Turkish **J**ournal of **M**athematics and **C**omputer **S**cience
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Investigation of Boundary Functionals for Renewal-Reward Process with a Generalized Reflecting Barrier

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Abstract. In this study, renewal - reward process with a generalized reflecting barrier ($X(t)$) and its three boundary functionals are mathematically constructed. Next, the asymptotic expansions are obtained for the first four moments of these boundary functionals of the process $X(t)$.

2010 AMS Classification: Primary: 60K15, Secondary: 60K05, 60K20.

Keywords: Renewal - reward process, Boundary functional, Asymptotic expansion, Reflecting barrier.

1. INTRODUCTION

A number of very interesting problems of queuing, reliability, stock control theory, stochastic finance, mathematical insurance, physics and biology are expressed by means of renewal - reward processes (e.g., Feller [7], Borovkov [3], Brown and Solomon [4], Patch [12], Khaniyev [8], Aliyev et. al. [1]). Moreover, many modifications of renewal - reward processes can be used for solutions of some problems in these fields, as well. These modifications are mostly given with various types of the barriers (e.g., absorbing, delaying, reflecting and elastic barriers) or a discrete interference of chance.

However, some interesting problems of physics (e.g., motion of the particle with high energy in a diluted environment) are expressed by means of the processes with reflecting barrier. In the literature, there are a number of interesting studies on stochastic processes with reflecting barriers (e.g., Feller [7], Borovkov [3], Khaniyev et. al. [9]). In these studies, authors generally have obtained the analytic results. Unfortunately, these results consist of highly complex mathematical structures.

Recently, for avoiding this difficulty, an asymptotic approach method has been begun to use. For this reason, in this study, the asymptotic expansions for the moments of the boundary functionals of the process are obtained. So as to give this results, we need to make a mathematical definition of renewal - reward process with a generalized reflecting barrier.

2. Mathematical Construction of the Process $X(t)$

Let $\{(\xi_n, \eta_n)\}, n = 1, 2, 3, ...,$ be a sequence of independent and identically distributed random variables defined on a same probability space $(\Omega, \mathcal{F}, P)$, such that the random variables ξ_n and η_n are also mutually independent and take only positive values. Suppose that the distribution functions of ξ_n and η_n are given and these are denoted by $\Phi(t)$ and $F(x)$, respectively, i.e.,

$$\Phi(t) = P\{\xi_n \le t\}, \quad F(x) = P\{\eta_n \le x\}; \quad t \ge 0 \quad x \ge 0, \quad n = 1, 2, ...$$

Here, $F(x)$ is a continuous distribution function such that $F(0) = 0$.

Define the renewal sequences $\{T_n\}$ and $\{S_n\}$ as follows:

$$T_0 = S_0 = 0, \quad T_n = \sum_{i=1}^{n} \xi_i, \quad S_n = \sum_{i=1}^{n} \eta_i, \quad n = 1, 2, ...,$$

and construct sequences of random variables $\{N_n\}$ and $\{\zeta_n\}$, $n = 0, 1, 2, ...$ as follows:

$$\begin{aligned} N_0 &= 0; \quad \zeta_0 = z \ge 0; \quad N_1 \equiv N_1(\lambda z) = \inf\{k \ge 1 : \lambda z - S_k < 0\}; \\ \zeta_1 &\equiv \zeta_1(\lambda z) = |\lambda z - S_{N_1}|; \\ N_n &\equiv N_n(\lambda\zeta_{n-1}) = \inf\left\{k \ge N_{n-1} + 1 : \lambda\zeta_{n-1} - \left(S_k - S_{N_{n-1}}\right) < 0\right\}; \\ \zeta_n &\equiv \zeta_n(\lambda\zeta_{n-1}) = \left|\lambda\zeta_{n-1} - \left(S_{N_n} - S_{N_{n-1}}\right)\right|, \quad n = 1, 2, 3, ... \end{aligned}$$

Here, $\lambda > 0$ is an arbitrary positive constant.

Using $\{N_n, \ n = 0, 1, 2, ...\}$, define the following sequence $\{\tau_n, \ n = 0, 1, 2, ...\}$:

$$\tau_0 \equiv 0; \ \ \tau_1 \equiv \tau_1(\lambda z) = \sum_{i=1}^{N_1} \xi_i; \ \ \tau_2 = \sum_{i=1}^{N_2} \xi_i; \ \ ... \ ; \ \ \tau_n = \sum_{i=1}^{N_n} \xi_i, \ \ n = 1, 2, ...$$

Moreover, let $\nu(t) = \max\{n \ge 1 : T_n \le t\}, t > 0,$.

We can now construct the desired stochastic process $X(t)$, as follows:

$$X(t) = \lambda\zeta_n - \left(S_{\nu(t)} - S_{N_n}\right), \ \ \tau_n \le t < \tau_{n+1}, \ \ n = 0, 1, 2...$$

The process $X(t)$ can be also rewritten as follows:

$$X(t) = \sum_{n=0}^{\infty} \left\{\lambda\zeta_n - \left(S_{\nu(t)} - S_{N_n}\right)\right\} I_{[\tau_n;\tau_{n+1})}(t).$$

Here $I_A(t)$ represents the indicator function of the set A, such that

$$I_A(t) = \begin{cases} 1, & t \in A \\ 0, & t \notin A \end{cases}$$

A trajectory of the process $X(t)$ is given as in Figure 1.

The process $X(t)$ is called renewal - reward process with a generalized reflecting barrier. In the case that $\lambda = 1$, the process $X(t)$ is known as renewal - reward process with reflecting barrier.

The main aim of this study is to investigate the asymptotic behaviours of some boundary functionals of the process $X(t)$, when $\lambda \to \infty$.

