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Série A. Mathématiques-Physique

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On The Bloch-Landau Constants

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Özet: Bloch-Landau Sabitleri hakkında. Bu yazı, Giriş hariç, iki kısımdan ibarettir. Birinci kısımda $\mathfrak{L}$ Sabiti hakkında evvelce elde edilmiş olan neticeler hatırlatılıyor. (K) Birim dairesinde, köşeleri çember üzerinde olan, kenarları çembere dik sıfır açılı dairesel üçgenler göz önüne alınıyor.

Bu üçgenler, kenarlarına nazaran simetrileri tam olarak w-düzlemini örten bir (T_w) üçgenine konform tasvir ediliyor. Bilindiği gibi üç tip (T_w) üçgeni vardır, bunlar eşkenar, iki kenarı eşit dik açılı ve bir eşkenar üçgenin yarısını teşkil eden üçgenlerdir. Bu üçgenleri sıra ile $(T_w)_1$, $(T_w)_2$, $(T_w)_3$ ile gösteriyoruz.

Yukarda tarifi verilen tasvirlerden mütevellit normalize fonksiyonların $\mathfrak{L}$ sınıfı için aşağıdaki kat'i netice elde ediliyor.

$\mathfrak{L}$ sınıfı dahilinde bir ve birtek extremal fonksiyon vardır. Bu fonksiyon eşkenar sıfır açılı üçgeni $(T_w)_1$ eşkenar üçgenine konform tasvir eder.

İkinci kısımda, sıfır açılı üçgenlerin, bunlara modüler üçgen de denir, yerine Schwarz üçgenleri gözönüne alınıyor. Schwarz üçgeni diye kenarları Birim çembere dik π/p, π/q, π/r, p, q, r pozitif tam sayı, ile gösterilen iç açıları $1/p + 1/q + 1/r < 1$ şartını sağlıyan üçgene denir. Schwarz üçgenlerinin $(T_w)_n$ $n = 1, 2, 3$ üçgenlerine konform tasvirlerinden mütevellit normalize analitik fonksiyonların sınıfına $\mathfrak{B}$ diyelim. Aynen yukardaki gibi gösteriyoruz ki $\mathfrak{B}$ sınıfı dahilinde bir ve birtek ekstremal fonksiyon vardır. Bu fonksiyon eşkenar $\pi/6$ açılı Schwarz üçgenini $(T_w)_1$ eşkenar üçgenine konform tasvir eder.

Bu teoremin ispatı yukardakinden farklı olarak Klasik Schwarz Lemmasının Birim dairesinde tarifli fakat orada üniform olmıyan analitik fonksiyonlara teşmili olan bir Lemmaya dayanmaktadır.

Introduction. The paper is divided into two parts. In the first part we consider the Class $\mathfrak{L}$ of normalized analytic functions which map conformally a zero-angled circular triangle, also called a modular triangle, inscribed in the unit circle onto a straight triangle whose repeated reflections in the sides just fill the whole plane. It is well known that there are three types of such triangles, viz., the equilateral triangle, the isoceles rectangular triangle an the half of an equilateral triangle. These triangles will be designated by $(T_w)_1$, $(T_w)_2$, $(T_w)_3$ respectively. The following *Uniqueness Theorem* is proved (§ 2.4). The normalized analytic function which maps conformally the modular equilateral triangle onto the straight equilateral triangle is the only extremal function within the Class $\mathfrak{L}$.

In the second part, the Class $\mathfrak{L}$ is replaced by the Class $\mathfrak{B}$ consisting of normalized analytic functions which map conformally a circular triangle orthogonal to and inside the unit circle with interior angles π/p, π/q. π/r satisfying the condition $1/p + 1/q + 1/r < 1$, where p, q, r are positive integers, also called a Schwarz' triangle, onto $(T_w)_n$, $n = 1,2,3$. We prove similarly that the normalized analytic function which maps conformally the equiangular Schwarz triangle with interior angles equal to $\pi/6$ onto the straight equilateral triangle is the only extremal function within the Class $\mathfrak{B}$. The proof of this theorem is based upon a generalization of the Classical Schwarz'Lemma to non uniform analytic functions defined in the unit circle and which is an extension of the results obtained by L. Ahlfors and Z. Nehari.

1. GENERALITIES

1.1. The Function Space of The Modular Triangles. We designate by (ABC) the Fundamental triangle in the τ-plane with boundary consisting of the two positive half-rays $BA : \Re\tau = 0$, $CA : \Re\tau = 1$ and the upper semi circle $BEC : |2\tau - 1| = 1$. The affixes of the vertices A, B, C being $\tau = \infty$, $\tau = 0$, $\tau = 1$ respectively.

Let $\alpha \in (ABC)$ be any interior or boundary point different from the vertices. To every α the substitution

$$(1.1) \qquad z = \frac{\tau - \alpha}{\tau - \bar{\alpha}}$$

associates in the unit circle $(K) : |z| < 1$, a Modular triangle $(ABC)_\alpha$ containing $z = 0$ in the interior or on the boundary and which is the transformed of (ABC) by (1.1), the vertex A corresponding to $\tau = \infty$ being at $z = 1$. Conversely to every such triangle $(ABC)_\alpha$ with the vertex A at $z = 1$ there corresponds by means of (1.1) a single point in (ABC) We thus obtain

Theorem 1.1 The totality of the Modular triangles inscribed in (K), with centre lying in the interior or on the boundary can, save for an arbitrary rotation round the centre, be represented on the Fundamental triangle.

Note that any of the Modular triangles $(ABC)_\alpha$ can be chosen as a Fundamental triangle. In this case (1.1) is replaced by a linear substitution which leaves fixed the unit circle (K). Such a Fundamental triangle is called a Function Space of the Modular triangles.

