COMMUNICATIONS

DE LA FACULTÉ DES SCIENCES
DE L'UNIVERSITÉ D'ANKARA

Série A: Mathématiques, Physique et Astronomie.

TOME 23 A ANNÉE 1974

On an Extremalization Method in The Theory of Analytic Functions

by

C. ULUÇAY

9

Faculté des Sciences de l'Université d'Ankara
Ankara, Turquie

Communications de la Faculté des Sciences de l'Université d'Ankara

La Revue "Communications de la Faculté des Sciences de l'Univereité d'Ankara" est un organe de publication englobant toutes les disciplines scientifiques représentées à la Faculté: Mathématiques pures et appliquées, Astronomie, Physique et Chimie théorique, expérimentale et techinique, Géologie, Botanique et Zoologie.

La Revue, á l'exception des tomes I, II, III, comprend de trois séries

Série A: Mathématiques, Physique et Astronomie.

Série B: Chimie.

Série C: Sciences naturelles.

En principe. la Revue est réservée aux mémoires originaux des membres de la Faculté. Elle accepte cependant, dans la mesure de la place disponible, les communications des auteurs étrangers. Les langues allemande, anglaise et française sont admises indifféremment. Les articles devront être accompagnés d'un bref sommaire en langue turque.

Adres: Fen Fakültesi Tebliğler Dergisi Fen Fakültesi, Ankara, Turquie

On an Extremalization Method in The Theory of Analytic Functions

C. ULUÇAY

Department of Mathematics, University of Ankara

(Received Sept. 3, 1974)

SUMMARY

In this paper we prove a general theorem (theorem 2) which is another form of the main theorem in [1]. New applications of this theorem are also given.

INTRODUCTION

In a paper, "A General Formulation of an Extremalization Problem..." we stated and proved the following general theorem [1].

Let E be the class of analytic functions represented by a Stieltjes integral

$$f(z) = \int_a^b g(z,t)d\mu(t)$$

where a, b are given real numbers, g(z,t) is a given function analytic in the unit disk K: (z, $|z|<1$) for $a \leqq t \leqq b$, and $\mu(t)$ runs through all possible non decreasing functions in $a \leqq t \leqq b$, subject to the condition

$$\int_a^b d\mu(t) = \mu(b) - \mu(a) = 1 .$$

Furthermore, we assume in the sequel that for a given point $z \in K$, g(z,t) is non constant analytic in some region containing the closed interval [a,b]: (t, $a \leqq t \leqq b$). We now state theorem 1 ([1], p. 13).

Theorem 1. For a given entire function F(w) and a given point $z \in K$, either of the functionals.

$$\mathrm{Re}\,\{F(\log f(z))\} \;,\; |F(\log f(z))|,\; f(z) \neq 0,$$

attains its maximum in E, only for functions of the form

$$f(z) = \sum_{k=1}^{n} \lambda_k g(z,t_k)$$

with $\lambda_k \geqq 0, \sum_{k=1}^{n} \lambda_k = 1$.

Here the logarithm is the principal value and we exclude from consideration the case in which for the extremal function, $F'(\log f(z)) = 0$.

For the geometrical meaning of the numbers λ_k, t_k the reader is refered to [1].

1. Recently several distortion theorems have been obtained for special classes of functions that are subclasses of E. Most of these theorems are immediate consequences of Theorem 1. Before going into details we find necessary for the benefit of the reader who might use this and the related paper [1] as a reference, to discuss the validity of the objections raised by Hummel while reviewing the paper in Zentralblatt, vol. 241-1973 p. 157.

In this review Hummel maintains that he cannot follow the argument leading to the conclusion ([1])

$$\int_{t_1}^{t_2} \mathrm{Re}\,\{F'(\log f(z))\; g_t(z,t)\,/f(z)\}\; |\mu(t) - c|\; dt = 0,$$

where t_1, t_2 are arbitrary numbers in the interval [a,b] (loc. cit. p. 14), since he says c depends on the sign of λ. Of course c depends on the sign of λ, but nevertheless the above conclusion holds, since the integral on the left of the equation has a fixed sign over any segment (t_1, t_2) by the first mean value theorem, the reviewer will see the same conclusion at several places in [2] (translation by A. W. Goodman, Amer. Math. Soc. Trans., II ser. 18, p. 5 formula (5′), p. 7 formula on line 13, etc.).

Hummel's second objection on my comment about Goluzin's theorem 1 ([2]), p. 13) is again not valid. Indeed, the numbers t_1, t_2 in the mentioned theorem, being the roots of the equation (9) ([2], p. 13), it follows from the variation (6) (loc. cit) that Re $\{\Phi'(f(z))s(z,t)\}$ (loc. cit.) has the same value at the points t_1, t_2 but then its derivative, i.e., equation (9) has still another root in the interval (t_1, t_2), and so (9) will have at least three different solutions, an impossibility, since (9) has at most two roots. Thus theorem 1 ([2]) is not correct as asserted in [1], but it would be correct if (9) had at most 4 roots instead of 2, so 4 is not a misprint as maintained by the reviewer.

