On the Geometrical Interpretation of the Integrals $\int q\,dt$ and $\int q_0\,dt$

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Özet: t reel parametresinin fonksiyonu olan

$$A(t) = a(t) + \varepsilon\, a_0(t)$$

dual vektörlerine tekabül eden regle yüzeylerin diferensiyel geometrisi W. Blaschke tarafından tetkik edilmiştir. W. Blaschke formüllerile Frenet formülleri arasındaki benzerlikten faydalanan L. Biran regle yüzeylerin eğriler teorisine benzeyen bir tetkikini yapmıştır. Bu çalışmada, L. Biran'ın çalışmalarından ilham alınarak W. Blaschke formüllerinde geçen integral invariyantlara geometrik bir mana verilmeğe çalışılmıştır.

F ve F^* gibi herhangi iki regle yüzeyin sırasile A_1 ve $-A_2^*$ yönlenmiş doğrularının t nin her değeri için çakışık olmaları şartını koşmak maksadımızı aydınlatmağa yetmektedir. Ayrıca F^* nin A_1^* ana doğrusunun daima A_1 in komşu ana doğrusunu kesmesi, F^* yüzeylerinin F ye F^* nin striksiyon çizgisi boyunca teğet olan tors yüzeyleri olmasını temin etmektedir.

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Introduction. W. Blaschke [1] (1) has made a survey of the differential geometry of the ruled surfaces corresponding to the dual vector

$$A(t) = a(t) + \varepsilon\, a_0(t)$$

which is a function of the real parameter t.

Using the resemblance of the Blaschke and Frenet formulae L. Biran [2] has made a study of the ruled surfaces similar to the theory of the space curves. In this paper, inspired by the studies of L. Biran we have tried to give a geometrical meaning

(1) Numbers in brackets refer to the bibliography at the end of the paper.

to the integrals invariants which are involved in the Blaschke formulae.

1. The ruled surfaces and the Blaschke formulae.

A ruled surface F can be represented by an unit dual vector which is a function of the real parameter t as

$$A(t) = a(t) + \varepsilon\, a_0(t)$$

satisfying the relation $A^2 = 1$. It shall be considered that its successive derivatives with respect to t exists to the highest order required.

A reference trihedral D attached to the ruled surface F will be defined as follows.

To the dual vector $A(t) = A_1(t)$ corresponds the first directed line of D, to the dual vector $A_2(t) = \frac{A'}{\sqrt{A'^2}}$ the second directed solid line of D, and to the third dual vector $A_3 = A_1 \times A_2$ corresponds the third directed solid line of D. Thus from the definition of D it is easy to derive that the directed solid lines A_1, A_2, A_3 of D intersect at a point N and form an orthogonal right handed system. D is called the Blaschke Trihedral and the point N is the central point of the surface belonging to the generator A_1. As the directed line $A(t)$ generates the surface F the point N draws a curve (N) on F which is called the line of Striction of the ruled surface. If the derivatives of the dual vectors $A_i(t)$ are expressed by means of the vectors $A_i(t)$, equations

$$\begin{aligned} A_1' &= P\, A_2 \\ A_2' &= -P\, A_1 + Q\, A_3 \\ A_3' &= -Q\, A_2 \end{aligned} \tag{1}$$

are derived. Separating the real and the dual parts of these equations we get the w. Blaschke Formulae.

where $P = \sqrt{A'^2} = p + \varepsilon\, p_0$

and $Q = \frac{(A,\, A',\, A'')}{A'^2} = q + \varepsilon\, q_0$

The resemblance of the Blaschke and the Frenet formulae are clear. If the parameter t is chosen as the arc-length of the curve (N), then P and Q are called the dualistic curvature and the torsion of the ruled surface F. In his paper L. Biran has proved that a ruled surface is uniquely determined except as to position in space when its dualistic curvature P and its torsion Q is given as the function of the arc-length of the line of Striction of the ruled surface F.

