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Screen Semi-Invariant Lightlike Submanifolds of Golden Semi-Riemannian Manifolds

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(Dedicated to the memory of **Prof. Dr. Aurel BEJANCU (1946 - 2020)**)

ABSTRACT

In this paper, we present screen semi-invariant lightlike submanifolds of golden semi-Riemannian manifolds. We research several properties of such submanifolds and get the conditions of integrability of distributions. We prove some results for totally umbilical screen semi-invariant lightlike submanifolds of golden semi-Riemannian manifolds. We also give an example.

Keywords: *Golden semi-Riemannian manifolds, golden structure, lightlike submanifolds, screen semi-invariant lightlike submanifolds.*

AMS Subject Classification (2020): 53C15; 53C40; 53C50.

1. Introduction

The theory of lightlike submanifolds of semi-Riemannian manifolds is one of important topics of differential geometry. Since the intersection of normal vector bundle and the tangent bundle is non-trivial, then in the study of lightlike submanifolds is more interesting and remarkably different from the study of non-degenerate submanifolds. Lightlike submanifolds is developed by Duggal and Bejancu [6] and Duggal and Şahin [9]. Moreover, many authors have studied the geometry of lightlike submanifolds [1, 4, 5, 16, 18, 22].

Duggal and Bejancu presented Cauchy-Riemann (CR)-lightlike submanifolds of Kaehler manifolds [6]. But CR-lightlike submanifolds exclude the complex and totally real submanifolds as subcases. Then Duggal and Şahin presented a new class of lightlike submanifolds which is called screen Cauchy–Riemann (SCR)-lightlike submanifolds of indefinite Kaehler manifolds [8]. Kılıç, Şahin and Keleş introduced screen semi-invariant (SSI) lightlike submanifolds of a semi-Riemannian product manifold and researched the geometry of such submanifolds [17]. Khursheed Haider, Thakur and Advin studied Screen Cauchy–Riemann lightlike submanifolds of a semi-Riemannian product manifold [15].

The number ϕ, which is the real positive root of the equation $x^2 - x - 1 = 0$ (thus, $\phi = \frac{1+\sqrt{5}}{2} \approx 1.618...$) is the golden ratio. Being inspired by the golden ratio, the notion of golden manifold was defined in [3]. Hretcanu and Crasmareanu worked invariant submanifolds of a golden Riemannian manifold [13]. They proved that a golden structure induces on every invariant submanifold a golden Structure, too [14]. Şahin and Akyol presented golden maps between golden Riemannian manifolds, give an example and show that such map is harmonic [21]. Özkan investigated complete and horizontal lifts of the golden structure in the tangent bundle [19]. Erdoğan and Yıldırım worked totally umbilical semi-invariant submanifold of golden Riemannian manifolds [10]. Gök, Keleş and Kılıç studied Schouten and Vrănceanu connections on golden manifolds [11]. Poyraz and Yaşar presented lightlike submanifolds of golden semi-Riemannian manifolds [20]. Acet studied screen pseudo-slant lightlike submanifolds of a golden semi-Riemannian manifold [2].

In this paper, we present screen semi-invariant lightlike submanifolds of golden semi-Riemannian manifolds. We research several properties of such submanifolds and get the conditions of integrability of

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distributions. We prove some results for totally umbilical screen semi-invariant lightlike submanifolds of golden semi-Riemannian manifolds. We also give an example.

2. Preliminaries

Let $\check{N}$ be an $m-$dimensional differentiable manifold. If a tensor field $\check{P}$ of type $(1,1)$ holds the following equation

$$\check{P}^2 = \check{P} + I \tag{2.1}$$

then $\check{P}$ is named a golden structure on $\check{N}$, where I is the identity transformation [12].

Let $(\check{N}, \check{g})$ be a semi-Riemannian manifold and $\check{P}$ be a golden structure on $\check{N}$. If $\check{P}$ holds the following equation

$$\check{g}(\check{P}U, V) = \check{g}(U, \check{P}V), \tag{2.2}$$

then $(\check{N}, \check{g}, \check{P})$ is called a golden semi-Riemannian manifold [19].

Let $(\check{N}, \check{g}, \check{P})$ be a golden semi-Riemannian manifold. Then the equation (2.2) is equivalent with

$$\check{g}(\check{P}U, \check{P}V) = \check{g}(\check{P}U, V) + \check{g}(U, V), \tag{2.3}$$

for any $U, V \in \Gamma(T\check{N})$.

Let $(\check{N}, \check{g})$ be a real $(m+n)-$dimensional semi-Riemannian manifold with index q, such that $m, n \geq 1$, $1 \leq q \leq m+n-1$ and $(\dot{N}, \dot{g})$ be an $m-$dimensional submanifold of $\check{N}$, where $\dot{g}$ is the induced metric of $\check{g}$ on $\dot{N}$. If $\check{g}$ is degenerate on the tangent bundle $T\dot{N}$ of $\dot{N}$ then $\dot{N}$ is named a lightlike submanifold of $\check{N}$. Then, for each tangent space $T_x\dot{N}$, $x \in \dot{N}$, we consider

$$T\dot{N}^{\perp} = \cup \left\{ V_x \in T_x\check{N} : \check{g}(V_x, U_x) = 0, \forall U_x \in T_x\dot{N}, x \in \dot{N} \right\},$$

which is a degenerate $n-$dimensional subspace of $T_x\check{N}$. Thus, both $T_x\dot{N}$ and $T_x\dot{N}^{\perp}$ are degenerate orthogonal subspaces but no longer complementary. Then, there exists a subspace $Rad(T_x\dot{N}) = T_x\dot{N} \cap T_x\dot{N}^{\perp}$ which is known as radical (null) space. If the mapping

$$Rad(T\dot{N}) : x \in \dot{N} \longrightarrow Rad(T_x\dot{N})$$

defines a smooth distribution on $\dot{N}$ of rank $r > 0$ then the submanifold $\dot{N}$ of $\check{N}$ is named an $r-$lightlike submanifold and $Rad(T\dot{N})$ is named the radical distribution on $\dot{N}$.