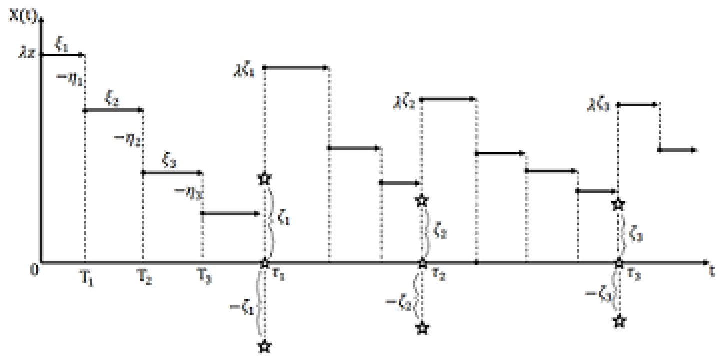

FIGURE 1. A trajectory of the process $X(t)$

3. Boundary Functionals of $X(t)$

Three important boundary functionals $N_1\left(\lambda\hat{\zeta}_\lambda\right)$, $\tau_1\left(\lambda\hat{\zeta}_\lambda\right)$, $S_{N_1\left(\lambda\hat{\zeta}_\lambda\right)}$ have been defined in Section 2, will be investigated in this section. Here, $\hat{\zeta}_\lambda$ is a random variable, the distribution function of which is $\pi_\lambda(z) \equiv \lim_{n\to\infty} P\left\{\zeta_n \leq z\right\}$. By definition, $\pi_\lambda(z)$ is the ergodic distribution of the Markov chain $\{\zeta_n\}$ which are the reflections of the process $X(t)$. The exact expressions and asymptotic expansions of them are obtained. Before giving these results, there need to be stated some propositions. First of all, let us give them.

3.1. **Proposition.** *(Aliyev et. al.[1]) Assume that the condition $\alpha_4 \equiv E(\xi_1^4) < \infty$ is satisfied. Then, the first four moments of the boundary functional $\tau_1(z)$ can be expressed by means of the boundary functionals $N_1(z)$ as follows:*

$$
\begin{aligned}
&1)\quad E\left(\tau_1(z)\right) &&= \alpha_1 E\left(N_1(z)\right);\\
&2)\quad E\left(\tau_1^2(z)\right) &&= \alpha_1^2 E\left(N_1^2(z)\right) + \left(\alpha_2-\alpha_1^2\right) E\left(N_1(z)\right);\\
&3)\quad E\left(\tau_1^3(z)\right) &&= \alpha_1^3 E\left(N_1^3(z)\right) + 3\alpha_1\left(\alpha_2-\alpha_1^2\right) E\left(N_1^2(z)\right)\\
& &&+\ \left(2\alpha_1^3-3\alpha_1^2\alpha_2+\alpha_3\right) E\left(N_1(z)\right);\\
&4)\quad E\left(\tau_1^4(z)\right) &&= \alpha_1^4 E\left(N_1^4(z)\right) + 6\alpha_1^2\left(\alpha_2-\alpha_1^2\right) E\left(N_1^3(z)\right)\\
& &&+\ \left(11\alpha_1^4-18\alpha_1^2\alpha_2+4\alpha_1\alpha_3+3\alpha_2^2\right) E\left(N_1^2(z)\right)\\
& &&+\ \left(\alpha_4+12\alpha_1^2\alpha_2-4\alpha_1\alpha_3-3\alpha_2^2-6\alpha_1^4\right) E\left(N_1^4(z)\right);
\end{aligned}
$$

Here $\alpha_k \equiv E\left(\xi_1^k\right)$, $k = 1,2,3,4$.

The boundary functional $N_1(z)$ is a renewal process. The asymptotic expansions for the moments of $N_1(z)$ exist in the literature [1]. It can be given as follows:

3.2. **Proposition.** *(Aliyev et. al. [1]) Suppose that $m_2 \equiv E\left(\eta_1^2\right) < \infty$. Then, the following two-term asymptotic expansions can be written for the first four moments* *of the boundary functional $N_1(z)$, when $z \to \infty$:*

$$E\left(N_1^n(z)\right) = \left(\frac{z}{m_1}\right)^n + \left[n^2 C_F - \frac{n(n-1)}{2}\right]\left(\frac{z}{m_1}\right)^{n-1} + o\left(z^{n-1}\right).$$

Here, $C_F = m_2/\left(2m_1^2\right)$ is coefficient of Feller and $m_k \equiv E\left(\eta_1^k\right)$, $k = 1, 2, 3, 4$.

The first aim of this study is to investigate the boundary functional $N_1\left(\lambda\hat{\zeta}_\lambda\right)$, when $\lambda \to \infty$. Before investigating the boundary functional $N_1\left(\lambda\hat{\zeta}_\lambda\right)$, let us give the following lemmas.

3.1. **Lemma.** *$g(x) : R^+ \to R$ is a measurable and bounded function and $\lim_{x\to\infty} g(x) = 0$. Then, the following relation can be written:*

$$\lim_{\lambda\to\infty} \int_0^\infty g(\lambda z) d\pi_\lambda(z) = 0.$$

Proof. Since $\lim_{x\to\infty} g(x) = 0$, for every $\varepsilon > 0$, there exist such a $x^*(\varepsilon) < \infty$ and for every $x \geq x^*(\varepsilon)$ the following inequality is hold:

$$|g(x)| \leq \frac{\varepsilon}{2}. \tag{3.1}$$

For instance, $x^*(\varepsilon)$ can be defined as follows:

$$x^*(\varepsilon) \equiv x^* = \sup\left\{x > 0 : |g(x)| > \frac{\varepsilon}{2}\right\}. \tag{3.2}$$

Then, for every $x \geq x^*$, $|g(x)| \leq \varepsilon/2$ will be hold. Let $\delta > 0$ be an arbitrary positive constant, for now. Then, $\int_0^\infty g(\lambda z) d\pi_\lambda(z)$ can be shown as follows:

$$\int_0^\infty g(\lambda z) d\pi_\lambda(z) = \int_0^\delta g(\lambda z) d\pi_\lambda(z) + \int_\delta^\infty g(\lambda z) d\pi_\lambda(z) \tag{3.3}$$