1.2. **The Modular Triangles**. By means of the circles, centre at $\tau = 0$, $\tau = 1$, with radius unity and the line $\Re\tau = \frac{1}{2}$ we may divide (ABC) into six subtriangles FAG, FGB, FBE, FEC, FCH, FHA read in the positive direction. Suppose α is in FAG. To α is associated by (1.1) the Modular triangle $(ABC)_\alpha$ with vertex A at $z = 1$. By rotating this triangle round $z = 0$ in the positive direction so as to carry the vertex B then C over into $z = 1$, we thus obtain the triangles $(A'B'C')_{\alpha'}$, $(A''B''C'')_{\alpha''}$ to which correspond in (ABC) the points

$$\alpha' = \frac{1}{1-\alpha} \qquad \alpha'' = \frac{\alpha - 1}{\alpha} \tag{1.2}$$

respectively Thus to the above rotations correspond in the τ-plane the substitutions (1.2) which leave fixed (ABC) and take FAG into FBE, FCH respectively. Hence together with the unit substitution the substitutions (1.2) form a Group which is isomorphic to the Group of rotations of the triangle $(ABC)_\alpha$ which take B then C onto $z = 1$ in such a way that these vertices would correspond each time to $\tau = \infty$ by means of (1.1).

Next, to the reflections of these triangles in the real axis correspond in order the points

$$1 - \bar{\alpha}, \ \frac{1}{\bar{\alpha}}, \ \frac{\bar{\alpha}}{\bar{\alpha} - 1}$$

Setting $1 - \bar{\alpha} = \beta$ we obtain again

$$\beta' = \frac{1}{1-\beta} \qquad \beta'' = \frac{\beta - 1}{\beta}$$

which transform FHA into FGB and FEC respectively.

We thus arrive at the following definition.

Definition. Two modular triangles $(ABC)_\alpha$, $(ABC)_{\alpha'}$ are said to be equivalent if and only if α and α' are related by one of the above substitutions.

On the other hand two Modular triangles are said to be equivalent in the larger sense if and only if by means of a finite number of reflections in the sides one obtains two Modular triangles $(ABC)_\alpha$, $(ABC)_{\alpha'}$ which are equivalent.

1.3. **Characterization of The Modular Triangles**

(i) α is an interior point of (ABC). In this case $z=0$ is an interior point of $(ABC)_\alpha$.

1. If α is on F, i.e., at the intersection of the lines which divide (ABC) into six subtriangles, then the three sides of $(ABC)_\alpha$ are equal. In the sequel this very special triangle will be denoted by $(ABC)_F$.

2. If α is on a line of division but is different from F then $(ABC)_\alpha$ has only two equal sides.

3. If for α neither one of the above cases hold then the three sides of $(ABC)_\alpha$ are unequal.

(ii) α is a boundary point of (ABC). In this case $z=0$ is on the boundary of $(ABC)_\alpha$.

1. α is on the line $\Re\tau=0$. In this case $z=0$ is on the side AB of $(ABC)_\alpha$. If α is above $\tau=i$ then the vertex C of $(ABC)_\alpha$ is on the arc of circle joining $z=-1$ and $z=-i$. If α is below $\tau=i$, C is on the arc of circle joining $z=-i$ and $z=1$. Finally if α coincides with $\tau=i$, C obviously coincides with $z=-i$. It is seen that in the last case $(ABC)_\alpha$ has two sides equal. This is in accordance with (*i*), 2. In short when α describes $R\tau=0$ then C describes the lower semi circle.

2. α is on the upper semi circle $|2\tau-1|=1$. In this case $z=0$ is on the side BC of $(ABC)_\alpha$.

3. α is on the line $\Re\tau=1$. In this case $z=0$ is on the side CA of $(ABC)_\alpha$.

2. PART I

2.1. **Introduction.** The Constant $\mathfrak{L}$ has been defined by Landau who gave at the same time an upper estimate [1]. He obtained the above value by mapping the modular triangle $(ABC)_G$ say, conformally onto the straight equilateral triangle $(T_\omega)_1$. Rademacher improved Landau's estimate by mapping $(ABC)_F$ conformally onto $(T_w)_1$ [2]. As a reason for this improvement he notices the full symmetry of $(ABC)_F$ over $(ABC)_G$.

The main object of this PART I is to prove Rademacher's conjecture. More precisely we shall solve the following extremal problem.

Consider the subclass $\mathfrak{L}_1 \subset \mathfrak{L}$ of normalized analytic functions obtained by mapping conformally a Modular triangle inscribed in the unit circle (K) onto the straight equilateral triangle $(T_w)_1$. It is clear that it will suffice to consider only Modular triangles such as $(ABC)_\alpha$. The problem consists in finding a triangle $(ABC)_\alpha$ which yields the extremal function within the Class $\mathfrak{L}_1$. Obviously in place of (ABC) as a Function Space we may chose any $(ABC)_\alpha$, say $(ABC)_F$. Accordingly we replace α by $\zeta \in (ABC)_F$. Then to every ζ there corresponds a definite Modular triangle $(ABC)_\zeta$, save for an arbitrary rotation. To $\zeta = 0$ will correspond of course $(ABC)_F$. Let

$$w = f^*(z) = z + \cdots$$

be the function which maps $(ABC)_F$ onto $(T_w)_1$. Then the function which maps $(ABC)_\zeta$ conformally onto $(T_w)_1$ is given by

$$w = f^*(\zeta) + |f^{*\prime}(\zeta)|(1 - |\zeta|^2)z + \cdots$$

The extremal problem consists therefore in finding the value ζ for which $|f^{*\prime}(\zeta)|(1 - |\zeta|^2)$ is maximal. We shall show that this expression is maximal for the single value $\zeta = 0$ In other words if the Function Space is (ABC) then the extremal function is unique and corresponds to the point $F \in (ABC)$.