Let $S(T\dot{N})$ be a screen distribution which is a semi-Riemannian complementary distribution of $Rad(T\dot{N})$ in $T\dot{N}$, i. e.

$$T\dot{N} = Rad(T\dot{N}) \perp S(T\dot{N}), \tag{2.4}$$

and $S(T\dot{N}^{\perp})$ is a complementary vector subbundle to $Rad(T\dot{N})$ in $T\dot{N}^{\perp}$. Let $tr(T\dot{N})$ and $ltr(T\dot{N})$ be complementary (but not orthogonal) vector bundles to $T\dot{N}$ in $T\check{N}_{|_{\dot{N}}}$ and $Rad(T\dot{N})$ in $S(T\dot{N}^{\perp})^{\perp}$, respectively. Thus we have

$$tr(T\dot{N}) = ltr(T\dot{N}) \perp S(T\dot{N}^{\perp}), \tag{2.5}$$

$$T\check{N} \mid_{\dot{N}} = T\dot{N} \oplus tr(T\dot{N}) = \{Rad(T\dot{N}) \oplus ltr(T\dot{N})\} \perp S(T\dot{N}) \perp S(T\dot{N}^{\perp}). \tag{2.6}$$

Theorem 2.1. *Let $(\dot{N}, \dot{g}, S(T\dot{N}), S(T\dot{N}^{\perp}))$ be an $r-$lightlike submanifold of a semi-Riemannian manifold $(\check{N}, \check{g})$. Suppose that U is a coordinate neighborhood of $\dot{N}$ and ξ_i, $i \in \{1, .., r\}$ is a basis of $\Gamma(Rad(T\dot{N})_{|_U})$. Then, there exist a complementary vector subbundle $ltr(T\dot{N})$ of $Rad(T\dot{N})$ in $S(T\dot{N})^{\perp}$ and a basis of $\Gamma(ltr(T\dot{N})_{|_U})$ consisting of smooth section $\{N_i\}$ of $S(T\dot{N}^{\perp})^{\perp}_{|_U}$ such that*

$$\check{g}(N_i, E_j) = \delta_{ij}, \quad \check{g}(N_i, N_j) = 0, \tag{2.7}$$

for any $i, j \in \{1, .., r\}$.

We say that a submanifold $(\dot{N}, \dot{g}, S(T\dot{N}), S(T\dot{N}^{\perp}))$ of $\check{N}$ is
Case 1: $r-$lightlike if $r < min\{m, n\}$,
Case 2: Coisotropic if $r = n < m$, $S(T\dot{N}^{\perp}) = \{0\}$,
Case 3: Isotropic if $r = m < n$, $S(T\dot{N}) = \{0\}$,
Case 4: Totally lightlike if $r = m = n$, $S(T\dot{N}) = \{0\} = S(T\dot{N}^{\perp})$.

Let $\check{\nabla}$ be the Levi-Civita connection on $\check{N}$. Then, the Gauss and Weingarten formulas are

$$\check{\nabla}_U V = \nabla_U V + h(U,V), \quad (2.8)$$
$$\check{\nabla}_U N = -A_N U + \nabla^t_U N, \quad (2.9)$$

for any $U,V \in \Gamma(T\dot{N})$ and $N \in \Gamma(tr(T\dot{N}))$, where $\{\nabla_U V, A_N U\}$ and $\{h(U,V), \nabla^t_U N\}$ belong to $\Gamma(T\dot{N})$ and $\Gamma(tr(T\dot{N}))$, respectively. ∇ and ∇^t are linear connections on $\dot{N}$ and on the vector bundle $tr(T\dot{N})$, respectively.

According to (2.5), considering the projection morphisms L and S of $tr(T\dot{N})$ on $ltr(T\dot{N})$ and $S(T\dot{N})$, respectively, from (2.8) and (2.9) we have

$$\check{\nabla}_U V = \nabla_U V + h^l(U,V) + h^s(U,V), \quad (2.10)$$
$$\check{\nabla}_U N = -A_N U + \nabla^l_U N + D^s(U,N), \quad (2.11)$$
$$\check{\nabla}_U W = -A_W U + \nabla^s_U W + D^l(U,W), \quad (2.12)$$

where $h^l(U,V) = Lh(U,V)$, $h^s(U,V) = Sh(U,V)$, $\{\nabla_U V, A_N U, A_W U\} \in \Gamma(S(T\dot{N}))$, $\{\nabla^l_U N, D^l(U,W)\} \in \Gamma(ltr(T\dot{N}))$ and $\{\nabla^s_U W, D^s(U,N)\} \in \Gamma(S(T\dot{N}^\perp))$. Then, considering (2.10)-(2.12) and taking into account that $\check{\nabla}$ is a metric connection, we derive

$$\check{g}(h^s(U,V),W) + \check{g}(V, D^l(U,W)) = \check{g}(A_W U, V), \quad (2.13)$$
$$\check{g}(D^s(U,N),W) = \check{g}(A_W U, N). \quad (2.14)$$

Let J be a projection of $T\dot{N}$ on $S(T\dot{N})$. Then, considering (2.4) we have

$$\nabla_U JV = \nabla^*_U JV + h^*(U,JV)E, \quad (2.15)$$
$$\nabla_U E = -A^*_E U + \nabla^{*t}_U E, \quad (2.16)$$

for any $U,V \in \Gamma(T\dot{N})$ and $E \in \Gamma(Rad(T\dot{N}))$, where $\{\nabla^*_U JV, A^*_E U\}$ and $\{h^*(U,JV), \nabla^{*t}_U E\}$ belong to $\Gamma(S(T\dot{N}))$ and $\Gamma(Rad(T\dot{N}))$, respectively.

By using above equtions, we obtain

$$\check{g}(h^l(U,JV),E) = \dot{g}(A^*_E U, JV), \quad (2.17)$$
$$\check{g}(h^*(U,JV),N) = \dot{g}(A_N U, JV), \quad (2.18)$$
$$\check{g}(h^l(U,E),E) = 0, \quad A^*_E E = 0. \quad (2.19)$$

Generally, ∇ on $\dot{N}$ is not metric connection. Since $\check{\nabla}$ is a metric connection, from (2.10) we derive

$$(\nabla_U \dot{g})(V,Z) = \check{g}(h^l(U,V),Z) + \check{g}(h^l(U,Z),V), \quad (2.20)$$

for any $U,V,Z \in \Gamma(T\dot{N})$. But, ∇^* is a metric connection on $S(T\dot{N})$.

