Some remarks on the analysis of light curves with the autocorrelation method.

by E. A. KREIKEN

(Department of Astronomy, of Ankara University)

Özet : Zaman serilerinin analizinde otokorelasyon metodu ve otokorolasyon fonksiyonunun "kuvvet spektrumu" na açılımı esaslı olarak incelenmiştir. Kuvvet spektrumunun aslı olmıyan ikinci derece periotlar çıkardığı bulunmuştur. Otokorelasyon analizi herhalde kıymetlidir. Kuvvet spektrumu analizi daha ziyade formal bir değeri haizdir.

Abstract : The autocorrelation method of analysing time series and the subsequent development of the autocorrelation function into a power spectrum is critically considered. It is found that the power spectrum may throw up secondary periodicities which are spurious. The autocorrelation analysis is valuable in all cases. The power spectrum analysis mostly has formal value only.

1. For the analysis of time series M. C. Kendall [1] has developed the powerful autocorrelation method. If the autocorrelation coëfficient is defined by

$$r_K = \frac{\Sigma\, u_i \cdot u_{i+K}}{\sqrt{\Sigma u_i^2 \cdot \Sigma\, u_i^2{}_{+K}}} \qquad \cdots\cdot(1)$$

and a second order autoregressive time series of the type $u_{t+2} + au_{t+1} + bu_t = \varepsilon_{t+2}$ (ε_{t+2} a random element) is considered, the theoretical correlogram generated by this equation is of the damped type

$$r_K = p^K \frac{\sin(k\theta + \Psi)}{\sin \Psi} \qquad \cdots\cdot(2)$$

where $2\pi/\theta$ is the autoregressive period of the regression equation and is given by $\cos a = -\theta/2\sqrt{b}$, but the typical series of this kind has no "period" in a strict sense. J. Ashbrook, R. L. Duncombe and A. J. J. van Woerkom [2] have with remarkable results applied this method for analysing the light curve of μ

Cepheï. They obtain a correlogram which is strongly damped. A disadvantage of the use of correlograms is that a secondary periodicity may be masked, if shorter than the dominant period. In order to overcome this disability, Ashbrook, Duncombe and van Woerkom transform the correlogram in the corresponding power spectrum, which by Kintchine's theorem [3] is the Fourier transform of the autocorrelation function, thus

$$\pi(f) = \frac{2}{N} \sum_{0}^{N} r_K \cdot \cos 2\pi fk \qquad \cdots\cdot(3)$$

where f is the reciprocal of the trial period.

For μ Cephei no secondary periodicities were found. Using their method, the present author analysed the light curve of the semi-regular variable Z Ursa Majoris [4]. For this light curve the damping coëfficient p in the correlogram 2 proved to be fairly small. The Fourier transform of the autocorrelation function indicates secondary periodicities of periods 1/2 P; 1/3 P..... suggesting the presence of overtones.

2. By applying the autocorrelogram and periodogram analyses to one and the same time-series, Kendall [5] finds that the periodogram indicates a far larger number of periodicities than the correlogram, so that the conclusion is inevitable that either the correlogram is insensitive or the periodogram is misleading. Kendall is able to prove that most of the periods thrown up for consideration by the periodogram are not significant. Consequently the question must be raised as to what extent any physical significance can be attached to the periods which turn up when from the light curve of a variable the autocorrelation function is computed while next this autocorrelattion function is developed into a power series.

For $k > N$ the numerical value of r_K is largely determined by sampling errors and has low weight. Therefore in their analysis of the light curve of μ Cephëi, Ashbrook, Duncombe and van Woerkom when applying the Fourier transform, only considered the r_K's for which $k \leq 25$. With the light curve of Z UMa also all terms r_K with $k \leq 25$ were omitted.

If a light curve is a pure harmonic, its correlogram also is a pure harmonic and in the power spectrum [3] all terms are zero except when $f = 1/\theta$ and in that case $\pi(f) = 1$. However, the series r_K cannot be interrupted at any arbitrary interval N. For a certain trial period $1/f$ all terms must be considered up to such a value of N for which N, $1/f$ and $P = \theta$ are commensurate. In other words, the part of the correlogram which is considered, must contain an integral number of times the trial period and an integral number of times the period P. It is easy to see that if this precaution is omitted and for all values f one and the same arbitrary value N is used, even with a pure harmonic the terms of the power series are $\pi(f) \neq 0$ and in the curve $\pi(f)$ several spurious secondary periods show up. Consequently with the light curve of both μ Cephei and *Z* UMa there is the possibility that the shape of the power spectrum $\pi(f)$ has been affected by the choice of the value *N*.

If a light curve is cyclical but not harmonic, in the power spectrum $\pi(f)$ for all trial periods $1/f$ which are not commensurate with the main period, the value $\pi(f) \equiv 0$. It is supposed of course that the correct values N are used. With a cyclical curve both the light curve and the periodogram are strictly periodical and the successive maxima and minima are identical. This is impossible if for any non commensurate trial period $1/f$ the term $\pi(f) \neq 0$. Obviously this same conclusion is reached by simple consideration of the mathematical properties of the Fourier series. On the other hand with a cyclical function some at least of the commensurate terms $\pi(f)$ must be $\neq 0$. If this were not the case the cyclical variation would be a pure harmonic. Consequently in the power spectrum secondary periods will appear with a frequency equal to 2, 3, 4... times the frequency of the fundamental period. Cyclical variations can result from the interference of several harmonic terms, this being the mathematical way of describing the shape of the curve in terms of a Fourier series. There is however no guarantee that all terms, set up by such an analysis, have real physical meaning. So we conclude that when analysing a cyclical variation by using the method of correlogram analysis and power spectrum, we are bound to obtain a series of "overtones". If, moreover, the series r_K is broken off at an arbitrary interval N, just as with a pure harmonic this may cause some additional values $\pi(f) \neq 0$ to appear and these latter certainly are spurious in this sense that they have no physical reality.