Now, define $\lambda_*(\varepsilon, \delta)$ as follows:

$$\lambda_*(\varepsilon, \delta) = \frac{x^*(\varepsilon)}{\delta}. \tag{3.4}$$

For every $\lambda \geq \lambda_*(\varepsilon, \delta)$, it will be $\lambda\delta \geq x^*(\varepsilon)$. Therefore, after computing, for each $\lambda \geq \lambda_*(\varepsilon, \delta)$, the following inequality is hold:

$$\begin{aligned} \left|\int_\delta^\infty g(\lambda z) d\pi_\lambda(z)\right| &\leq \int_\delta^\infty |g(\lambda z)|\, d\pi_\lambda(z) \leq \int_\delta^\infty \frac{\varepsilon}{2} d\pi_\lambda(z) \\ &= \frac{\varepsilon}{2}\int_\delta^\infty d\pi_\lambda(z) \leq \frac{\varepsilon}{2}\left(1 - \pi_\lambda(\delta)\right) \leq \frac{\varepsilon}{2} \end{aligned} \tag{3.5}$$

Moreover, according to conditions of Lemma 3.1, $g(x)$ is a bounded function and $\sup_{x\in R} |g(x)| \leq M < \infty$. Hence, for each $z \geq 0$, $|g(\lambda z)| \leq M$ is satisfied. Then, the following inequality can be written:

$$\left|\int_0^\delta g(\lambda z) d\pi_\lambda(z)\right| \leq \int_0^\delta |g(\lambda z)|\, d\pi_\lambda(z) \leq M\pi_\lambda(\delta). \tag{3.6}$$

Here, $\pi_\lambda(x)$ is a continuous distribution and $\pi_\lambda(0) = 0$. For this reason, there exist such a $\delta > 0$, so that

$$\pi_\lambda(\delta) \leq \frac{\varepsilon}{2M} \tag{3.7}$$

is satisfied. The greatest δ which is satisfied the Eq.(3.7), denote with δ^*. In other words, $\delta^* \equiv \delta^*(\varepsilon)$ can be define as follows:

$$\delta^* \equiv \delta^*(\varepsilon) = \sup\left\{\delta > 0 : \pi_\lambda(\delta) \leq \frac{\varepsilon}{2M}\right\} > 0.$$

Since $\pi_\lambda(z)$ is a non-decreasing monotone function with respect to parameter z, for every $\delta \leq \delta^*(\varepsilon)$, it will be $\pi_\lambda(\delta) \leq \varepsilon/(2M)$. In this case, it is followed that

$$\left|\int_0^\delta g(\lambda z) d\pi_\lambda(z)\right| \leq M\pi_\lambda(z) \leq M\frac{\varepsilon}{2M} = \frac{\varepsilon}{2}. \tag{3.8}$$

Thus, substituting Eq.(3.5) and Eq.(3.6) into Eq.(3.3), the following inequality is obtained:

$$\begin{aligned} \left|\int_0^{\delta^*(\varepsilon)} g(\lambda z) d\pi_\lambda(z)\right| &\leq \left|\int_0^{\delta^*(\varepsilon)} g(\lambda z) d\pi_\lambda(z)\right| + \left|\int_{\delta^*(\varepsilon)}^\infty g(\lambda z) d\pi_\lambda(z)\right| \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Therefore, it is concluded that

$$\lim_{\lambda\to\infty} \int_0^\infty g(\lambda z) d\pi_\lambda(z) = 0. \tag{3.9}$$

Thus, Lemma 3.1 is proved. □

3.2. **Lemma.** *Suppose that* $\lim_{x\to\infty} g(x) = 0$ *and* $\sup_x |g(x)| \leq M < \infty$. *Moreover,* $m_{n+1} \equiv E\left(\eta_1^{n+1}\right) < \infty$ *is satisfied. Then, the following relation is hold, when* $\lambda \to \infty$*:*

$$\lim_{\lambda\to\infty} \int_0^\infty z^n g(\lambda z) d\pi_\lambda(z) = 0, \quad n = 1, 2, ... \tag{3.10}$$

Proof. The integral in the relation (3.10) can be rewritten as follows:

$$\int_0^\infty z^n g(\lambda z) d\pi_\lambda(z) = \int_0^\delta z^n g(\lambda z) d\pi_\lambda(z) + \int_\delta^\infty z^n g(\lambda z) d\pi_\lambda(z). \tag{3.11}$$

Here, the constant $\delta > 0$ will be chosen by a special method which is going to be explained below. The aim is to satisfy $\lambda z \to \infty$, while investigating the asymptotic behavior of the integral $\int_\delta^\infty z^n g(\lambda z) d\pi_\lambda(z)$. For shortness, define the following notation:

$$J_{1n}(\delta) \equiv \int_0^\delta z^n g(\lambda z) d\pi_\lambda(z); \quad J_{2n}(\delta) \equiv \int_\delta^\infty z^n g(\lambda z) d\pi_\lambda(z).$$

The first step is to determine the δ. It is going to be determined according to the following rule for each $\varepsilon > 0$:

$$\delta^*(\varepsilon) \equiv \inf\left\{\delta > 0 : \pi_\lambda(\delta) \geq \frac{\varepsilon}{2M}\right\}. \tag{3.12}$$

Since the distribution of η_ns is continuous distribution, ζ_ns, which is the continuous function of them, will also be positive-valued and continuous random variables. Therefore, it should be $\pi_\lambda(0) = 0$. Then, $\delta^*(\varepsilon)$ can be derived from the following equation, directly:

$$\pi_\lambda\left(\delta^*(\varepsilon)\right) = \frac{\varepsilon}{2M}. \tag{3.13}$$

The solution of Eq.(3.13) exist and it is unique. Now, choose $\delta(\varepsilon)$ such as $\delta(\varepsilon) \equiv \min\{1, \delta^*(\varepsilon)\}$ and evaluate $J_{1n}\left(\delta(\varepsilon)\right)$:

$$\begin{aligned} \left|J_{1n}\left(\delta(\varepsilon)\right)\right| &\equiv \left|\int_0^{\delta(\varepsilon)} z^n g(\lambda z) d\pi_\lambda(z)\right| \leq \int_0^{\delta(\varepsilon)} z^n |g(\lambda z)| d\pi_\lambda(z) \\ &\leq M \int_0^{\delta(\varepsilon)} z^n d\pi_\lambda(z) \leq M\left[\delta(\varepsilon)\right]^n \pi_\lambda\left(\delta(\varepsilon)\right) \leq M\pi_\lambda\left(\delta(\varepsilon)\right) \\ &\leq M\pi_\lambda\left(\delta^*(\varepsilon)\right) \leq \frac{M\varepsilon}{2M} = \frac{\varepsilon}{2} \end{aligned} \tag{3.14}$$