Theorem 2.2. *Let $\check{N}$ be semi-Riemannian manifold and $\dot{N}$ be an r-lightlike submanifold of $\check{N}$. Then ∇ is a metric connection iff $Rad(T\dot{N})$ is a parallel distribution with respect to ∇ [6].*

Definition 2.1. A lightlike submanifold $\dot{N}$ of a semi-Riemannian manifold $\check{N}$ is called totally umbilical in $\check{N}$, if there is a smooth transversal vector field $H \in \Gamma(ltr(T\dot{N}))$ such that

$$h(U,V) = H\dot{g}(U,V), \quad (2.21)$$

for any $U,V \in \Gamma(T\dot{N})$ [7].

It is known that $\dot{N}$ is totally umbilical iff on each coordinate neighborhood U, there exists smooth vector fields $H^l \in \Gamma(ltr(T\dot{N}))$ and $H^s \in \Gamma(S(T\dot{N}^\perp))$ such that

$$h^l(U,V) = H^l\dot{g}(U,V), h^s(U,V) = H^s\dot{g}(U,V) \text{ and } D^l(U,W) = 0, \quad (2.22)$$

for any $U,V \in \Gamma(T\dot{N})$ and $W \in \Gamma(S(T\dot{N}^\perp))$.

3. Screen Semi-Invariant Lightlike Submanifolds

Definition 3.1. Let $\dot{N}$ be a lightlike submanifold of a golden semi-Riemannian manifold $\check{N}$. We say that $\dot{N}$ is SSI-lightlike submanifold of $\check{N}$ if the following statements are satisfied:

1) There exists a non-null distribution $\mu \subseteq S(T\dot{N})$ such that

$$S(T\dot{N}) = \mu \oplus \mu^{\perp}, \check{P}(\mu) = \mu, \check{P}(\mu^{\perp}) \subseteq S(T\dot{N}^{\perp}), \mu \cap \mu^{\perp} = \{0\}, \tag{3.1}$$

where $\mu^{\perp}$ is orthogonal complementary to μ in $S(T\dot{N})$.
2) $Rad(T\dot{N})$ is invariant with respect to $\check{P}$, i. e., $\check{P}(Rad(T\dot{N})) = Rad(T\dot{N})$.
Then we have

$$\check{P}(ltr(T\dot{N})) = ltr(T\dot{N}), \tag{3.2}$$

$$T\dot{N} = \mu' \oplus \mu^{\perp}, \mu' = \mu \perp Rad(T\dot{N}). \tag{3.3}$$

Thus it follows that μ' is also invariant with respect to $\check{P}$. We indicate the orthogonal complement to $\check{P}(\mu^{\perp})$ in $S(T\dot{N}^{\perp})$ by μ_0. Then we obtain

$$tr(T\dot{N}) = ltr(T\dot{N}) \perp \check{P}(\mu^{\perp}) \perp \mu_0. \tag{3.4}$$

If $\mu \neq \{0\}$ and $\mu^{\perp} \neq \{0\}$, then $\dot{N}$ is called a proper SSI-lightlike submanifold of $\check{N}$. Thus, for on proper $\dot{N}$, we have $dim(\mu) \geq 1$, $dim(\mu^{\perp}) \geq 1$, $dim(\dot{N}) \geq 3$ and $dim(\check{N}) \geq 5$. Moreover, there exists no proper SSI-lightlike hypersurface of a golden semi-Riemannian manifold.

If $\mu = \{0\}$, i. e., $\check{P}(S(T\dot{N})) \subset S(T\dot{N}^{\perp})$, then $\dot{N}$ is called screen anti-invariant lightlike submanifold.

Let $(\dot{N}, \dot{g}, S(T\dot{N}), S(T\dot{N}^{\perp}))$ be a lightlike submanifold of a golden semi-Riemannian manifold $\check{N}$. Then, for any $U \in \Gamma(T\dot{N})$ and $N \in \Gamma(tr(T\dot{N}))$ we can write

$$\check{P}U = PU + wU, \tag{3.5}$$

$$\check{P}N = BN + CN, \tag{3.6}$$

where PU, $BN \in \Gamma(T\dot{N})$ and wU, $CN \in \Gamma(tr(T\dot{N}))$. If $\dot{N}$ is a SSI-lightlike submanifold of $\check{N}$, then $PU \in \Gamma(\mu')$, $wU \in \Gamma(\check{P}(\mu^{\perp}))$, $BN \in \Gamma(\mu^{\perp})$ and $CN \in \Gamma(tr(T\dot{N}))$, respectively.

Lemma 3.1. *Let $\dot{N}$ be a screen semi-invariant lightlike submanifold of a golden semi-Riemannian manifold $\check{N}$. Then, one has*

$$P^2U = PU + U - BwU, \tag{3.7}$$

$$CwU = wPU, \tag{3.8}$$

$$PBN = BN - BCN, \tag{3.9}$$

$$C^2N = CN + N - wBN, \tag{3.10}$$

$$\dot{g}(PU, V) - \dot{g}(U, PV) = \dot{g}(U, wV) - \dot{g}(wU, V), \tag{3.11}$$

$$\begin{aligned} \dot{g}(PU, PV) = \; & \dot{g}(PU, V) + \dot{g}(U, V) + \dot{g}(wU, V) - \dot{g}(PU, wV) \\ & - \dot{g}(wU, PV) - \dot{g}(wU, wV), \end{aligned} \tag{3.12}$$

for any $U, V \in \Gamma(T\dot{N})$ and $N \in \Gamma(tr(T\dot{N}))$.

Proof. Applying $\check{P}$ to (3.5), considering (2.1) and taking tangential and transversal parts of the resulting equation, we derive (3.7) and (3.8). Similarly, applying $\check{P}$ to (3.6), using (2.1) and taking tangential and transversal parts of the resulting equation, we obtain (3.9) and (3.10). Considering (2.2), (2.3) and (3.5), we derive (3.11) and (3.12). □

Throughout this paper, we suppose that $\check{\nabla}\check{P} = 0$.