If the correlogram is a damped harmonic (2) similar effects are to be expected and the physical interpretation of the power spectrum will be difficult. In the next sections some examples are considered for the different curves.

3. It is supposed that some disturbance affects the atmosphere of a star in such a way as to cause oscillations in magnitude which can be represented by a damped harmonic. It is supposed that the disturbance repeats itself at regular inter-

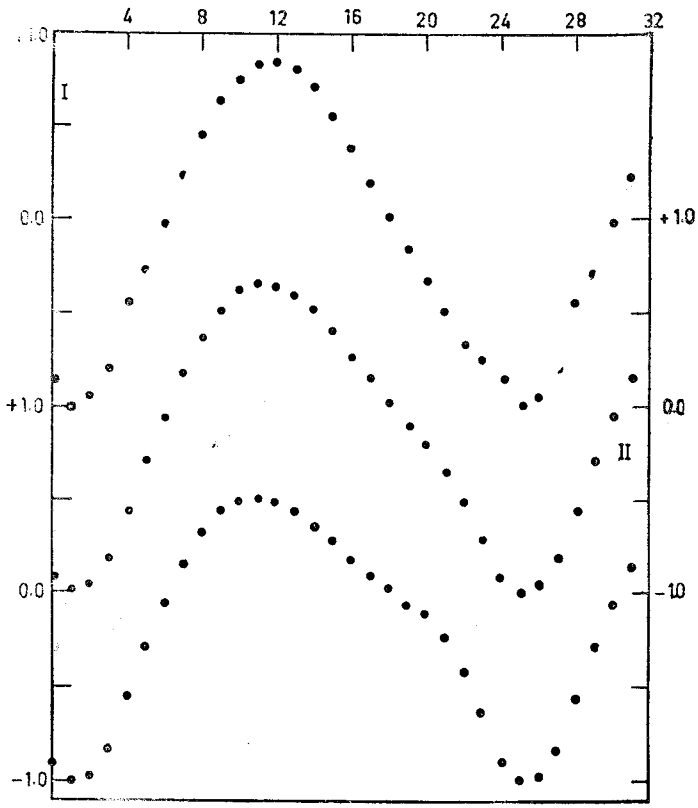

Fig. 1

vals of time of length P and at the moments $P = 0$; $P = 1$; $P = 2$ etc. causes an instantaneous decrease in magnitude of an amount a. (P is expressed in arbitrary units). For the present we take the length of the intervals to be equal to the "period" of the damped oscillation, but further on it will be shown that this latter assumption is not essential. The time interval P is divided into n equal subintervals. For practical reasons n is taken to be equal to 24. So the limits of the subintervals are at $t_0=0$; $t_1=1/n$ P; $t_2=2/n$ P$\cdots\frac{n-1}{n}$ P; $t_n=$ P. The magnitude disturbances occur at the moments $t=-n$P; $-(n-1)$P$\cdots\cdots\cdots-1$P; O; $+1$P; $+2$P etc. The damping during a complete period P $=1$ is represented by d and that during one of the subperiods by $p=d^{1/n}$. The total effect of all disturbances together at the instants t_0; t_n; $t_{2n}\cdots$is

$$u_0=a+ad+ad^2+\cdots=\frac{a}{1-d} \qquad \cdots(4)$$

and at the instants $t_1, t_{1+n}, t_{1+2n}\cdots$; t_2; t_{2+n}; $t_{2+2n}\cdots$

$$u_j=u_0p^j\cos 2\pi\, j/n \qquad \cdots(5)$$

When for d the values $d=1/2$; $d=1/4$ and $d=1/8$ are used, the resulting light curves are as indicated in table 1 in the columns under $\tau=0$. Up till now it was assumed that the influence of the disturbance is instantaneous or is at least concentrated within an infinitesimal amount of time $\tau\to 0$. This is hardly probable. The disturbing influence will cover some interval of time $\tau\neq 0$ and within this interval the influence of the disturbance rises from zero to maximum and decreases again from maximum to zero. The disturbance sets in before the instants $t_0=0$; $t_n=1$; $t_{2n}=3\cdots$ and also continues afterwards. At different times in the interval τ the influence of the disturbance on the magnitude is indicated by a_{-3}; a_{-2}; a_{-1}; a_0; a_{+1}; $a_{+2}\cdots$ Instead of (4) and (5) we now have

$$u_0=\frac{\left(a_0+a_{-1}p\cos 2\pi\frac{1}{n}+a_{-2}p^2\cos 2\pi\frac{2}{n}\cdots\right)}{(1-d)} \qquad \cdots(6)$$

and

$$u_j=\frac{a_0}{1-d}p^j\cos 2\pi\frac{j}{n}+\frac{1}{1-d}\left\{p^{j-1}a_{+1}\cos 2\pi\frac{(j-1)}{n}+\right.$$
$$\left.a_{-1}p^{j+1}\cos 2\pi\frac{(j+1)}{n}+a_{+2}p^{j-2}\cos 2\pi\frac{(j-2)}{n}\cdots\cdots\right\} \qquad \cdots(7)$$

There is no information available about the shape of the function $a(t)$. Therefore a simple relation was adopted viz. $a_0=1$;

$a_{-1} = a_{+1} = 3/4$; $a_{-2} = a_{+2} = 1/2$; $a_{-3} = a_{+3} = 1/4$; $a_{-4} = a_{+4} = a_{-5} = \cdots\cdots\cdots = 0$.