On the other hand if in place of $(T_w)_1$ we consider $(T_w)_2$ or $(T_w)_3$ we arrive at the same result within the subclasses $\mathfrak{L}_2$, $\mathfrak{L}_3$. i.e., in each case the extremal function occurs if and only if α is at F.

Finally designating by $(L_F)_1$, $(L_F)_2$, $(L_F)_3$ the radii of the "maximal„ circles circumscribed to $(T_w)_1$, $(T_w)_2$, $(T_w)_3$ respectively it is found that

$$(L_F)_1 < (L_F)_2 < (L_F)_3.$$

This solves completely the extremal problem within the Class $\mathfrak{L}$.

2.2. **The Mapping Function.** Let $(T_w)_1$ be a straight equilateral triangle in the w plane. The function $w = f(z)$ which realizes the conformal mapping of $(ABC)_\alpha$ onto $(T_w)_1$ in such a way that the vertices correspond may be obtained as follows:

Let

$$w = \mu \int_{\xi_0}^{\xi} \xi^{-2/3}(1-\xi)^{-2/3}\, d\xi$$

where $|\mu|$ is the positive scale constant of $(T_w)_1$, be the function which maps $\Im\xi > 0$ onto $(T_w)_1$. Let $k^2 = \lambda(\tau) = \xi$ be the elliptic modular function which maps (ABC) onto $\Im\xi > 0$. Finally let

$$z = \frac{\tau - \alpha}{\tau - \bar{\alpha}}$$

be the substitution which maps (ABC) onto $(ABC)_\alpha$.

Combining these three transformations we obtain the required analytic function which maps conformally $(ABC)_\alpha$ onto $(T_w)_1$. Clearly $w = f(z)$ depends on the two parameters α and μ.

As it is well known the mapping function can be continued analytically by reflections in the sides of $(ABC)_\alpha$. The repeated images of $(ABC)_\alpha$ will just fill the unit circle (K), whereas the corresponding repeated images of $(T_w)_1$ will build up an infinitely many sheeted (∞^2) Riemann surface over the w-plane in which the vertices of $(T_w)_1$ and its reflections form a regular point lattice where all these vertices become branch points of infinite order. Evidently the circumference of (K) is a natural boundary for $w = f(z)$. On the other hand the image of any circle interior and concentric to (K) cannot cover any of the branch points of the above Riemann surface over the w-plane. Hence $w = f(z)$ cannot cover a circle greater than the circle circumscribed to $(T_w)_1$.

2.3. **Normalization.** We now normalize the mapping by means of the condition $|f'(0)| = 1$ and obtain the condition between α and μ as follows:

Let K and iK' be the quarter periods of the Jacobian elliptic functions of modulus k. We have

$$(dw/d\xi)_{\xi = k^2} = \mu(k^2(1-k^2))^{-2/3}$$

$$(d\tau/d\xi)_{\xi = k^2} = \left(\frac{d\, iK'/K}{d\xi}\right)_{\xi = k^2} = \frac{\pi}{4iK^2k^2(1-k^2)}$$

$$(dz/d\tau)_{\tau = \alpha} = -i/(2\Im\alpha)$$

Carrying these expressions over into the identity

$$dw/dz = (dw/d\xi)\,(d\xi/d\tau)\,(d\tau/dz)$$

we obtain

$$\mu = -\frac{\pi}{8\Im\alpha K^2 (k^2(1-k^2))^{1/3}}$$

But

$$K^2 = \tfrac{1}{4}\pi^2 \prod_{n=1}^{\infty}((1-q^{2n})^4 (1+q^{2n-1})^8)$$

where

$$q = e^{\pi i \alpha}, \qquad |q| < 1$$

and

$$(k^2(1-k^2))^{1/3} = \frac{2(2q)^{1/3}}{\prod_{n=1}^{\infty}(1+q^{2n-1})^8}$$

together with the relation

$$\vartheta_1'(0 \mid \alpha) = \vartheta_2(0 \mid \alpha)\,\vartheta_3(0 \mid \alpha)\,\vartheta_4(0 \mid \alpha)$$

and

$$\prod_{n=1}^{\infty}(1-q^{2n})^4 = \frac{\vartheta_1'^{4/3}(0 \mid \alpha)}{2(2q)^{1/3}},$$

the expression for $|\mu|$ becomes simply

$$|\mu| = \frac{1}{2\pi\Im\alpha\,|\vartheta_1'(0 \mid \alpha)|^{4/3}}$$

Denoting by $(L_\alpha)_1$ the radius of the circle in the w-plane covered by $w = f(z)$ we obtain after normalization

$$(L_\alpha)_1 = \frac{3^{\frac{1}{2}}\,B(1/3,\ 1/2)}{6\pi\Im\alpha\,|\vartheta_1'(0 \mid \alpha)|^{4/3}} \tag{2.3}$$

which is a function of the position of α in (ABC).