Briefly, for each $\varepsilon > 0$, $\left|J_{1n}\left(\delta(\varepsilon)\right)\right| \leq \varepsilon/2$ is hold. Next, evaluate $J_{2n}\left(\delta(\varepsilon)\right)$:

$$\left|J_{2n}\left(\delta(\varepsilon)\right)\right| \leq \left|\int_{\delta(\varepsilon)}^{\infty} z^n g(\lambda z) d\pi_\lambda(z)\right| \leq \int_{\delta(\varepsilon)}^{\infty} z^n |g(\lambda z)| d\pi_\lambda(z) \tag{3.15}$$

According to definition of $\delta(\varepsilon)$, it is a positive number, i.e., $\delta(\varepsilon) \in (0, 1]$.In this case, when λ is chosen sufficiently large, then the condition $\lambda z \to \infty$ can be hold and $|g(\lambda z)| \leq \varepsilon/(2A)$ is true. Here, the positive constant A will be determined later. According to the conditions of Lemma 3.2, $\lim_{x\to\infty} g(x) = 0$. Hence, for every $\varepsilon > 0$ and $x \geq x^*$, there exist such a number x^* that can be written $|g(x)| \leq \varepsilon/(2A)$. The number $x^*(\varepsilon)$ can be determined as follows:

$$x^*(\varepsilon) \equiv \sup\left\{x > 0 : |g(x)| \geq \frac{\varepsilon}{2A}\right\}. \tag{3.16}$$

Then, for each $x \geq x^*(\varepsilon)$, $|g(x)| \leq \varepsilon/(2A)$. In this case, λ should be chosen as $\lambda z \leq x^* \equiv x^*(\varepsilon)$. For this, the smallest λ can be defined as follows:

$$\lambda_* \equiv \lambda_*(\varepsilon) = \frac{x^*(\varepsilon)}{\delta(\varepsilon)}. \tag{3.17}$$

For every $\lambda \geq \lambda_*$, since $\lambda z \geq \lambda\delta(\varepsilon) \geq \lambda^*(\varepsilon)\delta(\varepsilon) \geq x^*(\varepsilon)$ is hold, $g(\lambda z) \leq \varepsilon/(2A)$. Go back to Eq.(3.15), for every $\lambda \geq \lambda_*$, the following relation can be written:

$$\left|J_{2n}\left(\delta(\varepsilon)\right)\right| \leq \int_{\delta(\varepsilon)}^{\infty} z^n \frac{\varepsilon}{2A} d\pi_\lambda(z) \leq \frac{\varepsilon}{2A} \int_{\delta(\varepsilon)}^{\infty} z^n d\pi_\lambda(z) \leq \frac{\varepsilon}{2A} E\left(\hat{\zeta}_\lambda^n\right) \tag{3.18}$$

Here, $E\left(\hat{\zeta}_\lambda^n\right)$ is the n^{th} order moment of the ergodic distribution $\pi_\lambda(z)$. According to Rogozin [14], the following convergence is hold, when $\lambda \to \infty$:

$$\pi_\lambda(z) \to \pi_0(z) \equiv \frac{1}{m_1} \int_0^z \left(1 - F(v)\right) dv$$

Moreover, since $m_{n+1} < \infty$ is satisfied, according to convergence of moments (Feller [7], p. 251), the following convergence can be given, when $\lambda \to \infty$:

$$E\left(\hat{\zeta}_\lambda^n\right) \to \frac{m_{n+1}}{(n+1)m_1}. \tag{3.19}$$

Then, it can be possible to choose such a $\lambda_1 >> 1$ which satisfies the following expression:

$$E\left(\hat{\zeta}_{\lambda_1}^n\right) \leq \frac{2m_{n+1}}{(n+1)m_1} \equiv K < \infty \tag{3.20}$$

Now, choose $\lambda = \lambda^*(\varepsilon)$ which satisfies the following expression:

$$\lambda^*(\varepsilon) \equiv \max\left\{\lambda_*(\varepsilon), \lambda_1\right\}. \tag{3.21}$$

According to definition of $\lambda^*(\epsilon)$, for every $\lambda \geq \lambda^*(\varepsilon)$,
$E\left(\hat{\zeta}_\lambda^n\right) \leq 2m_{n+1}/\left[(n+1)m_1\right] \equiv K < \infty$ and

$$\left|J_{2n}\left(\delta(\varepsilon)\right)\right| \leq \frac{\varepsilon}{2K} E\left(\hat{\zeta}_\lambda^n\right) \leq \frac{\varepsilon}{2K}K = \frac{\varepsilon}{2}. \tag{3.22}$$

As a result, for every $\varepsilon > 0$, as long as $\lambda \geq \lambda^*(\varepsilon)$ is satisfied, $|J_{2n}(\delta(\varepsilon))| \leq \varepsilon/2$ is hold. Taking into account the Eq.(3.14) and Eq.(3.22), for every $\varepsilon > 0$ and $\lambda \geq \lambda^*(\varepsilon)$, the following inequality is satisfied:

$$\left|\int_0^\infty z^n g(\lambda z) d\pi_\lambda(z)\right| \leq \left|J_{1n}\left(\delta\left(\varepsilon\right)\right)\right| + \left|J_{2n}\left(\delta\left(\varepsilon\right)\right)\right| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \tag{3.23}$$

Thus, the following convergence is obtained, when $\lambda \to \infty$:

$$\int_0^\infty z^n g(\lambda z) d\pi_\lambda(z) \to \infty \tag{3.24}$$

Hence, Lemma 3.2 is proved. ☐

Using these results, it can be given the following theorem which expresses the asymptotic expansions for the first four moments of the boundary functional $N_1\left(\lambda\hat{\zeta}_\lambda\right)$.