Using these values of a we obtain the light curves which in table 1 are given in the columns under the heading $\tau = \Delta t$. They are easily derived from the columns preceding them. Each of the resulting curves was multiplied with a constant in order to reduce the largest negative deviation of the magnitude to $\Delta m = -1{,}000$. The three curves of table 1 are graphically represented in fig. 1. They bear a striking resemblance to some curves which have been published for Cepheid variables. There is nothing astonishing about that, because all constants were chosen in such a way, that the resulting curves should resemble Cepheid curves. Perhaps in a rough way it might be possible to consider Cepheid variations as the result of damped oscillations which are set up in the atmosphere by disturbances coming from the interior, while these disturbances are repeated at regular intervals. However, any attempt to identify the artificial curves of fig. 1 with Cepheid curves would be at least premature. We have assumed that the effect of a disturbance was such as to cause a magnitude defect of the amount a. If instead of this we had supposed that the disturbance was such as to cause an increase of magnitude, the curves in fig 1 would retain the same shape. They now would indicate a rapid decrease in magnitude to minimum and afterwards a slower increase to maximum.

Actually the curves in fig. 1 are purely artificial. They represent cyclical variations which have been made outwardly to resemble Cepheid light curves. They definitely contain but one period, e. g. P = 1 (or P = 24 when expressed in the subintervals as unit of time).

Before continuing it should be pointed out that if in (7) the period of disturbance P is not identical to the period of the vibration, the resulting curve will still be a cyclical one with period P, but in the terms $\cos 2\pi \frac{(j+1)}{n}$ the term n is no longer equal to 24. The period P having been divided into 24 equal subintervals, θ is not equal to 24 of these subintervals.

4. Using the relation (1) the autocorrelation functions corresponding to the curves in fig. 1 are computed. The results also appear in table 1 in the columns r_K. Only the values k up

to $k = 23$ have to be considered, while in the relation (1) the denominator is a constant. For obvious reasons the descending and the ascending branch of the autocorrelation function are found to be symmetrical. Next from (3) the power spectrum is computed. For reasons stated before $\pi(f) \equiv 0$ for all values for which the ratio $P : f$ is not an integer number. Consequently in table 1 only the values $\pi(f)$ corresponding to $f = 24$; $f = 12$; $f = 8$.... are given. In all three curves "overtones" are present. Our artificial curves definitely contain one period $P = 24$. No physical reality can be therefore attached to the overtones. In purely mathematical terms they describe the shape of the curves in figure 1. Still as long as no attempt is made to ascribe such a physical significance to them, these overtones may be valuable for the description of the artificial curves in fig. 1 and the observed light curves of variables.

On the other hand the autocorrelation function r_K does not uniquely determine the curve $u(t)$ and neither therefore does the fundamental and the overtones as derived from the autocorrelation function.

5. Next the case is considered where the time series is of the second order autoregressive type $u_{t+2} = au_{t+1} + bu_t + \varepsilon_{t+2}$ so that the corresponding autocorrelation function is given by equation (2). If for simplification the phase angle Ψ is taken $\Psi = \pm \frac{1}{2} \pi$, the equation (2) reduces to

$$r_K = p^K \cos \theta . k = p^K \cos 2\pi\, k/P$$

The period P is again divided into 24 equal subperiods and the damping coëfficient $d = p^{24}$ during a complete period is assumed to be $d = 1/2$; $d = 1/4$ and $d = 1/8$ respectively. The three resulting autocorrelation curves are given in table 2. In order to reduce the amount of tabulation in the table only the numerical values of r_K for even k's are given. Next the power spectrum corresponding to the different curves in table 2 is obtained by applying the relation (3). By adopting different values for N we can now study the influence which the choice of N has on the shape of the power function. For N the values $N = 30$; $N = 40$; $N = 50$; $N = 60$; N 24; $N = 48$ respectively are adopted. The resulting power spectra are tabulated in table 3. The first part of this table gives the values $\pi(f)$ as directly computed from the curves in table 2. In the second part of table 3 all curves have been reduced in such a way that for $f = 24$ the term $\pi(24) = +1.000$. Some of these latter curves are also graphically represented in figure 2. From table 3 and from fig. 2 it is clear that to a considerable extent the shape of the power spectrum depends on the choice of N, that is on the interval at which we break off the series r_K. With all curves there is but one fundamental period viz. $P = 24$ In the power spectrum the maximum corresponding to this fundamental is strongly broadened and several secondary maxima are thrown up resembling overtones. Both the position and the amplitude of these overtones systematically depend on the choice of N. It is not too difficult to realise how they originate and why they vary with N. When f, P and N are not commensurate and the autocorrelation function is broken off after the interval N, the part of the autocorrelation function which is considered contains $N + n'$ times the fundamental period and $m + m'$ times the trial period. Here n and m are integers and n' and m' fractions < 1 and > 0. With a given value of N the fraction n' remains a constant, but m' varies with different values of f. With any value of f a part of the harmonic curve representing the trial period is left out of consideration. Sometimes positive and sometimes negative amplitudes are omitted and of this the result appears in the numerical value of $\pi(f)$.

