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Numerical Algorithm for Solving General Linear Elliptic Quaternionic Matrix Equations

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Abstract

In this study, we develop a general method to solve the general linear elliptic quaternionic matrix equations by means of real representation of elliptic quaternion matrices. A pseudo-code for our approach that provides the solution of the linear elliptic quaternionic matrix equations is expressed. Moreover, we apply this method to the well-known Slyvester matrix equations and Kalman Yakubovich matrix equations over the elliptic quaternion algebra.

1. Introduction and Preliminaries

Real quaternions are a four-dimensional number system that was first expressed by Hamilton in 1843, based on the idea of generalizing complex numbers [1]. Hamilton first tried to describe the 3-dimensional number system as follows:

$$q = q_0 + q_1 i + q_2 j$$

where $q_0, q_1, q_2 \in \mathbb{R}$ and $i^2 = j^2 = -1$. However, he saw that this number system does not provide the closure property under multiplication. In this way, Hamilton saw that there could not be a system similar to any 3-dimensional complex number system and defined the 4-dimensional number system is known as the real quaternion in the following way:

$$\mathbb{K} = \{q = q_0 + q_1 i + q_2 j + q_3 k : \ q_0, q_1, q_2, q_3 \in \mathbb{R} \text{ and } i, j, k \notin \mathbb{R}\} \tag{1.1}$$

such that

$$i^2 = j^2 = k^2 = -1, \ ij = -ji = k, \ ik = -ki = -j, \ jk = -kj = i. \tag{1.2}$$

There are many applications of real quaternion algebra in different fields of the scientific world. The main areas are kinematics, mechanics, quantum physics, chemistry, image-signal restoration, and game development. For this reason, there are many studies related to real quaternions in literature [2]-[6].

On the other hand, Segre defined commutative quaternions in 1892 [7]. One of the most essential properties of a commutative quaternion is that it meets the commutative property of multiplication. The commutative quaternion algebra is a significant factor in fields such as Hopfield neural networks, digital signal, and image processing [8]-[11]. Therefore, commutative

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quaternion algebra theory has been increasingly important in recent years.

Elliptic quaternions are the generalized form of commutative quaternions. The set of elliptic quaternions is a commutative ring under a commutative law and combination law of a four-dimensional Clifford algebra. Moreover, this set contains non-trivial idempotents, nilpotent elements, and zero-divisors [8, 12, 13].
The set of elliptic quaternions with basic elements $1, i, j$ and k is represented as

$$\mathbb{H}_\alpha = \{a = a_0 + a_1 i + a_2 j + a_3 k :\ a_0,\ a_1,\ a_2,\ a_3 \in \mathbb{R} \text{ and } i, j, k \notin \mathbb{R}\} \tag{1.3}$$

which satisfy the equalities $i^2 = k^2 = \alpha,\ j^2 = 1,\ ij = ji = k,\ jk = kj = i,\ ki = ik = \alpha j,\ \alpha < 0,\ \alpha \in \mathbb{R}$, [8]. Addition of any two elliptic quaternions $a = a_0 + a_1 i + a_2 j + a_3 k,\ b = b_0 + b_1 i + b_2 j + b_3 k \in \mathbb{H}_\alpha$ is given by $a + b = (a_0 + b_0) + (a_1 + b_1)\, i + (a_2 + b_2)\, j + (a_3 + b_3)\, k$. Scalar multiplication of a elliptic quaternion $a \in \mathbb{H}_\alpha$ with a scalar $\lambda \in \mathbb{R}$ is expressed as $\lambda a = \lambda\, (a_0 + a_1 i + a_2 j + a_3 k) = \lambda a_0 + \lambda a_1 i + \lambda a_2 j + \lambda a_3 k$. In addition, the operation of the quaternionic multiplication of two elliptic quaternions $a, b \in \mathbb{H}_\alpha$ is expressed as

$$\begin{aligned} ab &= (a_0 b_0 + \alpha a_1 b_1 + a_2 b_2 + \alpha a_3 b_3) + (a_1 b_0 + a_0 b_1 + a_3 b_2 + a_2 b_3)\, i \\ &\quad + (a_0 b_2 + a_2 b_0 + \alpha a_1 b_3 + \alpha a_3 b_1)\, j + (a_3 b_0 + a_0 b_3 + a_1 b_2 + a_2 b_1)\, k. \end{aligned} \tag{1.4}$$