Example 3.1. Let $(\check{N}=\mathbb{R}^8_2,\check{g})$ be a 8-dimensional semi-Euclidean space with signature $(-,-,+,+,+,+,+,+)$ and $(x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8)$ be the standard coordinate system of $\check{N}$. If we define a mapping $\check{P}$ by $\check{P}(x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8)=(\phi x_1,\phi x_2,\phi x_3,(1-\phi)x_4,\phi x_5,\phi x_6,\phi x_7,\phi x_8)$ then $\check{P}^2=\check{P}+I$ and $\check{P}$ is a golden structure on $\check{N}$. Let $\dot{N}$ be a lightlike submanifold in $\check{N}$ given by the equations

$$\begin{aligned} x_1 &= u_1+u_2, x_2=-u_1+u_2, \\ x_3 &= u_5, x_4=\phi u_5, \\ x_5 &= u_1+u_3, x_6=-u_1+u_3, \\ x_7 &= u_2+u_4, x_8=u_2-u_4, \end{aligned}$$

where u_i, $1\leq i\leq 4$, are real parameters. Thus $T\dot{N}=Span\{U_1,U_2,U_3,U_4,U_5\}$, where

$$\begin{aligned} U_1 &= \frac{\partial}{\partial x_1}-\frac{\partial}{\partial x_2}+\frac{\partial}{\partial x_5}-\frac{\partial}{\partial x_6}, U_2=\frac{\partial}{\partial x_1}+\frac{\partial}{\partial x_2}+\frac{\partial}{\partial x_7}+\frac{\partial}{\partial x_8}, \\ U_3 &= \frac{\partial}{\partial x_5}+\frac{\partial}{\partial x_6}, U_4=\frac{\partial}{\partial x_7}-\frac{\partial}{\partial x_8}, U_5=\frac{\partial}{\partial x_3}+\phi\frac{\partial}{\partial x_4}. \end{aligned}$$

The radical distribution $Rad(T\dot{N})$ is spanned by $\{U_1,U_2\}$. Hence $\dot{N}$ is a 2-lightlike submanifold of $\check{N}$. $S(T\dot{N})$ and $S(T\dot{N}^{\perp})$ are spanned by $\{U_3,U_4,U_5\}$ and $\{W\}$, respectively, where

$$W=\phi\frac{\partial}{\partial x_3}-\frac{\partial}{\partial x_4}.$$

Furthermore, $ltr(T\dot{N})$ is spanned by

$$\begin{aligned} N_1 &= \frac{1}{4}(-\frac{\partial}{\partial x_1}+\frac{\partial}{\partial x_2}+\frac{\partial}{\partial x_5}-\frac{\partial}{\partial x_6}), \\ N_2 &= \frac{1}{4}(-\frac{\partial}{\partial x_1}-\frac{\partial}{\partial x_2}+\frac{\partial}{\partial x_7}+\frac{\partial}{\partial x_8}). \end{aligned}$$

Then $\mu=Span\{U_3,U_4\}$, $\mu^{\perp}=Span\{U_5\}$, $\mu_0=\{0\}$ and we can easily check that $Rad(T\dot{N})$, μ and $ltr(T\dot{N})$ are invariant distributions and $\check{P}(\mu^{\perp})=S(T\dot{N}^{\perp})$. Thus, $\dot{N}$ is a proper SSI-lightlike submanifold of $\check{N}$ with $\mu'=Span\{U_1,U_2,U_3,U_4\}$.

Theorem 3.1. *Let $\dot{N}$ be a SSI-lightlike submanifold of a golden semi-Riemannian manifold $\check{N}$. Then $S(T\dot{N})$ is integrable iff the following conditions are satisfied*

$$\dot{g}(A_NV,\check{P}U)=\dot{g}(A_NU,\check{P}V), U,V\in\Gamma(\mu), \tag{3.13}$$

$$\dot{g}(A_NV,\check{P}U)=\check{g}(D^s(U,N),\check{P}V), U\in\Gamma(\mu),V\in\Gamma(\mu^{\perp}), \tag{3.14}$$

$$\check{g}(D^s(U,N),\check{P}V)=\check{g}(D^s(V,N),\check{P}U), U,V\in\Gamma(\mu^{\perp}). \tag{3.15}$$

Proof. For any $U,V\in\Gamma(S(T\dot{N}))$ and $N\in\Gamma(ltr(T\dot{N}))$, we obtain

$$\begin{aligned} \check{g}([U,V],\check{P}N) &= \dot{g}(A_NU,\check{P}V)-\check{g}(D^s(U,N),\check{P}V) \\ &\quad -\dot{g}(A_NV,\check{P}U)+\check{g}(D^s(V,N),\check{P}U). \end{aligned} \tag{3.16}$$

Since $S(T\dot{N})=\mu\perp\mu^{\perp}$, letting $U,V\in\Gamma(\mu)$ in (3.16) we derive (3.13), taking $U\in\Gamma(\mu)$ and $V\in\Gamma(\mu^{\perp})$ in (3.16) we get (3.14) and letting $U,V\in\Gamma(\mu^{\perp})$ in (3.16) we get (3.15). Hence we have the assertion of the theorem. □

Theorem 3.2. *Let $\dot{N}$ be a SSI-lightlike submanifold of a golden semi-Riemannian manifold $\check{N}$. Then the following assertions are equivalent.*

(i) μ' is integrable.

(ii) $h(U,\check{P}V)=h(\check{P}U,V)$, for any $U,V\in\Gamma(\mu')$.

(iii) w vanishes on μ'.