If f increases, the part of the trial period which is left out of consideration changes, rapidly at first with small values of f and more slowly later on when f approaches P. So in the beginning several spurious maxima are thrown up close together, but later on near P the power spectrum gradually rises to the maximum representing the fundamental period. Also beyond P the decrease is a gradual one. When for N a slightly different value is selected, the terms n, n' and m will hardly be affected, but especially with low f values there is a considerable influence on m'. The result is a change of both the position and the intensity of the secondary maxima. With large changes of N all four terms n; n'; m and m' are affected and an altogether different power spectrum is thrown up. This latter also will have its maximum at the fundamental period P, but at both sides of the maximum the shape of the curve $\pi(f)$ will be different. In this case also secondary maxima are thrown up, but these may be

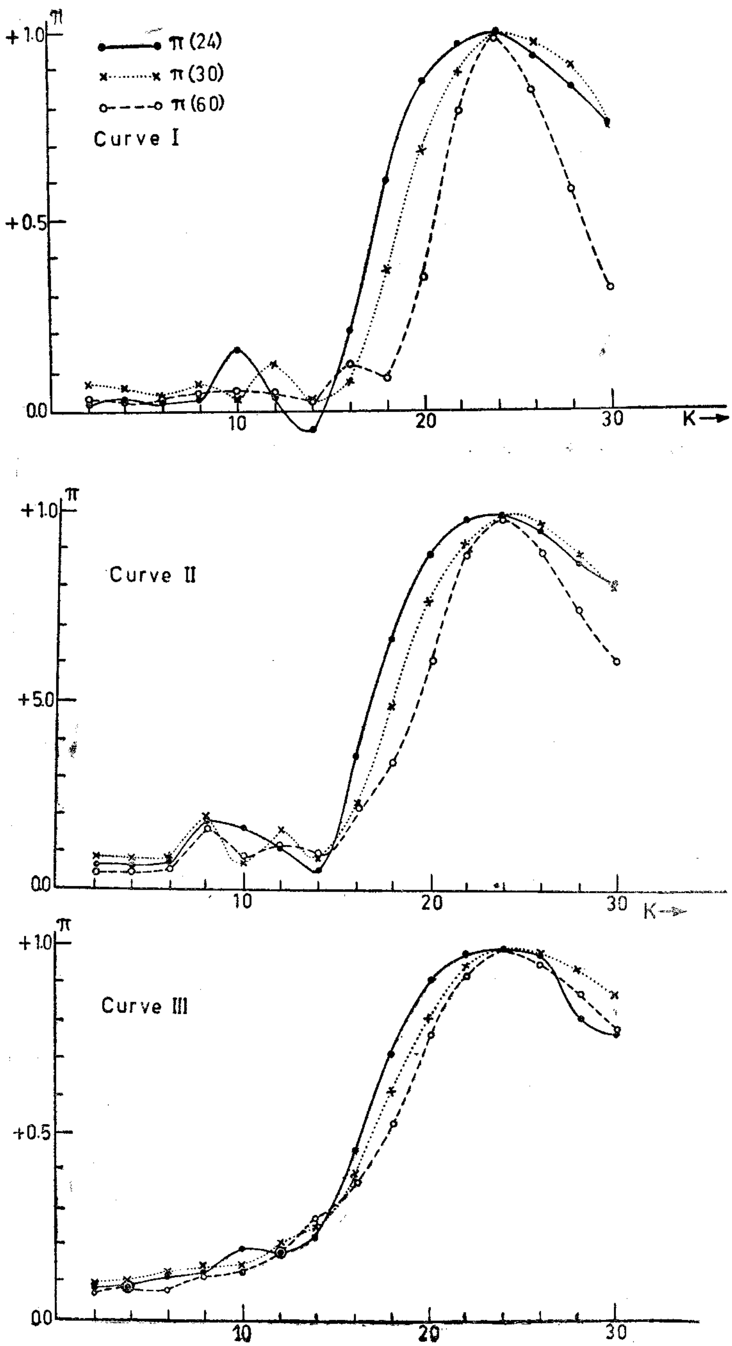

Fig. 2. Upper part $d = 1/2$, middle part $d = 1/4$; lower part $d = 1/8$

different from the previous ones. It is to be understood that N is small as compared to the total number of terms in the original time series.

The conclusions, which can be drawn from the curves in table 3 and in figure 2, may be summarised as follows:

a) For any given value of N a power spectrum is obtained which indicates a maximum corresponding to the fundamental period and in addition a number of secondary maxima.

b) The fundamental maximum is broadened out over a large range and the degree of broadening depends on the choice of N. On both sides of P the curves $\pi(f)$ become considerably steeper when larger numerical values of N are used.

c) This broadening of the fundamental maximum is most pronounced with the power spectra corresponding to the autocorrelation functions which are strongly damped. In a certain sense the degree of broadening can even be considered as a measure of the damping while with large damping the exact position of the maximum is rather undetermined.

d) The secondary maxima are most pronounced if small values of N are used. With increasing value of N these secondary maxima are displaced while at the same time they are flattened out. With $N = 60$ they have almost completely disappeared.

e) In the power spectrum corresponding to an autocorrelation curve with small damping coëfficient the secondary maxima have large intensities. In the spectrum of curves with large damping coëfficient the secondary maxima are almost non existent.

f) On the whole the influence of N is largest with the curves corresponding to a small damping coëfficient and smallest with those corresponding to a large damping coefficient.