Considering the formulae (1) we can deduce that A_3 intersects both generators A_1, and its consecutive $A_1 + dA_1$. Thus A_3 is in the tangent plane of the ruled surface F at the central point N. The plane determined by A_1 and A_3 is called the central plane of the surface at the central point N. A being perpendicular at the point N to the tangent plane of the surface F, is the normal of the surface at this point. The plane determined by A_1, A_2 is called the asymptotic plane of the surface at the point N.

Let x be the position vector of the central point N. The tangent vector x' of the curve (N) being in the tangent plane of F at the point N, must satisfy the relation

$$x' = \alpha\, a_1 + \beta\, a_3 .$$

It can be easily written

(2) $$x' = q_0\, a_1 + p_0 a_3 .$$

From (2)

(3) $$x'^2 = p_0^2 + q_0^2 = 1$$

can be derived and

(4) $$\int q_0\, dt = \int x' a_1\, dt$$

and

(4') $$\int p_0\, dt = \int x' a_3\, dt$$

can be deduced. The right hand side of (4) is the distance of the central point, measured along A_1 from a curve on F which intersects the generators A_1 orthogonally. The right hand side of (4') is the distance of a point of the same line of Striction measured on A_3 from a curve which intersects A_3 orthogonally.

Here it might be helpful to suggest that the integrals $\int q\, dt$ and $\int p\, dt$ need a geometrical interpretation.

2. Geometrical interpretation of the integrals $\int q\,dt$ and $\int q_0\,dt$.

To the ruled surface F generated by the unit dual vector

$$A^*(t^*) = a^* + \varepsilon\, a_0^*$$

we can apply all the results of 1 putting a star sign on the letters used.

The generators A_1 and A_1^* which generate the surfaces F and F* are functions of the parameters t and t^* which are the arc-lengths of the line of Strictions (N) and (N*) respectively.

Let us put forward this problem :

A ruled surface F is given by its curvature P(t) and its torsion Q(t). Find the function $t^* = t^*(t)$ so that $A_1 = -A_2^*$ for every value of t.

The condition we put for the generators corresponding to the unit dual vectors A_1 and $-A_2^*$ to coincide, makes it neces sary for the unit dual vectors A_2, A_3, A_1^*, A_3^* to be parallel to the same plane.

If Φ is the dual angle in between A_1^* and A_3 the relations

$$A_3 \cdot A_1^* = \cos\Phi \tag{5}$$

$$A_1^* = A_3 \cdot \cos\Phi + A_2 \cdot \sin\Phi \tag{6}$$

$$A_3^* = -A_2 \cdot \cos\Phi + A_3 \cdot \sin\Phi \tag{7}$$

can immediately be written. If we differentiate both sides of (6) with respect to t considering (1) we have

$$P^*\, t^{*\prime}\, A_2^* = -P \cdot \operatorname{Sin}\Phi\, A_1 + (\Phi' - Q)\cos\Phi\, A_2 - (\Phi' - Q)\operatorname{Sin}\Phi\, A_3.$$

Using the condition

$$A_1 = -A_2^*$$

$$\Phi' - Q = 0 \tag{8}$$

$$P^*\, t^{*\prime} = P \sin\Phi. \tag{9}$$

From (8)

$$\varphi = \int q\, dt \tag{10}$$

$$\varphi_0 = \int q_0\, dt \tag{11}$$

are derived. Differentiating (7) on both sides with respect to t, using (1) we have

$$-Q^* t^{*\prime} A_2^* = P \cos\Phi\, A_1 + (\Phi' - Q)\sin\Phi\, A_2 + (\Phi' - Q)\cos\Phi\, A_3 .$$

Considering the condition

$$A_1 = -A_2^*$$

we derive

$$Q^* t'^* = P\cos\Phi \tag{12}$$

besides (8) and (9).