Proof. Since $\check{\nabla}_U\check{P}V=\check{P}\check{\nabla}_UV$, for any $U,V\in\Gamma(\mu')$, from (2.8), (3.5) and (3.6) we obtain $h(U,\check{P}V)=w\nabla_UV+Ch(U,V)$, for any $U,V\in\Gamma(\mu')$. Since h is symmetric and ∇ is torsion free, we obtain $h(U,\check{P}V)-h(\check{P}U,V)=w[U,V]$. From this equation the proof is completed. □

Theorem 3.3. *Let $\dot{N}$ be a SSI-lightlike submanifold of a golden semi-Riemannian manifold $\check{N}$. Then the following assertions are equivalent.*
(i) $\mu^{\perp}$ is integrable.
(ii) $\check{g}(A_{\check{P}W}Z, N) = \check{g}(A_{\check{P}Z}W, N)$ and $\dot{g}(A_{\check{P}W}Z, U) = \dot{g}(A_{\check{P}Z}W, U)$, for any $U \in \Gamma(\mu)$, $Z, W \in \Gamma(\mu^{\perp})$, $N \in \Gamma(ltr(T\dot{N}))$.
(iii) A_N is self-adjoint on $\mu^{\perp}$, for any $N \in \Gamma(ltr(T\dot{N}))$ and $A_{\check{P}W}Z$ has no components in μ, for any $Z, W \in \Gamma(\mu^{\perp})$.

Proof. $\mu^{\perp}$ is integrable iff

$$\check{g}([Z, W], \check{P}N) = \dot{g}([Z, W], \check{P}U) = 0,$$

for any $Z, W \in \Gamma(\mu^{\perp})$, $U \in \Gamma(\mu)$ and $N \in \Gamma(ltr(T\dot{N}))$. Then from (2.2) and (2.12) we derive

$$\begin{aligned} \check{g}([Z, W], \check{P}N) &= \check{g}(\check{\nabla}_Z W, \check{P}N) - \check{g}(\check{\nabla}_W Z, \check{P}N) = \check{g}(\check{\nabla}_Z \check{P}W, N) - \check{g}(\check{\nabla}_W \check{P}Z, N) \\ &= -\check{g}(A_{\check{P}W}Z, N) + \check{g}(A_{\check{P}Z}W, N), \end{aligned} \tag{3.17}$$

$$\begin{aligned} \check{g}([Z, W], \check{P}U) &= \check{g}(\check{\nabla}_Z W, \check{P}U) - \check{g}(\check{\nabla}_W Z, \check{P}U) \\ &= \check{g}(\check{\nabla}_Z \check{P}W, U) - \check{g}(\check{\nabla}_W \check{P}Z, U) \\ &= -\dot{g}(A_{\check{P}W}Z, U) + \dot{g}(A_{\check{P}Z}W, U), \end{aligned} \tag{3.18}$$

for any $Z, W \in \Gamma(\mu^{\perp})$, $U \in \Gamma(\mu)$ and $N \in \Gamma(ltr(T\dot{N}))$. Thus we obtain (i)$\Longrightarrow$(ii).
Since $\check{g}(\check{P}W, N) = 0$ and $\check{\nabla}$ is a metric connection, we get

$$\check{g}(\check{\nabla}_Z \check{P}W, N) + \check{g}(W, \check{\nabla}_Z \check{P}N) = 0. \tag{3.19}$$

Using (2.11) and (2.12) in (3.19) we obtain

$$\check{g}(A_{\check{P}W}Z, N) = -\dot{g}(W, A_{\check{P}N}Z). \tag{3.20}$$

By replacing role of vector fields W and Z in (3.20)

$$\check{g}(A_{\check{P}Z}W, N) = -\dot{g}(Z, A_{\check{P}N}W). \tag{3.21}$$

Thus from (3.20), (3.21) and (ii) A_N is self-adjoint on $\mu^{\perp}$.
Considering (2.13) we get

$$\check{g}(h^s(Z, U), \check{P}W) = \dot{g}(A_{\check{P}W}Z, U), \tag{3.22}$$

for any $Z, W \in \Gamma(\mu^{\perp})$ and $U \in \Gamma(\mu)$. Considering (2.10) and (2.12) we derive

$$\check{g}(h^s(U, Z), \check{P}W) = -\dot{g}(A_{\check{P}Z}U, W). \tag{3.23}$$

Thus from (3.22) and (3.23) and using to symmetric of h^s we have

$$\dot{g}(A_{\check{P}W}Z, U) = -\dot{g}(A_{\check{P}Z}U, W). \tag{3.24}$$

Since $\check{\nabla}$ is a metric connection, $\check{g}(Z, \check{P}U) = 0$ and using to symmetric of h^s, we obtain

$$\dot{g}(A_{\check{P}Z}W, U) = \dot{g}(A_{\check{P}Z}U, W). \tag{3.25}$$

Since $\mu^{\perp}$ is integrable, considering (3.18), (3.24) and (3.25) we derive

$$\begin{aligned} \dot{g}([Z, W], \check{P}U) &= -\dot{g}(A_{\check{P}W}Z, U) + \dot{g}(A_{\check{P}Z}W, U) \\ &= 2\dot{g}(A_{\check{P}Z}W, U). \end{aligned} \tag{3.26}$$

From this we obtain $\dot{g}(A_{\check{P}W}Z, U) = 0$. Thus we get (ii)$\Longrightarrow$(iii).
Suppose that $A_{\check{P}W}Z$ has no components in μ and A_N is self-adjoint on $\mu^{\perp}$. Then from (3.26) we obtain $\dot{g}([Z, W], \check{P}U) = 0$, for any $Z, W \in \Gamma(\mu^{\perp})$ and $U \in \Gamma(\mu)$. Since $\check{\nabla}$ is a metric connection and A_N is self-adjoint on $\mu^{\perp}$ from (2.11) we get

$$\begin{aligned} \dot{g}([Z, W], \check{P}N) &= \check{g}(\check{\nabla}_Z W, \check{P}N) - \check{g}(\check{\nabla}_W Z, \check{P}N) \\ &= -\check{g}(W, \check{\nabla}_Z \check{P}N) + \check{g}(Z, \check{\nabla}_W \check{P}N) \\ &= \dot{g}(W, A_{\check{P}N}Z) - \dot{g}(Z, A_{\check{P}N}W) = 0. \end{aligned}$$

Thus $\mu^{\perp}$ is integrable. Thus we obtain (iii)$\Longrightarrow$(i). □

Theorem 3.4. *Let $\dot{N}$ be a SSI-lightlike submanifold of a golden semi-Riemannian manifold $\check{N}$. Then μ is integrable iff the following statements hold:*
(i) A_N is self adjoint on μ, for any $N \in \Gamma(ltr(T\dot{N}))$.
(ii) $\dot{g}(\check{P}V, A_{\check{P}Z}U) = \dot{g}(\check{P}U, A_{\check{P}Z}V)$, for any $U, V \in \Gamma(\mu)$ and $Z \in \Gamma(\mu^{\perp})$.