6. Ashbrook, Duncombe and van Woerkom found the light curve of μ Cephei to be due to stochastic rather than harmonic processes. As a specific instance they mention temporary disturbances on the surface of a rotating star. The actual observed light curve then consists of many superimposed short trains of damped oscillations, commencing at random distributed times. The autocorrelation diagram of such a light curve contains a "period" approximately equal to the period of rotation. The fun-

damental period appearing in the power spectrum must also correspond to the rotational period. This hypothesis for explaining the shape of the light curves of irregular variables seems to be very reasonable. On the other hand the power spectrum corresponding to the autocorrelation function of the semiregular variable Z UMa suggested the presence of various "overtones". This seems to indicate some sort of pulsation which is set up in the atmosphere in such a way that the atmosphere not only oscillates with the frequency of the fundamental but also simultaneously with that of some overtones. However, the results summarised at the end of the preceding section prove that the physical reality of the overtones thrown up by the power spectrum is highly doubtful. If the damping of the superimposed trains is fairly small, so will be the damping in the correlogram and the result is that several spurious overtones appear in the power spectrum. Their value is formal, e. g. in a purely mathematical way they describe the shape of the correlogram up to a certain interval. With the correlogram of Z UMa the damping was found to be small and spurious secondary overtones must turn up.

The periods of about 720 days and about 190 obtained from the correlograms of μ Cephei and Z UMa respectively may therefore very well correspond to the rotational periods of these stars. For a single star the limiting period of rotation is given by the relation $\log P = -1/2 \log \bar{\rho} - 0.936$ where P is the period in days and $\bar{\rho}$ the mean density of the sun $\rho_{\odot} = 1$. Using the periods mentioned before, the mean densities of μ Cephei and Z UMa can be estimated to be $\bar{\rho} = 2{\cdot}09 \times 10^{-8}$ and $\bar{\rho} = 3{\cdot}03 \times 10^{-7}$ respectively. These values are not unacceptable and consequently the rotational hypothesis does not violate other observational data. On the other hand these results do not prove the correctness of the hypothesis. For this it would be necessary (cumf note. 2.) that the separate trains be observed fotoelectrically.

7. If the effect on a stellar atmosphere is such that each disturbance apart from a fundamental damped oscillation sets up additional oscillations with larger frequencies, the overtones in the power spectrum might have real physical significance. How-

ever, it is doubtful whether such overtones could be traced either in the shape of the correlogram or in the corresponding power spectrum.

At any time the term u_t would be equal to the sum of several others so $u_t = u_t' + u_t'' + \cdots\cdots$ while we would have:

$u'_{t+2} = - a' u_{t+1} - b' u_t + \varepsilon'_{t+2}$; $u''_{t+2} = - a'' u_{t+1} - b'' u_t + \varepsilon''_{t+2}$

and consequently

$$u_{t+2} = - (a' + a'' \cdots) u_{t+1} - (b' + b'' \cdots) u_t + \varepsilon_{t+2}$$

Therefore the correlogram would have the normal shape 2 while

$$p = \sqrt{b' + b''} \quad \text{and} \quad \theta = \frac{-(a' + a'' + \cdots)}{2\sqrt{b' + b'' + \cdots}}$$

The representation of this correlogram would lead to exactly the same difficulties in the interpretation of the eventual secondary periodicities.

Another case is when apart from the fundamental periodicity a faint harmonic term is present. The obvious danger is that in the correlogram this secondary periodicity is masked by the dominant one. In order to see to what extent this might be the case we consider the artificial curve

$$u_{t+2} = 1{,}87 \ u_{t+1} - 0{,}94 \ u_t + \varepsilon_{t+2} \qquad \cdots\cdot(1.7)$$

If the random terms ε_{t+2} are taken equal to the two final digits of the consecutive numbers in the telephone directory, we readily obtain the artificial curve $u(t)$. In order to keep the numbers reasonably small, the values $u(t)$ obtained in this way were divided by 10. Next we assume an harmonic term to be present of constant amplitude $A = 20$ and period $P = 24$ and add the terms of this harmonic to those obtained from (1.7). The range of the numerical values of the first series is from about -60 to $+60$ and that of the harmonic from -20 to $+20$. The values u_t were computed for $t = 0$ to $t = 240$. I do not think it necessary to tabulate these values here. It is easy to construct such a table of numbers and apart from small differences due to sampling errors, the correlograms corresponding to any such curve must be the same. Due to the choice of the coëfficient b the damping coëfficient of the correlogram of the undisturbed curve, e. g. the curve unaffected by the harmonic curve must be nearly equal to the corresponding value of the first curve in table 3 and also the period $P = 2\pi/\theta$ must be of the same order of magnitude. From the disturbed curve used in the present paper the correlation coëfficients r_K as given in table 4 were obtained. This correlogram is graphically represented in fig. 3. The rapid decrease