2.4. **Reduction of the Function Space. Fundamental Lemma.** The symmetry character of $(T_w)_1$ together with § 1.2 shows that it will suffice to let vary $\alpha = u + iv$ in one of the six subtriangular regions, say FAG. In other words (2.3) is invariant under the substitutions

$$\alpha \to 1-\bar{\alpha},\quad \frac{1}{1-\alpha},\quad \frac{\bar{\alpha}}{\bar{\alpha}-1},\quad \frac{\alpha-1}{\alpha},\quad \frac{1}{\alpha}$$

Let $(L_\beta)_1$ be the radius of the circle corresponding to a point β

on FA ($u = \frac{1}{2}$, $v \geq 3^{\frac{1}{2}}/2$). For any given $v \geq 3^{\frac{1}{2}}/2$ we consider the ratio

$$(L_\beta)_1/(L_\alpha)_1 = \prod_{n=1}^{\infty} \frac{|1 - q^{2n}|^4}{(1-(-)^n q_0^{2n})^4} \qquad q = e^{\pi i u} q_0$$

If we put

$$|1 - q^{2n}|^2 = 1 - 2q_0^{2n} \cos 2n\pi u + q_0^{4n}$$

and make the change of variable $u = \frac{1}{2} - u^*$ we obtain

$$(L_\beta)_1/(L_\alpha)_1 = \prod_{n=1}^{\infty} \left(1 + (-)^n \frac{4q_0^{2n} \sin^2 n\pi u^*}{(1-(-)^n q_0^{2n})^2}\right)^2 = P^2$$

or what is the same thing

$$P = \prod_{n=1}^{\infty} \left(1 - \frac{4q_0^{4n-2} \sin^2(2n-1)\pi u^*}{(1+q_0^{4n-2})^2}\right)\left(1 + \frac{4q_0^{4n} \sin^2 2n\pi u^*}{(1-q_0^{4n})^2}\right)$$

Because of the inequality

$$|\sin 2n\pi u^*| \leq 2n \sin \pi u^*$$

it is clear that

$$P \leq \left(1 - \frac{4q_0^2 \sin^2 \pi u^*}{(1+q_0^2)^2}\right) \prod_{n=1}^{\infty} \left(1 + \frac{16n^2 q_0^{4n} \sin^2 \pi u^*}{(1-q_0^{4n})^2}\right)$$

For simplicity let us designate the first factor by $1 - a^2$ and the infinite product by $\prod_{n-1}^{\infty}(1 + b_{4n}^2)$, so that the above inequality becomes

$$P \leq (1 - a^2) \prod_{n=1}^{\infty} (1 + b_{4n}^2)$$

Now it is immediately seen that for $v = 3^{\frac{1}{2}}/2$ and à fortiori for $v > 3^{\frac{1}{2}}/2$ a^2/b_4^2 for $n=1$, is greater than 1 for all $0 < u^* \leq \frac{1}{2}$.

Hence

$$(1 - a^2)(1 + b_4^2) \leq 1 - a^2 + b_4^2 \leq 1$$

The equality occurs if and only if $u^* = 0$.

Thus

$$P \leq (1 - a^2 + b_4^2) \prod_{n=2}^{\infty} (1 + b_{4n}^2)$$

Similarly $(a^2 - b_4^2)/b_8^2 > 1$, $n = 2$. Hence

$$(1 - (a^2 - b_4^2))(1 + b_8^2) \leq 1 - (a^2 - b_4^2) + b_8^2 \leq 1$$

and

$$P \leq (1 - a^2 + b_4^2 + b_8^2) \prod_{n=3}^{\infty} (1 + b_{4n}^2)$$

In general

$$S_N = 1 - a^2 + b_4^2 + b_8^2 + \cdots + b_{4N}^2 \leq 1$$

and

$$P \leq (1 - a^2 + b_4^2 + b_8^2 + \cdots + b_{4N}^2) \prod_{n=N+1}^{\infty} (1 + b_{4n}^2)$$

For $N \to \infty$ we have

$$P \leq 1 - a^2 + \sum_{n=1}^{\infty} b_{4n}^2 \leq 1$$

since by construction any partial sum $S_N \leq 1$.

Obviously the equality, i. e., $P = 1$ occurs if and only if $u^* = 0$.

We have thus shown that the "maximal„ circle must necessarily correspond to some point β on FA. We maintain that it actually coincides with F. This is immediately seen by comparing

$$(L_F)_1/(L_\beta)_1 = \frac{q_{v_\beta}^{1/3} \prod_{n=1}^{\infty} (1 + q_{v_\beta}^{4n-2})(1 - q_{v_\beta}^{4n})}{q_{v_F}^{1/3} \prod_{n=1}^{\infty} (1 + q_{v_F}^{4n-2})(1 - q_{v_F}^{4n})}$$

In fact we remark that each factor $(1 + q_{v_\beta}^{4n-2})(1 - q_{v_\beta}^{4n})$ for any positive n (i. e., for $n=1$ and à fortiori for $n>1$) decreases toward 1 as v increases from a certain value $v_0 < v_F$. Consequently the infinite product

$$\prod_{n=1}^{\infty} (1 + q_{v_\beta}^{4n-2})(1 - q_{v_\beta}^{4n})$$

decreases toward 1 from the value $\prod_{n=1}^{\infty} (1 + q_{v_F}^{4n-2})(1 - q_{v_F}^{4n})$ as v increases from $v = v_F = 3^{\frac{1}{2}}/2$. Hence

$$\prod_{n=1}^{\infty} \frac{(1 + q_{v_\beta}^{4n-2})(1 - q_{v_\beta}^{4n})}{(1 + q_{v_F}^{4n-2})(1 - q_{v_F}^{4n})}$$

is less or equal to 1 and decreases steadily as v increases. On the other hand

$$q_{v_\beta}^{1/3} / q_{v_F}^{1/3} \leq 1$$

and decreases in the same interval $(v \geq 3^{\frac{1}{2}}/2)$.

Consequently $(L_F)_1/(L_\beta)_1 \leq 1$ and the equality occurs if and only if $v = 3^{\frac{1}{2}}/2$. We collect the results in the Fundamental Lemma stated as.