3.1. **Theorem.** *Assume that the conditions of Proposition 3.2, Lemma 3.1 and Lemma 3.2 are satisfied. Then, the following asymptotic expansions for the moments of boundary functional $N_1\left(\lambda\hat{\zeta}_\lambda\right)$ can be given, when $\lambda \to \infty$:*

$$E\left(N_1^n\left(\lambda\hat{\zeta}_\lambda\right)\right) = \left(\frac{\lambda}{m_1}\right)^n A_n + \left(\frac{\lambda}{m_1}\right)^{n-1} B_n A_{n-1} + o\left(\lambda^{n-1}\right).$$

Here,

$$A_n \equiv \frac{m_{n+1}}{(n+1)m_1}; \quad B_n \equiv n^2 C_F - \frac{n(n-1)}{2}; \quad m_n \equiv E\left(\eta_1^n\right),\ n = 1, 2, 3, 4$$

and C_F is the coefficient of Feller, i.e., $C_F \equiv m_2/\left(2m_1^2\right)$.

Proof. In Theorem 3.1, denote the coefficient of the second term with B_n, $n = 1, 2, 3, 4$, as follows:

$$B_n \equiv n^2 C_F - \frac{n(n-1)}{2},\ n = 1, 2, 3, 4. \tag{3.25}$$

Then, the following expansion can be rewritten for $E\left(N_1^n(z)\right)$, $n=1,2,3,4$:

$$E\left(N_1^n(z)\right)=\left(\frac{z}{m_1}\right)^n+B_n\left(\frac{z}{m_1}\right)^{n-1}+\left(\frac{z}{m_1}\right)^{n-1}g_n(z). \tag{3.26}$$

Here, $g_n(z)$ is a bounded function and $\lim_{z\to\infty}g_n(z)=0$. Integrate the both side of the Eq.(3.26) with respect to $\pi_\lambda(z)$, then the following expression is obtained:

$$\begin{aligned}
E\left(N_1^n\left(\lambda\hat{\zeta}_\lambda\right)\right) &\equiv \int_0^\infty E\left(N_1^n\left(\lambda z\right)\right)d\pi_\lambda(z)\\
&= \int_0^\infty\left(\frac{\lambda z}{m_1}\right)^n d\pi_\lambda(z)+B_n\int_0^\infty\left(\frac{\lambda z}{m_1}\right)^{n-1}d\pi_\lambda(z)\\
&+ \int_0^\infty\left(\frac{\lambda z}{m_1}\right)^{n-1}g_n\left(\lambda z\right)d\pi_\lambda(z)\\
&= \left(\frac{\lambda z}{m_1}\right)^n E\left(\hat{\zeta}_\lambda^n\right)+B_n\left(\frac{\lambda z}{m_1}\right)^{n-1}E\left(\hat{\zeta}_\lambda^{n-1}\right)+\left(\frac{\lambda z}{m_1}\right)^{n-1}I_n(\lambda) \qquad (3.27)
\end{aligned}$$

Here, $E\left(\hat{\zeta}_\lambda^n\right)=\int_0^\infty z^{n-1}d\pi_\lambda(z)$, $n=1,2,...$ and $I_n(\lambda)\equiv\int_0^\infty z^{n-1}g_n(\lambda z)d\pi_\lambda(z)$. From Lemma 3.2, $I_n(\lambda)\to 0$ is hold, when $\lambda\to\infty$. Consequently, $I_n(\lambda)=o(1)$.

Taking consideration Eq.(3.19) into the Eq.(3.27), the following expansion can be derived, when $\lambda\to\infty$:

$$E\left(N_1^n\left(\lambda\hat{\zeta}_\lambda\right)\right)=\left(\frac{\lambda}{m_1}\right)^n A_n+\left(\frac{\lambda}{m_1}\right)^{n-1}B_nA_{n-1}+o\left(\lambda^{n-1}\right) \tag{3.28}$$

Here,

$$\begin{aligned}
A_n &\equiv \frac{m_{n+1}}{(n+1)m_1};\quad B_n\equiv n^2C_F-\frac{n(n-1)}{2};\\
C_F &\equiv \frac{m_2}{2m_1^2};\quad m_n\equiv E(\eta_1^n),\ n=1,2,3,4.
\end{aligned}$$

Thus, Theorem 3.1 is proved. □

Now, investigate the boundary functional $\tau_1\left(\lambda\hat{\zeta}_\lambda\right)$. To obtain the moments of the boundary functional $\tau_1\left(\lambda\hat{\zeta}_\lambda\right)$, give the following lemma from literature.

3.3. **Lemma.** *(Aliyev et. al. [1]) Suppose that the conditions of Lemma 3.1. Then, the following asymptotic expansions for the first four moments of the boundary functional $\tau_1(z)$ can be written, when $\lambda\to\infty$:*

$$E\left(\tau_1^n(z)\right)=\alpha_1^n\left(\frac{z}{m_1}\right)^n+\frac{n^2}{2}\alpha_1^nD_n\left(\frac{z}{m_1}\right)^{n-1}+o(z^{n-1}) \tag{3.29}$$

Here, $D_n=C_v^2\left(\eta_1\right)+\frac{n-1}{n}C_v^2\left(\xi_1\right)+\frac{1}{n}$. $C_v\left(\eta_1\right)$ and $C_v\left(\xi_1\right)$ are the variation of coefficient of η_1 and ξ_1, respectively, i.e.,

$$\begin{aligned}
C_v\left(\eta_1\right) &\equiv \frac{\sigma_\eta}{m_1};\ \sigma_\eta\equiv Var\left(\eta_1\right);\ m_1\equiv E\left(\eta_1\right);\\
C_v\left(\xi_1\right) &\equiv \frac{\sigma_\xi}{\alpha_1};\ \sigma_\xi\equiv Var\left(\xi_1\right);\ \alpha_1\equiv E\left(\xi_1\right)\quad \square.
\end{aligned}$$

3.2. **Theorem.** *Suppose that* $E\left(\xi_1^2\right) < \infty$ *and* $E\left(\eta_1^{n+1}\right) < \infty$, $n = 1, 2, 3, 4$. *Then, the asymptotic expansion for the first four moments of* $\tau_1\left(\lambda\hat{\zeta}_\lambda\right)$ *can be written as follows, when* $\lambda \to \infty$*:*

$$E\left(\tau_1^n\left(\lambda\hat{\zeta}_\lambda\right)\right) = \alpha_1^n C_n \lambda^n + \frac{n^2}{2}\alpha_1^n C_{n-1} D_n \lambda^{n-1} + o\left(\lambda^{n-1}\right), \quad n = 1, 2, 3, 4. \tag{3.30}$$