On the other hand, we know that the elliptic quaternion $a \in \mathbb{H}_\alpha$ has three types of the conjugate: ${}^1\overline{a} = a_0 - a_1 i + a_2 j - a_3 k,\ {}^2\overline{a} = a_0 - a_1 i - a_2 j + a_3 k$ and ${}^3\overline{a} = a_0 + a_1 i - a_2 j - a_3 k$. Additionally, the norm of the elliptic quaternion $a \in \mathbb{H}_\alpha$ is

$$\|a\| = \sqrt[4]{a\, ({}^1\overline{a})\, ({}^2\overline{a})\, ({}^3\overline{a})} = \sqrt[4]{\left[(a_0 + a_2)^2 - \alpha (a_1 + a_3)^2\right]\ \left[(a_0 - a_2)^2 - \alpha (a_1 - a_3)^2\right]}. \tag{1.5}$$

If $a \in \mathbb{H}_\alpha$ and $\|a\| \neq 0$ then there exists multiplicative inverse of the elliptic quaternion a. So, multiplicative inverse of the elliptic quaternion a is $a^{-1} = \frac{({}^1\overline{a})({}^2\overline{a})({}^3\overline{a})}{\|a\|^4}$ [8, 12].

For

$$\mathbb{H}'_\alpha = \left\{ \begin{pmatrix} a_0 & \alpha a_1 & a_2 & \alpha a_3 \\ a_1 & a_0 & a_3 & a_2 \\ a_2 & \alpha a_3 & a_0 & \alpha a_1 \\ a_3 & a_2 & a_1 & a_0 \end{pmatrix} \in \mathbb{R}^{4\times 4} : a_0,\ a_1,\ a_2, a_3 \in \mathbb{R} \right\}, \tag{1.6}$$

$\mathbb{H}_\alpha$ is algebraically isomorphic to the matrix algebra $\mathbb{H}'_\alpha$ through the bijective map

$$\phi : \mathbb{H}_\alpha \to \mathbb{H}'_\alpha,\ \ \phi_a = \begin{pmatrix} a_0 & \alpha a_1 & a_2 & \alpha a_3 \\ a_1 & a_0 & a_3 & a_2 \\ a_2 & \alpha a_3 & a_0 & \alpha a_1 \\ a_3 & a_2 & a_1 & a_0 \end{pmatrix}. \tag{1.7}$$

Thus, every elliptic quaternion $a \in \mathbb{H}_\alpha$ has a real matrix representation

$$\phi_a = \begin{pmatrix} a_0 & \alpha a_1 & a_2 & \alpha a_3 \\ a_1 & a_0 & a_3 & a_2 \\ a_2 & \alpha a_3 & a_0 & \alpha a_1 \\ a_3 & a_2 & a_1 & a_0 \end{pmatrix} \tag{1.8}$$

in $\mathbb{H}'_\alpha$ [8].

Theorem 1.1. *([8, 12]). For $a, b \in \mathbb{H}_\alpha$ and $\lambda \in \mathbb{R}$, the following identities are satisfied:*

1. $a = b \Leftrightarrow \phi_a = \phi_b$,
2. $\phi_{(a+b)} = \phi_a + \phi_b$,
3. $\phi_{(ab)} = \phi_a \phi_b$,
4. $\phi_{(\phi_{(a)} b)} = \phi_a \phi_b$,
5. $\phi_{(\lambda a)} = \lambda \phi_a$,
6. $trace\,(\phi_a) = a + {}^1\overline{a} + {}^2\overline{a} + {}^3\overline{a}$,
7. $\|a\|^4 = |\det(\phi_a)|$.