Theorem 2.4. In the Class $\mathfrak{L}_1$ there exists a unique analytic function $w = f^*(z) = z + \cdots$ which covers in the w-plane the least maximal circle whose radius is

$$(L_F)_1 = \frac{B(1/3,\ 1/3)}{3\pi \mid \vartheta_1'^{4/3}(0 \mid e^{\pi i/3}) \mid} \tag{2.4}$$

2.5. **An expression for (2.4) in terms of Γ-Function.**

For $u = \frac{1}{2}(q = iq_0)$ we have

$$\mid \vartheta_1'(0 \mid \tfrac{1}{2},\ v) \mid^{4/3} = 2^{4/3} q_0^{1/3} \prod_{n=1}^{\infty} (1 - (iq_0)^{2n})^4$$

Put

$$S = \prod_{n=1}^{\infty} (1 - (i)^{2n} q_0^{2n}) \prod_{n=1}^{\infty} (1 - (-)^n q_0^{2n})$$

or what is the same thing

$$S = \prod_{n=1}^{\infty} (1 + q_0^{4n-2}) \prod_{n=1}^{\infty} (1 - q_0^{4n})$$

Letting $v = K'/K$ we have

$$\prod_{n=1}^{\infty} (1 - q_0^{4n})^4 = 2^{-2/3} \pi^{-2} q_0^{-2/3} k^{4/3} k'^{1/3} K^2$$

and

$$\prod_{n=1}^{\infty} (1 + q_0^{4n-2})^4 = 2^{1/3} q_0^{1/3} (1 - k')^{-1/3} k'^{-1/6} (1 + k')^{2/3}$$

The product of these two infinite products gives S^4. Hence,

$$\mid \vartheta_1'^{4/3} \mid = 2\pi^{-2} (1 - k')^{-1/3} k^{4/3} k'^{1/6} (1 + k')^{2/3} K^2$$

and therefore

$$(L_\beta)_1 = \frac{3^{\frac{1}{2}} B(1/3,\ 1/3)}{12 k^{2/3} k'^{1/6} (1 + k') KK'}$$

The expression in the denominator can be reduced further by means of Landen's transformation, viz,

$$2K'/K = M'/M, \quad 2K' = (1+m) M', \quad K = (1+m) M, \quad k' = \frac{1-m}{1+m}.$$

Hence,

$$(2.5) \qquad (L_\beta)_1 = \frac{\pi 3^{\frac{1}{2}} B(1/3, 1/3)}{3 \cdot 4^{4/3} (mm')^{1/3} MM'}$$

If in (2.5) we put $M'/M = 3^{\frac{1}{2}}$ and compute the complete integrals M and M′ in terms of Γ-Function we get

$$(L_F)_1 = 3^{\frac{1}{2}} \Gamma(2/3) \Gamma^2(1/3) \Gamma^{-2}(1/6)$$

Numerically we have

$$(L_F)_1 = .5432589\ldots$$

Using the well known identity

$$\Gamma(z)\Gamma(1-z) = \pi \operatorname{cosec} \pi z$$

$\Gamma(2/3)$ can be calculated in terms of $\Gamma(1/3)$ (just put $z = 1/3$ in the above identity). Replacing this value of $\Gamma(2/3)$ in $(L_F)_1$ by the one in term of $\Gamma(1/3)$ we obtain Rademacher's formula.

2.6. **Some Applications**. The radius of a circle corresponding to a point α on the side AB of (ABC) is easily seen to be

$$(L_\alpha)_1 = \frac{3^{\frac{1}{2}} \pi B(1/3, 1/3)}{24 KK' (kk')^{2/3}}$$

To obtain the radius of the circle corresponding to G we put $K'/K = 1$ and $k^2 = \frac{1}{2}$. In terms of Γ-Function it is

$$(L_G)_1 = 2^{2/3} \pi \Gamma^3(1/3) \Gamma^{-4}(1/4)$$

This is the formula obtained by Landau. On the other hand we know that

$$(L_G)_1 = (L_E)_1 = (L_H)_1$$

Hence putting in (2.3) $\alpha = \frac{1}{2} + i\frac{1}{2} = 2^{-1/2} e^{\pi i/4}$ and comparing with the above formula we get

$$|\vartheta_1'(0 | 2^{-\frac{1}{2}} e^{\pi i/4})| = \pi^{-3/2} 2^{-\frac{1}{2}} 3^{-3/8} \Gamma^{-3/4}(2/3) \Gamma^{-3/4}(1/3) \Gamma^3(1/4)$$

Similarly comparing the expression of $(L_F)_1$ in terms of Γ-Function with (2.4) we obtain

$$|\vartheta_1'(0 | e^{\pi i/3})| = \pi^{-3/4} 3^{-9/8} \Gamma^{-3/2}(2/3) \Gamma^{3/2}(1/6)$$