Here, $A_n \equiv m_{n+1}/((n+1)m_1)$, $C_n = A_n/m_1^n$ *and* $D_n = C_v^2(\eta_1) + ((n+1)/n)C_v^2(\xi_1) + 1/n$. *Moreover,* $C_v(\eta_1)$ *and* $C_v(\xi_1)$ *are variation coefficients of* η_1 *and* ξ_1*, respectively.*

Proof. By definition,

$$E\left(\tau_1\left(\lambda\hat{\zeta}_\lambda\right)\right) \equiv \int_0^\infty E\left(\tau_1\left(\lambda z\right)\right) d\pi_\lambda(z). \tag{3.31}$$

Here, $\pi_\lambda(z)$ is the ergodic distribution of Markov chain $\{\zeta_n\}$. On the other hand, according to relation in Eq.(3.29), the following representation can be written:

$$E\left(\tau_1^n(z)\right) = \alpha_1^n\left(\frac{z}{m_1}\right)^n + \frac{n^2}{2}\alpha_1^n D_n\left(\frac{z}{m_1}\right)^{n-1} + \left(\frac{z}{m_1}\right)^{n-1} g_n(z) \tag{3.32}$$

Here, $g_n(z)$ is a bounded function, besides $\lim_{z\to\infty} g_n(z) = 0$ and $\sup_{z\in R}|g_n(z)| = M_n < \infty$. Taking Eq.(3.32) into account of Eq.(3.31), the following expansion is derived:

$$\begin{aligned} E\left(\tau_1^n\left(\lambda\hat{\zeta}_\lambda\right)\right) &= \alpha_1^n\frac{\lambda^n}{m_1^n}\int_0^\infty z^n d\pi_\lambda(z) + \frac{n^2}{2}\alpha_1^n D_n\left(\frac{\lambda}{m_1}\right)^{n-1}\int_0^\infty z^{n-1} d\pi_\lambda(z) \\ &+ \left(\frac{\lambda}{m_1}\right)^{n-1}\int_0^\infty z^{n-1} g(\lambda z) d\pi_\lambda(z) \end{aligned} \tag{3.33}$$

Taking into account Lemma 3.2 in Eq.(3.33), the asymptotic expansions for the first four moments of $\tau_1\left(\lambda\hat{\zeta}_\lambda\right)$ can be obtained as follows:

$$\begin{aligned} E\left(\tau_1^n\left(\lambda\hat{\zeta}_\lambda\right)\right) &= \left(\frac{\alpha_1}{m_1}\right)^n \lambda^n E\left(\hat{\zeta}_\lambda^n\right) + \frac{n^2}{2}\alpha_1^n D_n\left(\frac{\lambda}{m_1}\right)^{n-1} E\left(\hat{\zeta}_\lambda^{n-1}\right) \\ &+ o\left(\left(\frac{\lambda}{m_1}\right)^{n-1}\right). \end{aligned} \tag{3.34}$$

Here, $E\left(\hat{\zeta}_\lambda^n\right)$ is the ergodic moments of the Markov chain $\{\zeta_n\}$.

Considering Eq.(3.19), the following asymptotic expansion is obtained from Eq.(3.34):

$$E\left(\tau_1^n\left(\lambda\hat{\zeta}_\lambda\right)\right) = \alpha_1^n C_n \lambda^n + \frac{n^2}{2}\alpha_1^n C_{n-1} D_n \lambda^{n-1} + o\left(\lambda^{n-1}\right).$$

This concludes the proof of Theorem 3.2. □

3.1. **Example.** η_1 has an exponential distribution with the parameter β and ξ_1 also has an exponential distribution with the parameter μ. Then, the followings can be derived:

$$\begin{aligned} E\left(\xi_1\right) &= \alpha_1=\frac{1}{\mu};\; E\left(\xi_1^2\right)=\frac{2}{\mu^2};\; \sigma_\xi^2=Var(\xi_1)=\frac{1}{\mu^2};\; \sigma_\xi=\frac{1}{\mu} \\ E\left(\eta_1\right) &= m_1=\frac{1}{\beta};\; E\left(\eta_1^2\right)=\frac{2}{\beta^2};\; \sigma_\eta^2=Var(\eta_1)=\frac{1}{\beta^2};\; \sigma_\eta=\frac{1}{\beta} \end{aligned}$$

The coefficient variations of ξ_1 and η_1 can be gotten as follows:

$$C_v\left(\xi_1\right)\equiv\frac{\sigma_\xi}{\alpha_1}=\frac{1/\mu}{1/\mu}=1;\; C_v\left(\eta_1\right)\equiv\frac{\sigma_\eta}{m_1}=\frac{1/\beta}{1/\beta}=1.$$

Now, the coefficients A_n, C_n and D_n can be calculated as follows:

$$\begin{aligned} A_n &\equiv \frac{m_{n+1}}{(n+1)m_1}=\frac{(n+1)!/\beta^{n+1}}{(n+1)/\beta}=\frac{n!}{\beta^n};\quad C_n=\frac{A_n}{m_1^n}=\frac{n!/\beta^n}{1/\beta^n}=n!; \\ D_n &\equiv C_v^2(\eta_1)+\left(\frac{n+1}{n}\right)C_v^2(\xi_1)+\frac{1}{n}=1+\frac{n-1}{n}+\frac{1}{n}=2. \end{aligned}$$

Therefore, the following result is hold:

$$\begin{aligned} E\left(\tau_1^n\left(\lambda\hat{\zeta}_\lambda\right)\right) &= \alpha_1^nC_n\lambda^n+\frac{n^2}{2}\alpha_1^nC_{n-1}D_n\lambda^{n-1}+o\left(\lambda^{n-1}\right) \\ &= \frac{n!}{\mu^n}\left\{\lambda^n+n\lambda^{n-1}+o\left(\lambda^{n-1}\right)\right\}. \end{aligned}$$