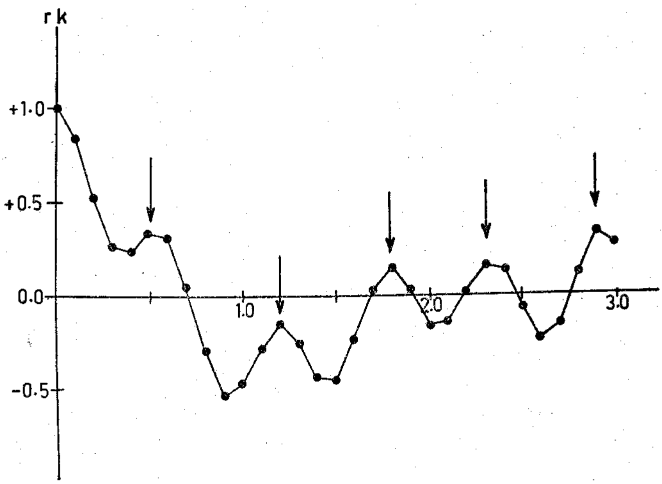

Fig. 3. Correlogram of certificial curve (second ordem autorogressive series plus harmonic term).

of the values r_K from $r_0 = 1.000$ to negative numbers is clearly indicated and also the subsequent rise to the maximum corresponding to the fundamental period.

At the same time the influence of the harmonic term on the shape of the correlogram is very evident. Obviously the harmonic term is only very partially masked by the fundamental periodicity.

The harmonic term is only partially suppressed in the part of the correlogram where K is small, but with the larger values of k the curve is largely determined by the oscillations due to the harmonic term. Even if we had not known about the existence of the harmonic term we would immediately have traced it in the correlogram. The power spectrum corresponding to this correlogram, appears in table 5, while its graph is given in figure 4. In the power spectrum also the harmonic period is clearly indicated. However, I doubt that the harmonic term is better indicated in the power spectrum than in the correlogram.

The present author would have more confidence in the correlogram.

As far as the correlogram is concerned, the case considered here is an unfavourable one. The damping in the correlogram is small. As regards the power spectrum the case considered here is a rather favourable one, because for the oscillations of the harmonic a rather large amplitude viz. A = 20 was adopted.

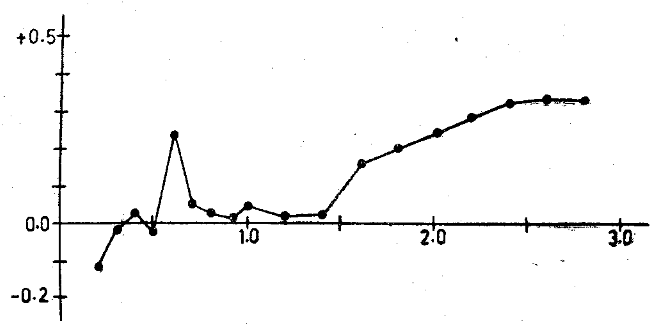

Fig. 4: Power spectrum corresponding to the curve in fig. 3.

8. Our conclusion is that while the autocorrelation analysis is a very reliable method for analysing the light curves of variables, it seems dubious whether much is gained by supplementing this method with a power series analysis of the autocorrelation function. In any case the results of this further analysis must be handled with utter caution. No physical importance at all can be attached to secondary periodicities which are thrown up, unless their reality is confirmed by evidence from other sources. This conclusion is not quite new. Kendall has emphatically warned against accepting as real all periods which turn up in a periodogram analysis. The aforegoing shows that his conclusions remain true also when the power spectrum is derived in this indirect way.

It does certainly not mean that the method is without merits, but merely that the secondary periodicities mainly have formal value only.

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Table 1

Artificial light curves with cyclical variation. The deviations Δm for $p = 1/2$ (curve I); $p = \frac{1}{4}$ (curve II) and $p = \frac{1}{8}$ (curve III) and for $\tau = 0$ and $\tau = \Delta t$. The autocorrelation functions r_K corresponding to these three curves and the power spectrum $\pi(t)$.