2.7. Solution of the Extremal Problem for the Cases $(T_w)_n$, $n = 2,3$. It is immediately obtained by Topological arguments, i. e., let

$$w = f_1(z) = z + \cdots$$

be the function which maps $(ABC)_F$ onto $(T_w)_1$. We have for all $\zeta \in (ABC)_F$

$$w = f_1(\zeta) + | f_1'(\zeta) | (1 - | \zeta |^2) z + \cdots$$

which maps $(ABC)_\zeta$ onto $(T_w)_1$. When normalized we have

$$(L_\zeta)_1 = \frac{(L_F)_1}{| f'(\zeta) | (1 - | \zeta |)^2}$$

Similar arguments and expressions hold when the index 1 is replaced by n. Now, as $(T_w)_1$ is deformed continuously into $(T_w)_n$, $(L_F)_1$ goes over into $(L_F)_n$ whereas $| f_1'(\zeta) | (1 - | \zeta |^2)$ is deformed continuously into $| f_n'(\eta) | (1 - | \eta |^2)$. Hence $(L_\zeta)_1$ is mapped topologically onto $(L_\eta)_n$. Hence the extremal function within the Class $\mathfrak{L}_n$ corresponds to a single point in $(ABC)_F$. To show that this point is F we may argue as follows. We know that the totality of the radii $(L_\zeta)_1$ remains unchanged say under the rotation φ of $(ABC)_F$ through an angle of $120°$ round the centre $\zeta = 0$. Consequently the totality of the radii $(L_\eta)_n$ must be also unchanged under a mapping ψ of $(ABC)_F$ onto itself in which the corners are mapped onto the corners since infinite radius must correspond to vertices, and also that ψ is the transformed of φ under the deformation. Thus ψ has a single fixed point which is necessarily $\eta = 0$ since ψ stands for a mapping of the set of triangles $(ABC)_\eta$ onto itself in which $(ABC)_F$ is mapped onto itself.

Finally it is clear that the minimum occurs for $\eta = 0$ for, being uniquely determined it should occur at a point which must be fixed under ψ.

We have shown at the same time that it will suffice to let vary the parameter α in one of the six subtriangles.

We may verify this result analytically as follows. The elliptic modular function $\xi = k^2 = \lambda(\tau)$ which maps (ABC) conformally onto the upper half plane $\Im \xi > 0$ turns the triangles FAG, FGB, FBE, FBC, FCH, FHA over into F*A*G*, F*G*B*, F*B*E*, F*B*C*, F*C*H*, F*H*A*. The boundary of these triangles consists of the real axis of the ξ-plane, the line $\Re\xi = \frac{1}{2}$ and the upper semi circles of radius 1 and centre at $\xi = 0$, $\xi = 1$. We may and shall suppose α varying in FHC. The vertices F, C, H correspond in order to the vertices F*, C*, H* with affixes $\xi = \frac{1}{2} + i3^{\frac{1}{2}}/2$, $\xi = \infty$, $\xi = -1$ respectively.

Clearly,

$$|k_\alpha^2| \geq |k_F^2| \tag{2.6.1}$$

The equality occurs if and only if α is on FH or k_α^2 is on F*H*, and

$$|1-k_\alpha^2| \geq |1-k_F^2| \tag{2.6.2}$$

The equality occurs if and only if α is at F.

Finally we may write

$$\frac{(L_F)_n}{(L_\alpha)_n} = \frac{v_\alpha}{v_F}\left|\frac{K_\alpha^2}{K_F^2}\right|\left|\frac{k_\alpha^2}{k_F^2}\right|^{\frac{1}{a_n}}\left|\frac{1-k_\alpha^2}{1-k_F^2}\right|^{\frac{1}{b_n}}$$

where for

$$n = 1, \quad a_1 = b_1 = 3$$
$$n = 2, \quad a_2 = b_2 = 4$$
$$n = 3, \quad a_3 = 6, \quad b_3 = 3$$

For $n = 1$ we know that

$$\frac{(L_F)_1}{(L_\alpha)_1} \leq 1$$

We wish to show that it is true for all n. In fact this follows at once from (2.6.1) and (2.6.2), since

$$\left|\frac{k_\alpha^2}{k_F^2}\right|^{\frac{1}{a_n}}\left|\frac{1-k_\alpha^2}{1-k_F^2}\right|^{\frac{1}{b_n}} \leq \left|\frac{k_\alpha^2}{k_F^2}\right|^{\frac{1}{3}}\left|\frac{1-k_\alpha^2}{1-k_F^2}\right|^{\frac{1}{3}}$$

By calculation it is found that

$$(L_F)_1 < (L_F)_2 < (L_F)_3$$

thus solving uniquely the extremal problem.

Our next goal is to establish similar results when Modular triangles are replaced by Schwarz' triangles.

3. PART II.

3.1. A Generalization of Schwarz' Lemma. The Classical Schwarz' Lemma has been generalized by L. Ahlfors [3] to analytic functions $\mathfrak{w} = f(z)$ from $|z| < 1$ to a Riemann surface R with a Riemannian metric $ds = \lambda \mid d\omega \mid$ of negative curvature ≤ -4 at every point $\mathfrak{w} \in R$. Here λ is a positive function of Class C_2 depending on the local complex parameter ω in such a way that ds remains invariant.

More recently Schwarz' Lemma has been generalized by Z. Nehari [4] to non uniform functions $w = f(z)$ analytic in $|z| < 1$ except for branch points of finite order with $|f(z)| \leq 1$ for all determinations $f(z)$.

By combining these two results i. e., releasing the last condition one obtains a generalization such as stated below.

Lemma. Let $w = f(z)$ be analytic in $|z| < 1$ except for branch point of finite order and satisfying the following conditions (i) $f'(z) < \infty$ in $|z| < 1$, (ii) at every point of the Riemann surface generated by w is defined a Riemannian metric $ds = \lambda \mid d\omega \mid$ with a Gaussian curvature ≤ -4. Now if $d\sigma_z$ is the hyperbolic metric of $|z| < 1$ then

$$ds \leq d\sigma_z$$

The proof is based essentially on a method of proof due to Ahlfors and offers no difficulty in adapting it to the present case.

In fact if we add the condition $|f(z)| \leq 1$ for all determinations $f(z)$ and replace $ds = \lambda \mid d\omega \mid$ by the hyperbolic metric $d\sigma_w$ of $|w| < 1$ we still have

$$d\sigma_w \leq d\sigma_z$$

Here the equality holds if and only if $f(z) = e^{i\theta} z$, θ real.