3.2. **Example.** Assume that random variable η_1 has uniform distribution with parameters $[0,1]$ and random variable ξ_1 has exponential distribution with parameter μ. Then, the followings can be given:

$$\begin{aligned} E\left(\xi_1\right) &= \alpha_1=\frac{1}{\mu};\; E\left(\xi_1^2\right)=\frac{2}{\mu^2};\; \sigma_\xi^2=Var(\xi_1)=\frac{1}{\mu^2};\; \sigma_\xi=\frac{1}{\mu}; \\ E\left(\eta_1^n\right) &= m_n=\int_0^1 x^n dx=\frac{1}{n+1},\; n=1,2,...; \\ E\left(\eta_1\right) &= \frac{1}{2};\; E\left(\eta_1^2\right)=m_2=\frac{1}{3};\; \sigma_\eta^2=Var(\eta_1)=\frac{1}{12};\; \sigma_\eta=\frac{1}{2\sqrt{3}}. \end{aligned}$$

The coefficient variations of ξ_1 and η_1 can be obtained as follows:

$$C_v(\xi_1)\equiv\frac{\sigma_\xi}{\alpha_1}=\frac{1/\mu}{1/\mu}=1;\; C_v(\eta_1)\equiv\frac{\sigma_\eta}{m_1}=\frac{1}{\sqrt{3}}.$$

Now, calculate the coefficient of $A_n,$ C_n and D_n as follows:

$$\begin{aligned} A_n &\equiv \frac{m_{n+1}}{(n+1)m_1}=\frac{1/(n+2)}{(n+1)/2}=\frac{2}{(n+1)(n+2)}\;; \\ C_n &\equiv \frac{A_n}{m_1^n}=\frac{2^{n+1}}{(n+1)(n+2)}; \\ D_n &\equiv C_v^2(\eta_1)+\left(\frac{n+1}{n}\right)C_v^2(\xi_1)+\frac{1}{n}=\frac{1}{3}+\frac{n-1}{n}+\frac{1}{n}=\frac{4}{3}. \end{aligned}$$

Then, the n^{th} $(n = 1, 2, 3, 4)$ order moment can be written as follows:

$$\begin{aligned} E\left(\tau_1^n\left(\lambda\hat{\zeta}_\lambda\right)\right) &\equiv \alpha_1^n C_n \lambda^n + \frac{n^2}{2}\alpha_1^n C_{n-1} D_n \lambda^{n-1} + o\left(\lambda^{n-1}\right) \\ &= \frac{2^{n+1}}{\mu^n}\left\{\frac{\lambda^n}{(n+1)(n+2)} + \frac{n\lambda^{n-1}}{3(n+1)} + o(\lambda^{n-1})\right\} \\ &= \frac{2^{n+1}}{(n+1)(n+2)\mu^n}\lambda^n + \frac{n2^{n+1}}{3(n+1)\mu^n}\lambda^{n-1} + o\left(\lambda^{n-1}\right). \end{aligned}$$

Now, investigate the boundary functional $S_{N_1\left(\lambda\hat{\zeta}_\lambda\right)}$.

According to the definition, $S_{N_1\left(\lambda\hat{\zeta}_\lambda\right)} = \sum_{i=1}^{N_1\left(\lambda\hat{\zeta}_\lambda\right)} \eta_i$. First, investigate asymptotic behaviour of the boundary functional $S_{N_1\left(\lambda\hat{\zeta}_\lambda\right)}$, when $z \to \infty$. For this reason, define the following Laplace transform:

$$\Psi(\lambda, k) \equiv \int_0^\infty e^{-\lambda z} E\left(e^{-kS_{N_1(z)}}\right) dz;\ \ \lambda > 0;\ \ k \geq 0.$$

Give the following lemma related with the Laplace transform written above.

3.4. **Lemma.** *(Aliyev et. al. [1]) Transform $\Psi(\lambda, k)$ is expressed by means of the Laplace-Stiltjes transform of η_1 as follows:*

$$\Psi(\lambda, k) = \frac{\varphi(k) - \varphi(\lambda + k)}{\lambda(1 - \varphi(\lambda + k))}$$

Here, $\varphi(\theta) \equiv E\left(\exp\left(-\theta\eta_1\right)\right),\ \ \theta \geq 0.$

It is possible to obtain a lot of useful information from Lemma 3.4. One of them can be given with the following lemma.

3.5. **Lemma.** *(Aliyev et. al [1]) Assume that the first three moment of η_1 is finite. Then the followings are true:*

$$\begin{aligned} a) \int_0^\infty e^{-\lambda z} E(S_{N_1(z)}) dz &= m_1 \tilde{U}_\eta(\lambda); \\ b) \int_0^\infty e^{-\lambda z} E(S_{N_1(z)}^2) dz &= m_2 \tilde{U}_\eta(\lambda) + 2m_1 \tilde{U}_\eta(\lambda) U_\eta^*(\lambda) D_1^*(\lambda); \\ c) \int_0^\infty e^{-\lambda z} E(S_{N_1(z)}^3) dz &= 6m_1 \tilde{U}_\eta(\lambda) U_\eta^{*2}(\lambda) D_1^*(\lambda) + 3m_1 \tilde{U}_\eta(\lambda) U_\eta^*(\lambda) D_2^*(\lambda) \\ &+ 3m_2 \tilde{U}_\eta(\lambda) U_\eta^*(\lambda) D_1^*(\lambda) + m_3 \tilde{U}_\eta(\lambda). \end{aligned}$$

Here, $m_n = E\left(\eta_1^n\right),\ \ D_n^*(\lambda) = E(\eta_1^n e^{-\lambda\eta_1}),\ \ n \geq 1;$ *and $U_\eta(z)$ is a renewal function generated by the random variables $\{\eta_n\}$. Moreover, $\tilde{U}_\eta(\lambda)$ and $U_\eta^*(\lambda)$ represents the Laplace and Laplace Stiltjes transforms of $U_\eta(z)$, respectively.*

The following result can be derived from Lemma 3.5.