Let's denote by $\mathbb{H}_{\alpha}^{m\times n}$ which is the set of all $m\times n$ type matrices with elliptic quaternion entries. $\mathbb{H}_{\alpha}^{m\times n}$ with the ordinary matrix summation and multiplication is a ring with identity. The conjugates of elliptic quaternion matrix $A=(a_{ij})\in\mathbb{H}_{\alpha}^{m\times n}$ which has three types of conjugate are given the following as:

$$^{1}\overline{A}=\left(^{1}\overline{a_{ij}}\right)\in\mathbb{H}_{\alpha}^{m\times n},\ ^{2}\overline{A}=\left(^{2}\overline{a_{ij}}\right)\in\mathbb{H}_{\alpha}^{m\times n}\ and\ ^{3}\overline{A}=\left(^{3}\overline{a_{ij}}\right)\in\mathbb{H}_{\alpha}^{m\times n}.$$

Also, elliptic quaternion matrix $A=(a_{ij})\in\mathbb{H}_{\alpha}^{m\times n}$ can be expressed as $A=A_0+A_1i+A_2j+A_3k$ where $A_0,\ A_1,\ A_2,\ A_3\in\mathbb{R}^{m\times n}$. Then, $^{1}\overline{A}=A_0-A_1i+A_2j-A_3k$, $^{2}\overline{A}=A_0-A_1i-A_2j+A_3k$ and $^{3}\overline{A}=A_0+A_1i-A_2j-A_3k$. A matrix $A^T\in\mathbb{H}_{\alpha}^{n\times m}$ is transpose of $A\in\mathbb{H}_{\alpha}^{m\times n}$. Also $A^{*_s}=\left(^{s}\overline{A}\right)^T\in\mathbb{H}_{\alpha}^{m\times n},\ s=1,2,3,$ is called conjugate transpose with respect to the s^{th} conjugate of $A\in\mathbb{H}_{\alpha}^{m\times n}$, [12].

Theorem 1.2. *([12]) Let's assume that A and B are elliptic quaternion matrices of appropriate sizes. Then the following expressions are provided:*

1. $\left(^{s}\overline{A}\right)^T={}^{s}\overline{\left(A^T\right)}$,
2. $(AB)^{*_s}=B^{*_s}A^{*_s}$,
3. $(AB)^T=B^TA^T$,
4. ${}^{s}\overline{(AB)}=\left(^{s}\overline{A}\right)\left(^{s}\overline{B}\right)$,
5. *If* A^{-1} *and* B^{-1} *exist then* $(AB)^{-1}=B^{-1}A^{-1}$,
6. *If* A^{-1} *exists* $\left(A^{*_s}\right)^{-1}=\left(A^{-1}\right)^{*_s}$,
7. $\left(^{s}\overline{A}\right)^{-1}={}^{s}\overline{\left(A^{-1}\right)}$.

For any elliptic quaternion matrix $A=A_0+A_1i+A_2j+A_3k\in\mathbb{H}_{\alpha}^{m\times n}$, the real representation Φ_A of the elliptic quaternion matrix A were given in [13] as follows,

$$\Phi_A=\begin{pmatrix} A_0 & \alpha A_1 & A_2 & \alpha A_3\\ A_1 & A_0 & A_3 & A_2\\ A_2 & \alpha A_3 & A_0 & \alpha A_1\\ A_3 & A_2 & A_1 & A_0\end{pmatrix}\in\mathbb{R}^{4m\times 4n}$$

in here $A_0,A_1,A_2,A_3\in\mathbb{R}^{m\times n},\ \alpha\in\mathbb{R}$ and $\alpha<0$.