Proof. μ is integrable iff

$$\check{g}([U,V], N) = \dot{g}([U,V], Z) = 0,$$

for any $U, V \in \Gamma(\mu)$, $Z \in \Gamma(\mu^{\perp})$ any $N \in \Gamma(ltr(T\dot{N}))$. Since $\check{\nabla}$ is a metric connection, considering (2.3), (2.10), (2.11) and (2.12) we derive

$$\check{g}([U,V], N) = \dot{g}(A_N U, V) - \dot{g}(U, A_N V), \tag{3.27}$$

$$\begin{aligned} \dot{g}([U,V], Z) &= \check{g}(\check{\nabla}_U V, Z) - \check{g}(\check{\nabla}_V U, Z) \\ &= \check{g}(\check{\nabla}_U \check{P}V, \check{P}Z) - \check{g}(\check{\nabla}_U V, \check{P}Z) - \check{g}(\check{\nabla}_V \check{P}U, \check{P}Z) + \check{g}(\check{\nabla}_V U, \check{P}Z) \\ &= \check{g}(\check{\nabla}_U \check{P}V, \check{P}Z) - \check{g}(\check{\nabla}_V \check{P}U, \check{P}Z) \\ &\quad -\check{g}(h^s(U,V) - h^s(V,U), \check{P}Z) \\ &= -\check{g}(\check{P}V, \check{\nabla}_U \check{P}Z) + \check{g}(\check{P}U, \check{\nabla}_V \check{P}Z) \\ &= \dot{g}(\check{P}V, A_{\check{P}Z}U) - \dot{g}(\check{P}U, A_{\check{P}Z}V), \end{aligned} \tag{3.28}$$

for any $U, V \in \Gamma(\mu)$, $Z \in \Gamma(\mu^{\perp})$ any $N \in \Gamma(ltr(T\dot{N}))$. Hence from (3.27) and (3.28) we obtain (i) and (ii), respectively. Conversely, (i) and (ii) are satisfied. From (3.27) and (3.28), we have $[U,V] \in \Gamma(\mu)$, for any $U, V \in \Gamma(\mu)$. □

Theorem 3.5. *Let $\dot{N}$ be a SSI-lightlike submanifold of a golden semi-Riemannian manifold $\check{N}$. Then $Rad(T\dot{N})$ is integrable iff*

$$\check{g}(h^l(E,U), E') = \check{g}(h^l(E',U), E), \tag{3.29}$$
$$\check{g}(h^s(E, \check{P}E'), \check{P}Z) = \check{g}(h^s(E', \check{P}E), \check{P}Z), \tag{3.30}$$

for any $E, E' \in \Gamma(Rad(T\dot{N}))$, $U \in \Gamma(\mu)$ and $Z \in \Gamma(\mu^{\perp})$.

Proof. $Rad(T\dot{N})$ is integrable iff

$$\dot{g}([E,E'], U) = \dot{g}([E,E'], Z) = 0,$$

for any $E, E' \in \Gamma(Rad(T\dot{N}))$, $U \in \Gamma(\mu)$ and $Z \in \Gamma(\mu^{\perp})$. Since $\check{\nabla}$ is a metric connection, considering (2.3) and (2.10) we derive

$$\begin{aligned} \dot{g}([E,E'], U) &= \check{g}(\check{\nabla}_E E', U) - \check{g}(\check{\nabla}_{E'} E, U) = -\check{g}(E', \check{\nabla}_E U) + \check{g}(E, \check{\nabla}_{E'} U) \\ &= -\check{g}(h^l(E,U), E') + \check{g}(h^l(E',U), E), \end{aligned} \tag{3.31}$$

$$\begin{aligned} \dot{g}([E,E'], Z) &= \check{g}(\check{\nabla}_E E', Z) - \check{g}(\check{\nabla}_{E'} E, Z) \\ &= \check{g}(\check{\nabla}_E \check{P}E', \check{P}Z) - \check{g}(\check{\nabla}_E E', \check{P}Z) - \check{g}(\check{\nabla}_{E'} \check{P}E, \check{P}Z) + \check{g}(\check{\nabla}_{E'} E, \check{P}Z) \\ &= \check{g}(h^s(E, \check{P}E'), \check{P}Z) - \check{g}(h^s(E,E'), \check{P}Z) \\ &\quad -\check{g}(h^s(E', \check{P}E), \check{P}Z) + \check{g}(h^s(E',E), \check{P}Z) \\ &= \check{g}(h^s(E, \check{P}E'), \check{P}Z) - \check{g}(h^s(E', \check{P}E), \check{P}Z). \end{aligned} \tag{3.32}$$

Thus the proof follows from (3.31) and (3.32). □

Theorem 3.6. *Let $\dot{N}$ be a SSI-lightlike submanifold of a golden semi-Riemannian manifold $\check{N}$. Then μ' is integrable iff*

$$\dot{g}(A_{\check{P}Z}U, \check{P}V) = \dot{g}(A_{\check{P}Z}V, \check{P}U), \tag{3.33}$$

for any $U, V \in \Gamma(\mu')$ and $Z \in \Gamma(\mu^{\perp})$.

