *and*

$$\lim_{x\to+\infty}\int_0^{\sqrt{\ln\frac{x}{a}}}\xi^2e^{-2\xi^2}d\xi=\lim_{x\to\infty}\left[-\frac{\xi e^{-2\xi^2}}{4}\Big|_0^{\sqrt{\ln\frac{x}{a}}}+\frac{1}{4}\int_0^{\sqrt{\ln\frac{x}{a}}}e^{-2\xi^2}d\xi\right]=0+\frac{1}{4}\frac{\sqrt{2\pi}}{4}=\frac{\sqrt{2\pi}}{16},$$

we know that $\lim_{x\to+\infty}\int_0^{\sqrt{\ln\frac{x}{a}}}\xi^2e^{-2\xi^2}\cos(\xi^2)d\xi$ *and* $\lim_{x\to+\infty}\int_0^{\sqrt{\ln\frac{x}{a}}}\xi^2e^{-2\xi^2}\sin(\xi^2)d\xi$ *are convergent. Thus, we can set*

$$\lim_{x\to+\infty}\int_0^{\sqrt{\ln\frac{x}{a}}}\xi^2e^{-2\xi^2}\cos(\xi^2)d\xi=K \text{ and } \lim_{x\to+\infty}\int_0^{\sqrt{\ln\frac{x}{a}}}\xi^2e^{-2\xi^2}\sin(\xi^2)d\xi=L.$$

Selecting sequence $\{x_k\}=\{e^{\frac{5\pi}{2}+\frac{\pi}{4}+2k\pi-\arctan\frac{-L}{K}}\}$, $\lim_{k\to\infty}x_k=\infty$, *then we calculate*

$$\begin{aligned}
\lim_{k\to\infty}\Bigg\{(\ln x_k)^{-1/2}x_k\Bigg[\frac{2M+1}{\sqrt{\pi}}\int_0^{\sqrt{\ln\frac{x_k}{a}}}\xi^2e^{-\xi^2}d\xi+\frac{\sqrt{2}x_k}{\sqrt{\pi}}\Bigg(\sin\left(\ln x_k-\frac{\pi}{4}\right)\int_0^{\sqrt{\ln\frac{x_k}{a}}}\xi^2e^{-2\xi^2}\cos(\xi^2)d\xi\\
-\cos\left(\ln x_k-\frac{\pi}{4}\right)\int_0^{\sqrt{\ln\frac{x_k}{a}}}\xi^2e^{-2\xi^2}\sin(\xi^2)d\xi\Bigg)\Bigg]\Bigg\}.
\end{aligned}$$

Firstly, let compute the following limit.

$$\begin{aligned}
&\lim_{k\to\infty}\left(\sin\left(\ln x_k-\frac{\pi}{4}\right)\int_0^{\sqrt{\ln\frac{x_k}{a}}}\xi^2e^{-2\xi^2}\cos(\xi^2)d\xi-\cos\left(\ln x_k-\frac{\pi}{4}\right)\int_0^{\sqrt{\ln\frac{x_k}{a}}}\xi^2e^{-2\xi^2}\sin(\xi^2)d\xi\right)\\
&= K\cdot\lim_{k\to\infty}\sin\left(\frac{5\pi}{2}+2k\pi-\arctan\frac{-L}{K}\right)-L\cdot\lim_{k\to\infty}\cos\left(\frac{5\pi}{2}+2k\pi-\arctan\frac{-L}{K}\right)\\
&= K\sin\left(\frac{5\pi}{2}-\arctan\frac{-L}{K}\right)-L\cos\left(\frac{5\pi}{2}-\arctan\frac{-L}{K}\right)\\
&= \sqrt{K^2+L^2}.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
&\lim_{k\to\infty}\Bigg\{(\ln x_k)^{-1/2}x_k\Bigg[\frac{2M+1}{\sqrt{\pi}}\int_0^{\sqrt{\ln\frac{x_k}{a}}}\xi^2e^{-\xi^2}d\xi+\frac{\sqrt{2}x_k}{\sqrt{\pi}}\Bigg(\sin\left(\ln x_k-\frac{\pi}{4}\right)\int_0^{\sqrt{\ln\frac{x_k}{a}}}\xi^2e^{-2\xi^2}\cos(\xi^2)d\xi\\
&\qquad -\cos\left(\ln x_k-\frac{\pi}{4}\right)\int_0^{\sqrt{\ln\frac{x_k}{a}}}\xi^2e^{-2\xi^2}\sin(\xi^2)d\xi\Bigg)\Bigg]\Bigg\}\\
&= (+\infty)\left[\frac{2M+1}{\sqrt{\pi}}\frac{\sqrt{\pi}}{4}+(+\infty)\sqrt{K^2+L^2}\right]\\
&= \infty.
\end{aligned}$$

Then we obtain

$$\limsup_{x\to+\infty}(\Psi(x))^{1-\eta}\, I_{T^+}^{\mu;\Psi}\left(\frac{M}{V(x)}+\frac{1}{V(x)}\int_{x_0}^{x} g(t)V(t)dt\right)=\infty.$$

Similarly, selecting the sequence $x_l=\{e^{\frac{3\pi}{2}+\frac{\pi}{4}+2l\pi-\arctan\frac{-L}{K}}\}$, *we can obtain*

$$\liminf_{x\to+\infty}(\Psi(x))^{1-\eta}\, I_{T^+}^{\mu;\Psi}\left(\frac{M}{V(x)}+\frac{1}{V(x)}\int_{x_0}^{x} g(t)V(t)dt\right)=-\infty.$$

Therefore, all solutions of (3.7) *are oscillatory by Theorem 3.1.*

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Author's contributions

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