K	Curve I				Curve II				Curve III			
(t)	$\tau = 0$	$\tau = \Delta t$	r_K	$\pi(t)$	$\tau = 0$	$\tau = \Delta t$	r_K	$\pi(t)$	$\tau = 0$	$\tau = \Delta t$	r_K	$\pi(t)$
0	−1.00	− .83	+1.000		−1.00	− .91	+1.000		−1.00	− .90	+1.000	
1	− .94	−1.00	+ .961		− .91	−1.00	+ .960		− .89	−1.00	+ .954	
2	− .82	− .94	+ .856		− .77	− .96	+ .847		− .73	− .98	+ .825	
3	− .65	− .79	+ .693		− .60	− .80	+ .673		− .55	− .82	+ .631	
4	− .44	− .43	+ .486		− .40	− .55	+ .456		− .35	− .55	+ .398	
5	− .22	− .28	+ .246		− .19	− .29	+ .216		− .17	− .29	+ .155	
6	− .00	− .02	− .010	+.001	.00	− .04	− .030	+.001	.00	− .05	+ .083	+.001
7	+ .21	+ .23	− .262		+ .17	+ .19	− .265		+ .14	+ .16	− .299	
8	+ .40	+ .46	− .495	+.002	+ .31	+ .38	− .477	+.007	+ .25	+ .32	− .484	+.01[illegible]
9	+ .55	+ .64	− .694		+ .42	+ .52	− .653		+ .33	+ .44	− .633	
10	+ .64	+ .76	− .847		+ .49	+ .62	− .785		+ .36	+ .50	− .741	
11	+ .71	+ .83	− .942		+ .51	+ .66	− .867		+ .38	+ .51	− .806	
12	+ .71	+ .85	− .975	+.010	+ .50	+ .64	− .897	+.041	+ .35	+ .50	− .828	+.081
13	+ .67	+ .80	− .942		+ .45	+ .60	− .867		+ .31	+ .45	− .806	
14	+ .58	+ .70	− .847		+ .39	+ .52	− .785		+ .26	+ .37	− .741	
15	+ .46	+ .56	− .694		+ .30	+ .40	− .653		+ .19	+ .29	− .633	
16	+ .31	+ .39	− .495		+ .20	+ .28	− .265		+ .12	+ .19	− .484	
17	+ .16	+ .20	− .262		+ .10	+ .18	− .477		+ .06	+ .10	− .299	
18	.00	+ .01	− .010		.00	+ .02	− .030		.00	+ .02	− .083	
19	− .15	− .16	+ .216		− .09	− .10	+ .216		− .05	− .06	+ .155	
20	− .28	− .32	+ .488		− .16	− .20	+ .456		− .09	− .11	+ .398	
21	− .38	− .49	+ .693		− .21	− .33	+ .673		− .11	− .24	+ .631	
22	− .46	− .66	+ .856		− .24	− .50	+ .847		− .13	− .41	+ .825	
23	− .49	− .73	+ .961		− .25	− .70	+ .960		− .14	− .64	+ .954	
24	−1.00	− .83	+1.000	+.985	−1.00	− .91	+1.000	+.945	−1.00	− .90	+1.000	+.905

Table 2

Curves $r_K = p^K \cos \theta K$ for $d = p^{24} = \frac{1}{2}$ (curve I); $d = \frac{1}{4}$ (curve II) and $d = 1/8$ (curve III). The corresponding power spectra appear in table 3.

K	r_K			K	r_K		
	I	II	III		I	II	III
0	+1.000	+1.000	+1.000	32	—.197	—.078	—.031
2	+ .817	+ .772	+ .728	34	—.323	—.121	—.046
4	+ .445	+ .396	+ .354	36	—.352	—.125	—.044
6	.000	.000	.000	38	—.287	—.096	—.032
8	— .396	— .314	— .250	40	—.156	—.049	—.015
10	— .643	— .485	— .365	42	.000	.000	.000
12	— .706	— .500	— .354	44	+.140	+.039	+.011
14	— .578	— .386	— .257	46	+.228	+.061	+.016
16	— .314	— .198	— .125	48	+.248	+.062	+.016
18	.000	.000	.000	50	+.203	+.048	+.011
20	+ .280	+ .157	+ .088	52	+.110	+.025	+.005
22	+ .457	+ .243	+ .129	54	.000	.000	.000
24	+ .500	+ .250	+ .125	56	—.098	—.019	—.004
26	+ .407	+ .192	+ .091	58	—.161	—.030	—.006
28	+ .222	+ .099	+ .044	60	—.176	—.031	—.006
30	.000	.000	.000				

Table 3

Power spectra $\pi(f)$ corresponding to the curves I, II and III in table 2. For N in the Relation (3) the values N = 30; 40; 50; 60; 24 and 48 have been used.

Curve I

N	30	40	50	60	24	48	30	40	50	60	24	48
2	+.041	+.023	+.024	+.017	+.021	+.015	+ .058	+ .036	+ .043	+ .034	+ .028	+ .027
4	+.036	+.024	+.018	+.015	+.022	+.015	+ .051	+ .038	+ .032	+ .030	+ .030	+ .027
6	+.029	+.033	+.027	+.019	+.023	+.017	+ .041	+ .052	+ .048	+ .038	+ .031	+ .030
8	+.045	+.037	+.038	+.023	+.031	+.022	+ .064	+ .058	+ .068	+ .046	+ .042	+ .039
10	+.019	+.053	+.024	+.024	+.119	+.005	+ .027	+ .083	+ .043	+ .048	+ .160	+ .009
12	+.085	+.015	+.044	+.023	+.031	+.024	+ .120	+ .024	+ .078	+ .046	+ .042	+ .043
14	+.021	+.079	+.034	+.010	—.045	+.062	+ .030	+ .124	+ .061	+ 020	— .060	+ .111
16	+.047	+.025	+.068	+.073	+.167	+.042	+ .067	+ .039	+ .121	+ .146	+ .224	+ .075
18	+.252	+.101	+.045	+.040	+.449	+.052	+ .356	+ .159	+ .080	+ .080	+ .602	+ .093
20	+.487	+.345	+.243	+.172	+.646	+.291	+ .688	+ .542	+ .432	+ .345	+ .866	+ .521
22	+.618	+.536	+.461	+.396	+.734	+.475	+ .873	+ .843	+ .820	+ .795	+ .984	+ .850
24	+.708	+.636	+.562	+.498	+.746	+.559	+1.000	+1.000	+1.000	+1.000	+1.000	+1.000
26	+.691	+.594	+.497	+.422	+.705	+.498	+ .976	+ .934	+ .884	+ .848	+ .945	+ .891
28	+.625	+.486	+.368	+.287	+.641	+.383	+ .921	+ .764	+ .655	+ .576	+ .859	+ .685
30	+.534	+.358	+.236	+.169	+.565	+.265	+ .754	+ .563	+ .420	+ .339	+ .758	+ .474