Proof. μ' is integrable iff $\dot{g}([U,V],Z)=0$, for any $U,V\in\Gamma(\mu')$ and $Z\in\Gamma(\mu^{\perp})$. Then from (2.3), (2.10) and (2.12) we derive

$$\begin{aligned}\dot{g}([U,V],Z) &= \check{g}(\check{\nabla}_U V,Z)-\check{g}(\check{\nabla}_V U,Z)\\ &= \check{g}(\check{\nabla}_U\check{P}V,\check{P}Z)-\check{g}(\check{\nabla}_U V,\check{P}Z)-\check{g}(\check{\nabla}_V\check{P}U,\check{P}Z)+\check{g}(\check{\nabla}_V U,\check{P}Z)\\ &= \check{g}(\check{\nabla}_U\check{P}V,\check{P}Z)-\check{g}(\check{\nabla}_V\check{P}U,\check{P}Z)-\check{g}(h^s(U,V)-h^s(V,U),\check{P}Z)\\ &= -\check{g}(\check{P}V,\check{\nabla}_U\check{P}Z)+\check{g}(\check{P}U,\check{\nabla}_V\check{P}Z)\\ &= \dot{g}(A_{\check{P}Z}U,\check{P}V)-\dot{g}(A_{\check{P}Z}V,\check{P}U),\end{aligned}\tag{3.34}$$

for any $U,V\in\Gamma(\mu')$ and $Z\in\Gamma(\mu^{\perp})$. From (3.34) the proof is completed. □

Theorem 3.7. *Let $\dot{N}$ be a SSI-lightlike submanifold of a golden semi-Riemannian manifold $\check{N}$. Then the following assertions are equivalent.*

(i) $h^s(U,\check{P}V)$ has no components in $\check{P}(\mu^{\perp})$, for any $U,V\in\Gamma(\mu')$.

(ii) $\dot{g}(\check{P}E,D^l(U,\check{P}Z))=0$ and $A_{\check{P}Z}U$ has no components in μ, for any $U\in\Gamma(\mu')$, $E\in\Gamma(Rad(T\dot{N}))$ and $Z\in\Gamma(\mu^{\perp})$.

(iii) μ' defines totally geodesic foliation on $\dot{N}$.

Proof. From (2.13) we derive

$$\check{g}(h^s(U,\check{P}V),\check{P}Z)+\check{g}(\check{P}V,D^l(U,\check{P}Z))=\check{g}(A_{\check{P}Z}U,\check{P}V),$$

for any $U,V\in\Gamma(\mu')$ and $Z\in\Gamma(\mu^{\perp})$. Replacing V by $E\in\Gamma(Rad(T\dot{N}))$ in this equation we have $\check{g}(h^s(U,\check{P}E),\check{P}Z)=-\check{g}(\check{P}E,D^l(U,\check{P}Z))$. Using (i) in this equation we derive $\check{g}(\check{P}E,D^l(U,\check{P}Z))=0$. If we take $V\in\Gamma(\mu)$ in (2.13) we get $\check{g}(h^s(U,\check{P}V),\check{P}Z)=\dot{g}(A_{\check{P}Z}U,\check{P}V)$. Thus, $A_{\check{P}Z}U$ has no components in μ. Thus we get (i)$\Longrightarrow$(ii).

Since $\check{\nabla}$ is a metric connection, considering (2.3) and (2.12) we derive

$$\begin{aligned}\dot{g}(\nabla_U V,Z) &= \check{g}(\check{\nabla}_U V,Z)=\check{g}(\check{\nabla}_U\check{P}V,\check{P}Z)-\check{g}(\check{\nabla}_U V,\check{P}Z)\\ &= -\check{g}(\check{P}V,\check{\nabla}_U\check{P}Z)+\check{g}(V,\check{\nabla}_U\check{P}Z)\\ &= \dot{g}(A_{\check{P}Z}U,\check{P}V)-\check{g}(D^l(U,\check{P}Z),\check{P}V)-\dot{g}(A_{\check{P}Z}U,V)+\check{g}(D^l(U,\check{P}Z),V),\end{aligned}$$

for any $U,V\in\Gamma(\mu')$ and $Z\in\Gamma(\mu^{\perp})$. Thus we derive $\nabla_U V\in\mu$ and we obtain (ii)$\Longrightarrow$(iii). Considering (2.3) and (2.10) we derive

$$\begin{aligned}\dot{g}(\nabla_U V,Z) &= \check{g}(\check{\nabla}_U V,Z)=\check{g}(\check{\nabla}_U\check{P}V,\check{P}Z)-\check{g}(\check{\nabla}_U\check{P}V,Z)\\ &= \check{g}(h^s(U,\check{P}V),\check{P}Z)+\dot{g}(\nabla_U\check{P}V,Z),\end{aligned}$$

for any $U,V\in\Gamma(\mu')$ and $Z\in\Gamma(\mu^{\perp})$. Since $\nabla_U V\in\mu'$ and μ' is invariant, $\nabla_U\check{P}V\in\mu'$. Thus we obtain (iii)$\Longrightarrow$(i). □

Theorem 3.8. *Let $\dot{N}$ be a SSI-lightlike submanifold of a golden semi-Riemannian manifold $\check{N}$. Then the following assertions are equivalent.*

(i) $A_{\check{P}V}U$ has no components in μ', for any $U,V\in\Gamma(\mu^{\perp})$.

(ii) $h^s(U,\check{P}Z)$ and $D^s(U,\check{P}N)$ have no components in $\check{P}(\mu^{\perp})$, for any $U\in\Gamma(\mu^{\perp})$, $Z\in\Gamma(\mu)$ and $N\in\Gamma(ltrT(\dot{N}))$.

(iii) $\mu^{\perp}$ defines totally geodesic foliation on $\dot{N}$.

Proof. (i)$\Longrightarrow$(ii). From (2.13) we obtain

$$\dot{g}(A_{\check{P}V}U,\check{P}Z)=\check{g}(h^s(U,\check{P}Z),\check{P}V),$$

for any $U,V\in\Gamma(\mu^{\perp})$ and $Z\in\Gamma(\mu)$. Hence $h^s(U,\check{P}Z)$ has no components in $\check{P}(\mu^{\perp})$. Also, using (2.14) we derive

$$0=\dot{g}(A_{\check{P}V}U,\check{P}N)=\check{g}(D^s(U,\check{P}N),\check{P}V).$$

Thus $D^s(U,\check{P}N)$ has no components in $\check{P}(\mu^{\perp})$ which implies (ii).