It is in this form that the Lemma will be used in the solution of the extremal problem. Later we shall give a direct proof of the solution.

3.2. The Solution of the Extremal Problem. Let $(ABC)_0$, with corresponding angles π/p, π/q, π/r respectively, be a Schwarz triangle containing $z = 0$, say in its interior and which is mapped by

$$w = f(z) = z + \cdots$$

conformally onto $(T_w)_1$. Then for all $\zeta \in (ABC)_0$ we have

$$w = f(\zeta) + \frac{ds}{d\sigma} z + \cdots \tag{3.2.1}$$

where $ds = |dw|$ and $d\sigma$ is the hyperbolic metric of $|\zeta| < 1$.

Let p^*, q^*, r^* be another set of integers and $(A^*B^*C^*)_0$ be the conformal map of $(ABC)_0$ in which vertices with the same letter and the origin correspond Hence for a ζ^* which is the image of ζ (3.2.1) can be written as

$$w = f^*(\zeta^*) + \frac{ds}{d\sigma^*} z^* + \cdots$$

Now assuming $p^* \geqq p$, $q^* \geqq q$, $r^* \geqq r$ we have by the previous Lemma

$$\frac{d\sigma}{d\sigma^*} \leqq 1 \tag{3.2.2}$$

where the equality holds if and only if $p = p^*$, $q = q^*$, $r = r^*$. Hence we may state this first result as.

Theorem 3.2.1. The extremal function is in the Class determined by the normalized functions which map an equiangular Schwarz' triangle with interior angles equal to $\pi/6$ onto $(T_w)_1$.

The theorem says that the extremal function belongs to the Class of normalized analytic functions whose Riemann surfaces possess branch points of least order.

Finally because of the symmetry character of the above argument there is a one to one correspondence between the radii $(B_\zeta)_1$ and $(B_{\zeta^*})_1$ of the circles circumscribed to $(T_w)_1$ and covered univalently by the values of the corresponding normalized functions.

In fact in the conformal mapping of $(ABC)_0$ onto $(A^*B^*C^*)_0$ let $e \in (ABC)_0$ and $e^* \in (A^*B^*C^*)_0$ be the set of points such that at these points

$$\frac{d\sigma}{d\sigma^*} = \frac{d\sigma_0}{d\sigma_0^*} \tag{3.2.3}$$

Let $h \subset e$ be the set of points such that after normalization each corresponding radius is equal to $(B_0)_1$. Thus

(3.2.4) $$\frac{ds}{d\sigma} = \frac{ds_0}{d\sigma_0}$$

On the other hand if $h^* \subset e^*$ is the image of h, $(T_w)_1$ remaining fixed in this mapping, we have

(3.2.5) $$ds^* = ds, \quad ds_0^* = ds_0$$

Hence (3.2.3) together with (3.2.5) can be written as

$$\frac{ds/d\sigma}{ds^*/d\sigma^*} = \frac{ds_0/d\sigma_0}{ds_0^*/d\sigma_0^*}$$

Because of (3.2.4) we have

$$\frac{ds^*}{d\sigma^*} = \frac{ds_0^*}{d\sigma_0^*}$$

which means that after normalization, radii corresponding to points of h^* are equal.

Hence making $p = q = r = \infty$ and comparing the associated Modular triangle with the $\pi/6$ equiangular Schwarz' triangle the one to one correspondence between the radii $(L_{\zeta^*})_1$ and $(B_\zeta)_1$ just obtained yields the final result, i.e..

Theorem 3.2.2. The extremal function within the Class $\mathfrak{B}_1$ is unique and corresponds to the $\pi/6$ equiangular equilateral Schwarz' triangle.

Analoguous results hold when $(T_w)_1$ is replaced by $(T_w)_2$, $(T_w)_3$. In each case the extremal function is unique and is determined by mapping conformally the equilateral Modular triangle onto the Schwarz' triangle yielded by the analogue of Theorem 3.2.1 in such a way that the origins correspond.

Numeral comparison solves completely the extremal problem stated for the case of a Schwarz' triangle.

Note that the present solution together with the results obtained on Bloch functions [5] (1) give a partial answer to the conjecture made by Ahlfors and Grunsky [6].

(1) There follows evidently that a Bloch function of the first or second kind is an Automorphic function (Fuchsoid). Accordingly the size $\mathfrak{B}$ or $\mathfrak{L}$ must be unaltered under the Group of Conformal maps of the associated Riemann surface onto itself. It will be shown in a subsequent paper that this is possible whenever the group of the Conformal maps is generated by the repeated symmetries of a triangle $(T_w)_n$, $n = 1, 2, 3$, with respect to the sides, thereby yielding in conjunction with the present work the exact values of these constants.

3.3. Second Proof. In this section we are going to give a direct proof of (3.2.2). More precisely we shall prove the Lemma for the particular case of a function which maps conformally a Schwarz' triangle onto another with suitably chosen interior angles.

Let be given in the unit circle (K) two Schwarz' triangles (ABC), (A′B′C′) with interior angles

$$\frac{\pi}{p}, \quad \frac{\pi}{q}, \quad \frac{\pi}{r}$$

and

$$\frac{\pi}{p}, \quad \frac{\pi}{q}, \quad \frac{\pi}{r'}$$

respectively such that

$$r' > r$$

By means of two non Euclidean displacements the triangles (ABC), (A′B′C′) can be replaced by (obc), $(o'b'c')$ in such a way that the angles at o coincide and ob, oc are colinear with ob', oc' respectively. An elementary geometrical consideration shows that the third side bc with a bigger angle π/r at c will be, except for the end points, completely inside the triangle $(ob'c')$.