Theorem 1.3. *([13]) Let* $A,\ B\in\mathbb{H}^{m\times n},\ C\in\mathbb{H}_{\alpha}^{n\times p}$ *and* $\lambda\in\mathbb{R}$ *be given. In that case, following identities for the elliptic quaternion matrix are satisfied:*

1. $A=B\Leftrightarrow\Phi_A=\Phi_B,\ \Phi_{A+B}=\Phi_A+\Phi_B$,
2. $\Phi_{AC}=\Phi_A\Phi_C,\ \Phi_{\lambda A}=\lambda\Phi_A$,
3. $A=\frac{1}{2-2\alpha}E_{4m}\Phi_A\left(^{1}\overline{E}_{4n}\right)^T$ *where* $E_{4t}=\begin{pmatrix}I_t & iI_t & jI_t & kI_t\end{pmatrix}\in\mathbb{H}^{t\times 4t}$,
4. *If A is a nonsingular matrix of size m, then*

$$\Phi_{A^{-1}}=\Phi_A^{-1},\quad A^{-1}=\frac{1}{2-2\alpha}E_{4m}\Phi_A^{-1}\left(^{1}\overline{E}_{4n}\right)^T,$$

5. $\Phi_{A^-}=\Phi_A^-,\ \ A^-=\frac{1}{2-2\alpha}E_{4m}\Phi_A^-\left(^{1}\overline{E}_{4n}\right)^T$ *are generalized inverse of* Φ_A *and* A, *respectively,*
6. $\Phi_A=R_{4m}^{-1}\Phi_AR_{4n},\ \Phi_A=S_{4m}^{-1}\Phi_AS_{4n}$ *and* $\Phi_A=T_{4m}^{-1}\Phi_AT_{4n}$ *where*

$$R_{4t}=\begin{pmatrix}0 & \alpha I_t & 0 & 0\\ I_t & 0 & 0 & 0\\ 0 & 0 & 0 & \alpha I_t\\ 0 & 0 & I_t & 0\end{pmatrix},\ S_{4t}=\begin{pmatrix}0 & 0 & I_t & 0\\ 0 & 0 & 0 & I_t\\ I_t & 0 & 0 & 0\\ 0 & I_t & 0 & 0\end{pmatrix},\ T_{4t}=\begin{pmatrix}0 & 0 & 0 & \alpha I_t\\ 0 & 0 & I_t & 0\\ 0 & \alpha I_t & 0 & 0\\ I_t & 0 & 0 & 0\end{pmatrix}.$$

2. On solutions of general linear elliptic quaternionic matrix equations

In this section, we study the solutions of the equations

$$A_1XB_1+\cdots+A_lXB_l=C \tag{2.1}$$

by means of the real representations of elliptic quaternion matrices, where $A_s\in\mathbb{H}_{\alpha}^{m\times n}$, $B_s\in\mathbb{H}_{\alpha}^{p\times q}$, $C\in\mathbb{H}_{\alpha}^{m\times q}$ and $s=1,2,3,\ldots,l$.

Theorem 2.1. *The elliptic quaternionic matrix equation given by (2.1) has a solution X if and only if the real matrix equation*

$$\Phi_{A_1} Y \Phi_{B_1} + \cdots + \Phi_{A_l} Y \Phi_{B_l} = \Phi_C \tag{2.2}$$

has a solution $Y \in \mathbb{R}^{4n\times 4p}$, in which case, if $Y \in \mathbb{R}^{4n\times 4p}$ is a solution of the real matrix equation (2.2), then the matrix

$$X = \frac{1}{2-2\alpha} E_{4n} Y' \left({}^{1}\overline{E}_{4p}\right)^T \tag{2.3}$$

is a solution of (2.1) where

$$Y' = \frac{1}{4}\left(Y + R_{4n} Y R_{4p}^{-1} + S_{4n} Y S_{4p}^{-1} + T_{4n} Y T_{4p}^{-1}\right) \tag{2.4}$$

and

$$E_{4t} = \begin{pmatrix} I_t & iI_t & jI_t & kI_t \end{pmatrix} \in \mathbb{H}_\alpha^{t\times 4t},\ t = n, p.$$