Curve II

2	+.033	+.023	+.020	+.015	+.031	+.019	+ .064	+ .054	+ .057	+ .050	+ .054	+ .051
4	+.038	+.026	+.021	+.017	+.034	+.021	+ .073	+ .061	+ .059	+ .057	+ .059	+ .058
6	+.036	+.031	+.024	+.019	+.037	+.023	+ .070	+ .073	+ .060	+ .063	+ .064	+ .064
8	+.096	+.078	+.064	+.052	+.105	+.062	+ .186	+ .184	+ .181	+ .173	+ .182	+ .171
10	+.038	+.038	+.029	+.024	+.099	+.025	+ .074	+ .089	+ .082	+ .080	+ .171	+ .069
12	+.079	+.049	+.040	+.035	+.061	+.047	+ .150	+ .115	+ .113	+ .117	+ .106	+ .130
14	+.045	+.057	+.039	+ 029	+.024	+.047	+ .087	+ .134	+ .110	+ .096	+ .041	+ .130
16	+.121	+.085	+.079	+.069	+.205	+.076	+ .234	+ .200	+ .223	+ .230	+ .354	+ .210
18	+.256	+.163	+.121	+.102	+.386	+.128	+ .495	+ .384	+ .342	+ .340	+ .667	+ .354
20	+.396	+.292	+.225	+.182	+.511	+.241	+ .766	+ .687	+ .636	+ .607	+ .883	+ .666
22	+.488	+.389	+.320	+.268	+.568	+.332	+ .935	+ .915	+ .904	+ .893	+ .981	+ .917
24	+.517	+.425	+.354	+.300	+.579	+.362	+1.000	+1.000	+1.000	+1.000	+1.000	+1.000
26	+.506	+.405	+.330	+.267	+.556	+.338	+ .979	+ .953	+ .953	+ .890	+ .960	+ .934
28	+.461	+.350	+.274	+.226	+.509	+.285	+ .892	+ .824	+ .774	+ .753	+ .879	+ .787
30	+.421	+.299	+.226	+.184	+.477	+.240	+ .814	+ .706	+ .639	+ .613	+ .824	+ .663

Curve III

2	+.034	+.026	+.021	+.017	+.038	+.021	+ .085	+ .083	+ .082	+ .079	+ .081	+ .078
4	+.037	+.027	+.022	+.018	+.040	+.022	+ .092	+ .086	+ .086	+ .085	+ .085	+ .082
6	+.039	+.031	+.021	+.015	+.044	+.025	+ .097	+ .099	+ .082	+ .071	+ .094	+ .094
8	+.047	+.035	+.029	+.024	+.051	+.029	+ .127	+ .112	+ .114	+ .112	+ .109	+ .108
10	+.054	+.045	+.034	+.029	+.090	+.035	+ .135	+ .144	+ .134	+ .136	+ .192	+ .131
12	+.082	+.056	+.046	+.038	+.083	+.047	+ .204	+ .179	+ .180	+ .178	+ .177	+ .175
14	+.099	+.083	+.069	+.056	+.109	+.074	+ .247	+ .265	+ .271	+ .262	+ .232	+ .276
16	+.155	+.113	+.093	+.078	+.220	+.096	+ .387	+ .361	+ .365	+ .365	+ .469	+ .458
18	+.249	+.176	+.138	+.115	+.341	+.144	+ .621	+ .562	+ .541	+ .537	+ .727	+ .537
20	+.332	+.249	+.197	+.163	+.423	+.207	+ .819	+ .796	+ .773	+ .762	+ .902	+ .773
22	+.381	+.296	+.239	+.200	+.459	+.249	+ .951	+ .942	+ .941	+ .935	+ .979	+ .929
24	+.401	+.314	+.255	+.214	+.469	+.268	+.1000	+1.000	+1.000	+1.000	+1.000	+1.000
26	+.393	+.303	+.244	+.204	+.455	+.258	+ .980	+ .968	+ .957	+ .953	+ .970	+ .944
28	+.379	+.285	+.226	+.188	+.379	+.236	+ .945	+ .910	+ .886	+ .879	+ .808	+ .881
30	+.349	+.254	+.200	+.166	+.314	+.210	+ .870	+ .812	+ .784	+ .776	+ .771	+ .784

Table 4

The autocorrelation coefficients corresponding to a time series for which the individual terms are obtained from the second order autoregressive scheme (1.7) after addition of an harmonic term

K	r_k
0	+1.000
1	+ .846
2	+ .519
3	+ .286
4	+ .257
5	+ .348
6	+ .330
7	+ .070
8	— .291
9	— .513
10	— .487
11	— .280
12	— .139
13	— .230
14	— .412
15	— .425
16	— .219
17	+ .029
18	+ .177
19	+ .023
20	— .166
21	— .155
22	+ .005
23	+ .185
24	+ .145
25	— .066
26	— .208
27	— .137
28	+ .120
29	+ .339

Table 5.

The power spectrum corresponding to the autocorrelation function in table 4

K	$\pi(t)$
2	— .113
3	— .011
4	+ .032
5	— .016
6	+ .242
7	+ .051
8	+ .034
9	+ .027
10	+ .057
12	+ .022
14	+ .026
16	+ .162
18	+ .205
20	+ .258
22	+ .299
24	+ .330
26	+ .345
28	+ .348
30	+ .343