We can verify the results by the help of (8), (9) and (12) differentiating the condition

$$A_1 = -A_2^* .$$

Taking the reel parts of (6) and (7) we have

$$a_1^* = a_2 \sin\varphi + a_3 \cos\varphi \tag{13}$$

$$a_3^* = -a_2 \cos\varphi + a_3 \sin\varphi \tag{14}$$

and the dual parts of (9) and (12) we have

$$p_0^*\, t'^* = p\,\varphi_0 \cos\varphi + p \sin\varphi \tag{15}$$

$$q_0^*\, t'^* = -p\,\varphi_0 \sin\varphi + p_0 \cos\varphi . \tag{16}$$

The position vector of N* being x

$$x'^* = (q_0^*\, a_1^* + p_0^*\, a_3^*)\, t$$

$$x'^* = -p\,\varphi_0\, a_2 + p_0\, a_3 \tag{17}$$

can be derived. From (17) considering (3) we have

$$t^{*\prime} = (p^2 \varphi_0^2 + p_0^2)^{1/2} \quad \longrightarrow \quad t^* = \int (p^2 \varphi_0^2 + p_0^2)^{1/2}\, dt$$

with these, the invariants P* and Q* the curvature and the torsion belonging to the ruled surface F*, can be found as

$$P^* = \frac{P \sin\Phi}{(p^2\varphi_0^2 + p_0^2)^{1/2}}$$

$$Q^* = \frac{P \cos\Phi}{(p^2\varphi_0^2 + p_0^2)^{1/2}} .$$

Thus if the ruled surface F is given then F* can be determined uniquely except as to position in space.

Considering (5) the shortest distance between the generators A_1^* and A_3 is nothing else but the distance of the central points situated on the coinciding generators of the surfaces F and F*.

With this we have given a new meaning to the integral

$$\varphi_0 = \int q_0 \, dt.$$

Again from (5) it can be deduced that the asymptotic plane of F* at the point N makes the angle φ with the central plane of F at the point N.

Thus, because of (10) the integral

$$\varphi = \int q \, dt$$

also gets a geometrical meaning.

We are now putting forward a second condition to be fulfilled for the problem in 2. This condition which must be satisfiedt too is

$$D(A_1^* , A_1') = 0.$$

This means that generator A_1^* intersects the consecutive generator A_1' of A_1.

With this new supposition let us consider the properties of the ruled surface F*. Because of

$$A_1 = - A_2^*$$

the new condition becomes

(19) $$D(A_1^* \cdot A_2^{*\prime}) = 0,$$

but since

$$A_2^{*\prime} = (-P^* A_1^* + Q^* A_3^*) \, t^{*\prime}$$

(19) becomes

(20) $$D(P^* t^{*\prime}) = 0.$$

From (9), (20) becomes

(21) $$D(P \sin \Phi) = 0.$$

Now we can deduce the following properties

a) $$A_1 \cdot A_1^* = 0$$

b) $$D(A_1^* \cdot A_1') = 0$$

c) $$D\,(A_1^{*\prime}\cdot A_1^{*\prime})=0$$

d) $$A_1\times A_1^{*}=A_1^{*}\times A_1^{*\prime}.$$

(a) is clear from (6). (b) is the last condition supposed. (a) and (b) express that the generator A_1^* is in the tangent plane of F at the point N*. From (21) and (8), (c) can easily be seen. This shows us that the consecutive generators of F intersect. So the ruled surface F* is a torse. By (8) and (21), (d) can be verified. This last property shows that the ruled surfaces F and F* have the same tangent plane at the point N*. Thus the ruled surface F* is a torse which is tangent to the ruled surface F along the line of Striction (N*).

References

[1]. W. Blaschke. Differentialgeometrie, Bd I, (3rd ed.), Berlin, 1930, p. 261.

[2]. L. Biran. Revue de la Faculté des Sciences de l'Université d'Istanbul, Tome VI, Fasc. 3-4 Série A, 1942.

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