3.1 Corollary. Under the conditions of Lemma 3.5, the following explicit expressions for the first three moments of the boundary functional $S_{N_1(z)}$ can be given:

$$\begin{aligned} 1) E(S_{N_1(z)}) &= m_1 U_\eta(z); \\ 2) E(S^2_{N_1(z)}) &= m_2 U_\eta(z) + 2m_1 U_\eta(z) * U_\eta(z) * D_1(z); \\ 3) E(S^3_{N_1(z)}) &= 6m_1 U_\eta^{*3}(z) * D_1^{*2}(z) \\ &+ 3U_\eta^{*2}(z) * [m_1 D_2(z) + m_2 D_1(z)] + m_3 U_\eta(z). \end{aligned}$$

Here, $D_1(z) = \int_0^z s dF(s)$; $D_2(z) = \int_0^z s^2 dF(s)$ (Khaniyev [8]) □.

Using Lemma 3.5 and Corollary 3.1, the following lemma can be obtained:

3.6. **Lemma.** *Suppose that $E\left(\eta_1^3\right) < \infty$ is satisfied, then the following expansions for the first three moment of the boundary functional $S_{N_1(z)}$, when $z \to \infty$:*

$$E\left(S_{N_1^n(z)}\right) = z^n + n\frac{m_2}{2m_1} z^{n-1} + o\left(z^{n-1}\right), \; n = 1, 2, 3. \tag{3.35}$$

From Lemma 3.6, the following result can be derived by induction.

3.2 Corollary. Under the condition $E\left(\eta_1^2\right) < \infty$, the moments of the boundary functional $S_{N_1(z)}$ can be represented as follows:

$$E\left(S^n_{N_1(z)}\right) = z^n + n\frac{m_2}{2m_1} z^{n-1} + z^{n-1} g_n(z), \; n = 1, 2, ... \tag{3.36}$$

Here, $g_n(z)$ is a bounded function and $\lim_{z\to\infty} g_n(z) = 0$. □

According to definition of $\hat{\zeta}_\lambda$ and using the Corollary 3.2, the following theorem can be given for the moments of the boundary functional $S_{N_1(\lambda\hat{\zeta}_\lambda)}$.

3.3. **Theorem.** *For all $n = 1, 2, 3, 4$, $E(\eta_1^{n+1}) < \infty$ is satisfied. Then, the asymptotic expansions for the moments of boundary functional $S_{N_1(\lambda\hat{\zeta}_\lambda)}$ can be written as follows, when $\lambda \to \infty$:*

$$E\left(S^n_{N_1(\lambda\hat{\zeta}_\lambda)}\right) = \frac{m_{n+1}}{(n+1)m_1}\lambda^n + \frac{m_2 m_n}{2m_1^2}\lambda^{n-1} + o(\lambda^{n-1}).$$

Here, $m_n \equiv E(\eta_1^n)$, $n = 1, 2, 3, 4$.

Proof. Recall the following definition:

$$E\left(S^n_{N_1(\lambda\hat{\zeta}_\lambda)}\right) \equiv \int_0^\infty E(S^n_{N_1(\lambda z)}) d\pi_\lambda(z) \tag{3.37}$$

Here, $\pi_\lambda(z)$ is the distribution of $\hat{\zeta}_\lambda$. Substituting λz instead of z in the Eq.(3.36), the following expression can be derived:

$$\begin{aligned} E\left(S^n_{N_1(\lambda\hat{\zeta}_\lambda)}\right) &= \int_0^\infty (\lambda z)^n d\pi_\lambda(z) + n\frac{m_2}{2m_1}\int_0^\infty (\lambda z)^{n-1} d\pi_\lambda(z) \\ &+ \int_0^\infty (\lambda z)^{n-1} g_n(\lambda z) d\pi_\lambda(z) \\ &= \lambda^n E\left(\hat{\zeta}_\lambda^n\right) + n\frac{m_2}{2m_1}\lambda^{n-1} E\left(\hat{\zeta}_\lambda^{n-1}\right) + \lambda^{n-1}\int_0^\infty z^{n-1} g_n(\lambda z) d\pi_\lambda(z) \qquad (3.38) \end{aligned}$$

According to Lemma 3.1 and Lemma 3.2, for each $n = 1, 2, ...$,
$\lim_{\lambda\to\infty} \int_0^\infty z^{n-1} g(\lambda z) d\pi_\lambda(z) = 0$ is satisfied. Then, Eq.(3.38) can be rewritten as an asymptotic expansion as follows:

$$E\left(S^n_{N_1(\lambda\hat{\zeta}_\lambda)}\right) = \lambda^n E\left(\hat{\zeta}^n_\lambda\right) + n\frac{m_2}{2m_1}\lambda^{n-1}E\left(\hat{\zeta}^{n-1}_\lambda\right) + o(\lambda^{n-1}) \tag{3.39}$$

Considering the relation in Eq.(3.19) into asymptotic expansion Eq. (3.39), for each $n = 1, 2, ...$, the following asymptotic expansion is obtained, when $\lambda \to \infty$:

$$E\left(S_{N_1(\lambda\hat{\zeta}_\lambda)}\right) = \frac{m_{n+1}}{(n+1)m_1}\lambda^n + \frac{m_1 m_n}{2m_1^2}\lambda^{n-1} + o(\lambda^{n-1}).$$

Thus, Theorem 3.3 is proved. □

4. Summary and Conclusion

In this study, a renewal - reward process with a generalized reflecting barrier is considered and its three important boundary functionals $N_1\left(\lambda\hat{\zeta}_\lambda\right)$, $\tau_1\left(\lambda\hat{\zeta}_\lambda\right)$, $S_{N_1\left(\lambda\hat{\zeta}_\lambda\right)}$ are investigated. The asymptotic expansions for their first-four moments are obtained. These formulas are especially expressed by means of some characteristics of the residual waiting time generated by η_n. Hence, we observed that there is a connection between the moments of these there boundary functionals and some numerical characteristics of residual waiting time generated by η_n. This connection can form a basis for applying some important results related to residual waiting time (e.g., Feller [7], Smith [15], Rogozin [14], etc.) to solution of various similar problems.

Although the obtained formulas in this study, are approximated, they are both simple and sufficiently accurate for many applied problems. Therefore, this asymptotic approach can also be applied to stationary characteristics of the process and it can be obtained simple and approximated formulas which have high accuracy for them. Moreover, improving this approach, it can be applied to random walks with a generalized reflecting barrier in the future studies.

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