(ii)$\Longrightarrow$(iii). Since μ and $ltr(T\dot{N})$ are invariant distribution, for any $Z\in\Gamma(\mu)$ and $N\in\Gamma(ltr(T\dot{N}))$, we have $\check{P}Z\in\Gamma(\mu)$ and $\check{P}N\in\Gamma(ltr(T\dot{N}))$, respectively. Then the distribution $\mu^{\perp}$ defines totally geodesic foliation on $\dot{N}$ iff

$\dot{g}(\nabla_U V, \check{P}Z) = \check{g}(\nabla_U V, \check{P}N) = 0$, for any $U, V \in \Gamma(\mu^\perp)$, $Z \in \Gamma(\mu)$ and $N \in \Gamma(ltr(T\dot{N}))$. Then taking into account that $\check{\nabla}$ is a metric connection and using (2.2), (2.10) and (2.11) we have

$$\begin{aligned}\dot{g}(\nabla_U V, \check{P}Z) &= \check{g}(\check{\nabla}_U V, \check{P}Z) = \check{g}(\check{\nabla}_U \check{P}V, Z)\\ &= -\check{g}(\check{P}V, \check{\nabla}_U Z) = -\check{g}(h^s(U, Z), \check{P}V),\end{aligned}$$

$$\begin{aligned}\check{g}(\nabla_U V, \check{P}N) &= \check{g}(\check{\nabla}_U V, \check{P}N) = \check{g}(\check{\nabla}_U \check{P}V, N)\\ &= -\check{g}(\check{P}V, \check{\nabla}_U N) = -\check{g}(D^s(U, N), \check{P}V),\end{aligned}$$

which implies (iii).

(iii)$\Longrightarrow$(i). Since $\mu^\perp$ defines totally geodesic foliation on $\dot{N}$, $\dot{g}(\nabla_U V, \check{P}Z) = \check{g}(\nabla_U V, \check{P}N) = 0$, for any $U, V \in \Gamma(\mu^\perp)$, $Z \in \Gamma(\mu)$ and $N \in \Gamma(ltr(T\dot{N}))$. Considering (2.2) and (2.12) we have

$$\begin{aligned}\dot{g}(\nabla_U V, \check{P}Z) &= \check{g}(\check{\nabla}_U V, \check{P}Z) = \check{g}(\check{\nabla}_U \check{P}V, Z)\\ &= -\dot{g}(A_{\check{P}V} U, Z)\end{aligned}$$

$$\begin{aligned}\check{g}(\nabla_U V, \check{P}N) &= \check{g}(\tilde{\nabla}_U V, \check{P}N) = \dot{g}(\tilde{\nabla}_U \check{P}V, N)\\ &= -\dot{g}(A_{\check{P}V} U, N),\end{aligned}$$

which implies (i). □

Corollary 3.1. *Let $\dot{N}$ be a totally umbilical SSI-lightlike submanifold of a golden semi-Riemannian manifold $\check{N}$. Then $\mu^\perp$ is totally geodesic in $\dot{N}$.*

Proof. For any $U, V \in \Gamma(\mu^\perp)$, we have

$$\nabla_U V = \bar{\nabla}_U V + \bar{h}(U, V)$$

where $\bar{\nabla}_U V \in \Gamma(\mu^\perp)$ and $\bar{h}(U, V) \in \Gamma(\mu')$. Since μ' is an invariant distribution, for any $Z \in \Gamma(\mu')$, we have $\check{P}Z \in \Gamma(\mu')$. Then taking into account that $\check{\nabla}$ is a metric connection and using (2.2) and (2.10), it can be easily calculated

$$\begin{aligned}\dot{g}(\bar{h}(U, V), \check{P}Z) &= \dot{g}(\nabla_U V, \check{P}Z) = \check{g}(\check{\nabla}_U V, \check{P}Z) = \check{g}(\check{\nabla}_U \check{P}V, Z)\\ &= -\check{g}(\check{P}V, \check{\nabla}_U Z) = -\check{g}(\check{P}V, h^s(U, Z)).\end{aligned} \tag{3.35}$$

Using (2.22) we derive

$$h^s(U, Z) = H^s \dot{g}(U, Z) = 0,$$

for any $U \in \Gamma(\mu^\perp)$ and $Z \in \Gamma(\mu')$ and we have assertion of corollary. □

Theorem 3.9. *Let $\dot{N}$ be a totally umbilical SSI-lightlike submanifold of a golden semi-Riemannian manifold $\check{N}$. Then following assertions are equivalent.*

(i) μ' is totally geodesic in $\dot{N}$.
(ii) $A_{\check{P}Z}$ is $\mu^\perp-$valued, for any $Z \in \Gamma(\mu^\perp)$.
(iii) $H^s \in \Gamma(\mu_0)$.

Proof. For any $U, V \in \Gamma(\mu')$, we have

$$\nabla_U V = \nabla'_U V + h'(U, V)$$

where $\nabla'_U V \in \Gamma(\mu')$ and $h'(U, V) \in \Gamma(\mu^\perp)$. Then taking into account that $\check{\nabla}$ is a metric connection and using (2.2) and (2.10) it can be easily calculated

$$\begin{aligned}\dot{g}(h'(U, \check{P}V), Z) &= \dot{g}(\nabla_U \check{P}V, Z) = \check{g}(\check{\nabla}_U \check{P}V, Z) = \check{g}(\check{\nabla}_U V, \check{P}Z)\\ &= -\check{g}(V, \check{\nabla}_U \check{P}Z) = \check{g}(A_{\check{P}Z} U, V),\end{aligned}$$

for any $U, V \in \Gamma(\mu')$ and $Z \in \Gamma(\mu^\perp)$. Thus we obtain (i)$\Longrightarrow$(ii).

From (2.13) and (2.22) we derive

$$\check{g}(A_{\check{P}Z} U, V) = \check{g}(h^s(U, V), \check{P}Z) - \check{g}(V, D^l(U, \check{P}Z)) = \check{g}(U, V) g(H^s, \check{P}Z).$$

Thus we get (ii)$\Longrightarrow$(iii).

Since $\check{\nabla}$ is a metric connection, considering (2.2), (2.10) and (2.22) we derive

$$\begin{aligned}\dot{g}(h'(U, \check{P}V), Z) &= \dot{g}(\nabla_U \check{P}V, Z) = \check{g}(\check{\nabla}_U \check{P}V, Z) = \check{g}(\check{\nabla}_U V, \check{P}Z)\\ &= \check{g}(h^s(U, V), \check{P}Z) = \check{g}(U, V) \check{g}(H^s, \check{P}Z).\end{aligned}$$

Thus we obtain (iii)$\Longrightarrow$(i). □

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