Proof. Suppose that the real matrix

$$Y = \begin{pmatrix} Y_{11} & Y_{12} & Y_{13} & Y_{14} \\ Y_{21} & Y_{22} & Y_{23} & Y_{24} \\ Y_{31} & Y_{32} & Y_{33} & Y_{34} \\ Y_{41} & Y_{42} & Y_{43} & Y_{44} \end{pmatrix},\ Y_{uv} \in \mathbb{R}^{n\times p},\ u,v = 1,2,3,4 \tag{2.5}$$

is a solution to the equation (2.2), then, we say that the matrix given in (2.3) is a solution to equation (2.1). According to Theorem 1.3, we get

$$\Phi_{A_s} = R_{4m}^{-1}\Phi_{A_s}R_{4n},\ \ \Phi_{B_s} = R_{4p}^{-1}\Phi_{B_s}R_{4q} \text{ and } \ \Phi_C = R_{4m}^{-1}\Phi_C R_{4q},$$

$$\Phi_{A_s} = S_{4m}^{-1}\Phi_{A_s}S_{4n},\ \ \Phi_{B_s} = S_{4p}^{-1}\Phi_{B_s}S_{4q} \text{ and } \ \Phi_C = S_{4m}^{-1}\Phi_C S_{4q},$$

$$\Phi_{A_s} = T_{4m}^{-1}\Phi_{A_s}T_{4n},\ \ \Phi_{B_s} = T_{4p}^{-1}\Phi_{B_s}T_{4q} \text{ and } \ \Phi_C = T_{4m}^{-1}\Phi_C T_{4q}.$$

where $s = 1,2,3,\ldots,l$. Substituting them into (2.2), respectively, and simplifying the corresponding equation, we have three equations as follows,

$$\begin{aligned} &\Phi_{A_1}\left(R_{4n}YR_{4p}^{-1}\right)\Phi_{B_1} + \cdots + \Phi_{A_l}\left(R_{4n}YR_{4p}^{-1}\right)\Phi_{B_l} = (\Phi_C), \\ &\Phi_{A_1}\left(S_{4n}YS_{4p}^{-1}\right)\Phi_{B_1} + \cdots + \Phi_{A_l}\left(S_{4n}YS_{4p}^{-1}\right)\Phi_{B_l} = (\Phi_C), \\ &\Phi_{A_1}\left(T_{4n}YT_{4p}^{-1}\right)\Phi_{B_1} + \cdots + \Phi_{A_l}\left(T_{4n}YT_{4p}^{-1}\right)\Phi_{B_l} = (\Phi_C). \end{aligned} \tag{2.6}$$

This equation express that if Y is a solution of the equation given by (2.2), then $R_{4n}YR_{4p}^{-1}$, $S_{4n}YS_{4p}^{-1}$ and $T_{4n}YT_{4p}^{-1}$ are also solutions of the real matrix equation defined by (2.2). Thus the undermentioned real matrix:

$$Y' = \frac{1}{4}\left(Y + R_{4n} Y R_{4p}^{-1} + S_{4n} Y S_{4p}^{-1} + T_{4n} Y T_{4p}^{-1}\right) \tag{2.7}$$

is a solution to (2.2). By substituting (2.5) in (2.7) and making necessary simplifications, it can easily be written by

$$Y' = \begin{pmatrix} Z_0 & \alpha Z_1 & Z_2 & \alpha Z_3 \\ Z_1 & Z_0 & Z_3 & Z_2 \\ Z_2 & \alpha Z_3 & Z_0 & \alpha Z_1 \\ Z_3 & Z_2 & Z_1 & Z_0 \end{pmatrix}$$

where

$$Z_0 = \frac{1}{4}\left(Y_{11}+Y_{22}+Y_{33}+Y_{44}\right),\ Z_1 = \frac{1}{4}\left(\frac{Y_{12}}{\alpha}+Y_{21}+\frac{Y_{34}}{\alpha}+Y_{43}\right),$$
$$Z_2 = \frac{1}{4}\left(Y_{13}+Y_{24}+Y_{31}+Y_{42}\right),\ Z_3 = \frac{1}{4}\left(\frac{Y_{14}}{\alpha}+Y_{23}+\frac{Y_{32}}{\alpha}+Y_{41}\right). \tag{2.8}$$

Thus, we get $\Phi_X = Y'$. From Theorem 1.3, we obtain

$$X = \frac{1}{2-2\alpha}\left(I_n\ iI_n\ jI_n\ kI_n\right)Y'\begin{pmatrix} I_p \\ -iI_p \\ jI_p \\ -kI_p \end{pmatrix} = Z_0+Z_1 i+Z_2 j+Z_3 k. \tag{2.9}$$

Moreover, since $\Phi_X = Y'$ the elliptic quaternionic matrix equation given in (2.1) has a solution if and only if the real matrix equation given in (2.2) has a solution.