Let

(3.3.1) $$z = \psi(z')$$

be the function which maps conformally $(ob'c')$ onto (obc) with the vertices o, b', c' corresponding to o, b, c respectively. By means of repeated reflections in the sides of the triangles round the origin we obtain two regions F and F′ such that F is completely inside F′. On the other hand because of Schwarz' Principle of symmetry (3.3 1) maps F′ conformally onto F.

The origin being a fixed point in this transformation we have (see e.g., Principes géometriques d'analyse, 1930 by G. Julia)

(3.3.2) $$\lim_{z' \to 0} \left| \frac{\psi(z')}{z'} \right| = \alpha < 1$$

We may now prove

Lemma 1. Let $z = \psi(z')$ and $r' \geq r$ then

$$D(o, z) \leq D(o, z')$$

The equality holds if and only if $r' = r$.

Proof. Let (obc) and $(ob'c')$ be any two neighbouring triangles with $r' = r + \varepsilon \;\; \varepsilon > 0$. Consider two corresponding points ζ, ζ' on the corresponding sides bc, $b'c'$. It is clear that for ε sufficiently small

$$|\zeta| < |\zeta'| .$$

Hence the inequality holds for the sides bc, $b'c'$ of any two such triangles whenever $r' > r$.

Now, $\psi(z')/z'$ is analytic in F' except perhaps at $z' = 0$. But because of (3.3.2) the origin is a removable singularity. Hence $\psi(z')/z'$ is analytic in the interior of F' without exception. Since $|\psi(z')/z'| < 1$ on the boundary of F', it follows by the Maximum Principle that it is true in the interior. Hence $|z| < |z'|$, or in terms of non euclidean distances,

$$D(o, z) < D(o, z')$$

Obviously $D(o, z) = D(o, z')$ if and only if $r = r'$

Lemma 2. Let z' and z be any two corresponding points under $z = \psi(z')$ with $d\sigma'$ and $d\sigma$ as the values of the hyperbolic metric of the unit circle at these points respectively. Then for $r' \geq r$.

$$d\sigma \leq d\sigma'$$

The equality holds if and only if $r' = r$.

Proof. By Lemma 1

$$\frac{1+|z|}{1-|z|} \leq \frac{1+|z'|}{1-|z'|}$$

Hence

$$\tfrac{1}{2}\log\frac{1+|z|}{1-|z|} \leq \tfrac{1}{2}\log\frac{1+|z'|}{1-|z'|}$$

Along the radial path $\overline{o-z}$

$$\tfrac{1}{2}\log\frac{1+|z|}{1-|z|} = \int_{\overline{o-z}} d\sigma$$

Hence

$$\int_{\overline{o-z}} d\sigma \leq \int_{\overline{o-z'}} d\sigma'$$

But if the arc $\widehat{o-z'}$ is, say, the image of $\overline{o-z}$, then clearly

$$\int_{\overline{o-z'}} d\sigma' \leq \int_{\widehat{o-z'}} d\sigma'$$

Hence

$$\int_{\overline{o-z}} d\sigma \leq \int_{\widehat{o-z'}} d\sigma'$$

But then it is well known that

$$d\sigma \leq d\sigma'.$$

Lemma 2 may now be extended to the general case

$$p' \geq p, \quad q' \geq q, \quad r' \geq r$$

by repeated applications to a chain of mappings as follows.

Let

$$\frac{\pi}{p'}, \ \frac{\pi}{q'}, \ \frac{\pi}{r},$$

$$\frac{\pi}{p'}, \ \frac{\pi}{q}, \ \frac{\pi}{r}$$

be the interior angles of two intermediary Schwarz' triangles. The chain of conformal mappings I, II, III of one Schwarz' triangle onto another is indicated schematically below.

$$\begin{array}{c|ccc}
\text{I} \downarrow & \frac{\pi}{p'} & \frac{\pi}{q'} & \frac{\pi}{r'} \\
 & \frac{\pi}{p'} & \frac{\pi}{q'} & \frac{\pi}{r} \\
\text{II} \downarrow & & & \\
 & \frac{\pi}{p'} & \frac{\pi}{q} & \frac{\pi}{r} \\
\text{III} \downarrow & & & \\
 & \frac{\pi}{p} & \frac{\pi}{q} & \frac{\pi}{r}
\end{array}$$

We have successively,

$$d\sigma_{\mathrm{I}} \leq d\sigma_{\mathrm{I}}', \quad d\sigma_{\mathrm{II}} \leq d\sigma_{\mathrm{I}}, \quad d\sigma_{\mathrm{III}} \leq d\sigma_{\mathrm{II}},$$

Hence

$$d\sigma_{\mathrm{III}} \leq d\sigma_{\mathrm{I}}'$$

which was the assertion.

Since the hyperbolic metric is invariant with respect to reflections it follows that the above inequality holds when the mapping is extended by reflections to the whole unit circle.

Literature

[1] E. Landau, Über die Blochsche Konstante..., MZ, vol. 30, 1929.
[2] H. Rademacher, On the Bloch-Landau Constant, Amer. Journal of Math. 1943.
[3] L. Ahlfors, An extension of Schwarz's Lemma, Transactions of the Amer. Math. Soc. vol. 43, 1938.
[4] Z. Nehari, A generalization of Schwarz' Lemma, Duke Math. Journal vol. 14, 1947.
[5] C. Uluçay, Bloch functions and the definition of a new Constant, Proceedings of the International Mathematical Congress Amsterdam, Sept. 1954.
[6] Ahlfors-Grunsky, Über die Blochsche Konstante, MZ, 1936.

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