□

3. Numerical algorithm

Considering the discussions in the previous section, now, we provide numerical Algorithm for solving general linear elliptic quaternionic matrix equation

$$A_1XB_1+\cdots+A_lXB_l = C$$

where $A_s \in \mathbb{H}_\alpha^{m\times n}$, $B_s \in \mathbb{H}_\alpha^{p\times q}$, $C \in \mathbb{H}_\alpha^{m\times q}$ and $s = 1,2,3,\ldots,l$.

Algorithm 1 Numerical Algorithm for Solving General Linear Elliptic Quaternionic Matrix Equations

1: **Begin**
2: Input $A_s \in \mathbb{H}_\alpha^{m\times n}$, $B_s \in \mathbb{H}_\alpha^{p\times q}$ and $C \in \mathbb{H}_\alpha^{m\times q}$ where $1 \le s \le l$.
3: Form Φ_{A_s}, Φ_{B_s} and Φ_C.
4: Compute Y and $Y' = \frac{1}{4}\left(Y+R_{4n}YR_{4p}^{-1}+S_{4n}YS_{4p}^{-1}+T_{4n}YT_{4p}^{-1}\right)$.
5: Calculate $X = \frac{1}{2-2\alpha}\left(I_n\ \ iI_n\ \ jI_n\ \ kI_n\right)Y'\begin{pmatrix} I_p \\ -iI_p \\ jI_p \\ -kI_p \end{pmatrix}$.
6: **End**

4. Numerical examples

For $l = 2$, the special case of (2.1) is given by

$$A_1XB_1+A_2XB_2 = C \tag{4.1}$$

where $A_1, A_2 \in \mathbb{H}_\alpha^{m\times n}$, $B_1, B_2 \in \mathbb{H}_\alpha^{p\times q}$ and $C \in \mathbb{H}_\alpha^{m\times q}$. If $B_1 = I_p$, $A_2 = -I_n$, $m = n$, $p = q$ are taken in (4.1), we have elliptic quaternionic Sylvester matrix equation $AX - XB = C$. Similarly, $A_1 = I_n$, $B_1 = I_p$, $m = n$, $p = q$, $A_2 = -A$ and $B_2 = B$ are taken in (4.1) we have elliptic quaternionic Kalman-Yakubovich matrix equation $X - AXB = C$.

In the literature, the equations $AX - XB = C$ and $X - AXB = C$ are known as the Sylvester matrix equation and the Kalman-Yakubovich matrix equation, respectively. These equations play an important role in control theory, signal processing, filtering, image restoration, decoupling techniques for ordinary and partial differential equations, and block-diagonalization of matrices, [14]-[18]. In this section, we obtain the solutions of the given elliptic quaternionic matrix equations $AX - XB = C$ and $X - AXB = C$ according to our Algorithm.

Note that all computations in the rest of the paper are performed on an Intel i7-3630QM@2.40 GHz/16GB computer using MATHEMATICA 9 software.

Let's take $\alpha = -2$ specifically to solve the elliptic quaternionic Kalman Yakubovich matrix equation

$$X - \begin{pmatrix} 1+k & i \\ j-k & 1-j \end{pmatrix} X \begin{pmatrix} j & 1+2i \\ k & i+j \end{pmatrix} = \begin{pmatrix} 3+i+3j+k & 2+2i+7j+k \\ 5+2i-6j+k & -7-2i-2j+8k \end{pmatrix}.$$

Real representation of given equation is

$$Y-\begin{pmatrix} 1 & 0 & 0 & -2 & 0 & 0 & -2 & 0 \\ 0 & 1 & 0 & 0 & 1 & -1 & 2 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 1 & -1 \\ 0 & 0 & -2 & 0 & 1 & 0 & 0 & -2 \\ 1 & -1 & 2 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ -1 & 0 & 1 & -1 & 0 & 0 & 0 & 1 \end{pmatrix} Y \begin{pmatrix} 0 & 1 & 0 & -4 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & 1 & -2 & 0 \\ 0 & 2 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & -4 \\ 0 & 1 & -2 & 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & 0 & 2 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 3 & 4 & -2 & 6 & 1 & 7 & 0 & -2 \\ 7 & -12 & 0 & 6 & -7 & 3 & -4 & -18 \\ 1 & -3 & 3 & 4 & 0 & 1 & 1 & 7 \\ 0 & -3 & 7 & -12 & 2 & 9 & -7 & 3 \\ 1 & 7 & 0 & -2 & 3 & 4 & -2 & 6 \\ -7 & 3 & -4 & -18 & 7 & -12 & 0 & 6 \\ 0 & 1 & 1 & 7 & 1 & -3 & 3 & 4 \\ 2 & 9 & -7 & 3 & 0 & -3 & 7 & -12 \end{pmatrix}.$$

If we solve this equation, we have

$$Y = \begin{pmatrix} 1 & 0 & 2 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & -2 & -4 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 2 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & -2 & -4 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 1 & 0 \\ 1 & 2 & 0 & 1 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Then

$$X = \frac{1}{24}\begin{pmatrix} I_2 & iI_2 & jI_2 & kI_2 \end{pmatrix}\left(Y + R_8 Y R_8^{-1} + S_8 Y S_8^{-1} + T_8 Y T_8^{-1}\right)\begin{pmatrix} I_2 \\ -iI_2 \\ jI_2 \\ -kI_2 \end{pmatrix}$$
$$= \begin{pmatrix} 1-i & j \\ 1+k & j+2k \end{pmatrix}.$$

Similarly, let's take $\alpha = -5$ specifically to solve the elliptic quaternionic Sylvester matrix equation

$$\begin{pmatrix} 1+i & i+3j+2k \\ 3k & 2 \end{pmatrix} X - X \begin{pmatrix} i & j+2k \\ 5+i & 2-3j \end{pmatrix} = \begin{pmatrix} -46+13i-19j+k & -19+6i-35j+15k \\ 25-22i-8j+7k & 48-6i+21k \end{pmatrix}.$$

The solution of real representation of given elliptic quaternionic Sylvester matrix equation is

$$Y = \begin{pmatrix} 1 & 2 & -5 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -25 & 1 & 3 & -20 & 0 \\ 1 & 0 & 1 & 2 & 0 & 0 & 1 & 0 \\ 0 & 5 & 0 & 0 & 4 & 0 & 1 & 3 \\ 1 & 0 & 0 & 0 & 1 & 2 & -5 & 0 \\ 1 & 3 & -20 & 0 & 0 & 0 & 0 & -25 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 2 \\ 4 & 0 & 1 & 3 & 0 & 5 & 0 & 0 \end{pmatrix}.$$

Thus, we get

$$X = \frac{1}{24}\begin{pmatrix} I_2 & iI_2 & jI_2 & kI_2 \end{pmatrix}\left(Y + R_8 Y R_8^{-1} + S_8 Y S_8^{-1} + T_8 Y T_8^{-1}\right)\begin{pmatrix} I_2 \\ -iI_2 \\ jI_2 \\ -kI_2 \end{pmatrix}$$
$$= \begin{pmatrix} 1+i+j & 2 \\ j+4k & 5i+3j \end{pmatrix}.$$

5. Conclusion

In this study, we established the solution of general linear elliptic quaternionic matrix equations with the help of the real representation of elliptic quaternion matrices and expressed an Algorithm for the solutions of these equations. In addition, we investigated solutions of elliptic quaternionic Sylvester and Kalman Yakubovich matrix equations, which are essential applications in various areas of science. Actually, general linear matrix equations over the complex field form a special class of general linear elliptic quaternionic matrix equations. Thus, the obtained results extend, generalize and complement the scope of general linear matrix equations known in the literature.

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Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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