2. Let us show once more that Goluzin's theorems and other's are easy consequences of the following general theorem which is quite similar to theorem 1 in [1].

Theorem 2. For a given entire function $F(w)$ and a given point $z \in K$, either of the functionals

$$\mathrm{Re}\,\{F(f(z))\}, \quad |F(f(z))|,$$

attains its maximum in E, only for functions of the form

$$f(z) = \sum_{k=1}^{n} \lambda_k g(z,t_k)$$

with $\lambda_k \geqq 0$, $\sum_{k=1}^{n} \lambda_k = 1$.

Here we exclude from consideration the case in which for the given point, $F'(f(z)) = 0$.

Proof. It is exactly the same as in theorem 1 ([1]). It is sufficient of course to prove the theorem for the first functional. As in [1] we consider the two variational formulas ([1], p. 12) and arrive at the relations

$$J_f^* = J_f + \lambda \int_{t_1}^{t_2} \mathrm{Re}\,\{F'(f(z))\,g_t(z,t)\}\,|\,\mu(t) - c\,|\,dt + 0(\lambda^2)$$

$$J_f^{**} = J_f + \lambda\,\mathrm{Re}\,\{F'(f(z))\,(g(z,t_1) - g(z,t_2))\} + 0(\lambda^2)$$

where $J_f = \text{Re}\{F(f(z))\}$, λ is a small real number, and c a constant depending on the sign of λ. In the first formula t_1 and t_2 are arbitrary with $a \leqq t_1 < t_2 \leqq b$. In the second formula t_1 and t_2 are points of discontinuity of μ (t) with $a \leqq t_1 < t_2 \leqq b$. We see as before that if f(z) is an extremal function, then μ (t) = constant between any two roots t_1, t_2 of the equation

$$\Phi(t) = \text{Re}\ \{ F'(f(z))\ g_t(z, t) \},$$

while the second variational formula yields the condition

$$\text{Re}\ \{ F'(f(z))\ (g(z, t_1) - g(z, t_2)) \} = 0.$$

This condition shows that

$$\text{Re}\{F'(f(z))\ g(z,t)\}$$

has the same value for t_1 and t_2. Hence its derivative, that is Φ (t) has still another root somewhere between t_1 and t_2. Thus the maximum number of jump points of μ (t) is half of the number of the roots of $\Phi(t) = 0$ (if the number of roots is odd, say 2n–1, then the maximum number of jump points is n). This fact may be used as a criterion in the extremalization. And the theorem follows.

The following applications of theorem 2 are now considered.

3. Class T_r of typically real functions. A representation for a function f(z) of class T_r is ([1])

$$f(z) = \int_{-1}^{1} \frac{z}{1-2tz+z^2}\ d\mu(t).$$

Now, $\mu(t)$ = constant between any two roots of

$$\text{Re}\ \{ F'(f(z))\ 2z^2\ (1-2t\bar{z}+\bar{z}^2)^2\} = 0.$$

But this equation has for a fixed z at most two roots if $z \neq 0$ and $F'(f(z)) \neq 0$ (compare this equation with (9) in [2]), and by the above criterion μ (t) has at most one discontinuity point in $-1 \leqq t \leqq 1$. And accordingly,

Theorem 3. For a given entire function F(w) and a given point $0 \neq z \in K$, either of the functionals

$$\mathrm{Re}\,\{F(f(z))\}\ ,\ |F(f(z))|$$

attains its maximum in T_r only for a function of the form

$$f(z)=\frac{z}{1-2tz+z^2}$$

Here, we exclude from consideration the case in which for the extremal function $F'(f(z)) = 0$. (Again compare with theorem 1, p. 13, [2]).

4. Class $S_\alpha\ (\gamma)$ of spiral-like functions. These are functions f(z) analytic in K with $f(0) = 0$, $f'(0) = 1$ and satisfying the condition

$$\mathrm{Re}\ \{e^{i\gamma}zf'(z)\ /\ f(z)\} > \alpha\cos\gamma,\ |\gamma|<\frac{\pi}{2},\ 0\ \leqq\ \alpha\ <\ 1.$$

We have the representation

$$f(z)=z\exp\left\{-2c\int_{-\pi}^{\pi}\log(1-ze^{-it})\ d\mu(t)\right\},\quad c=(1-\alpha)\cos\gamma e^{-i\gamma}\ .$$

Here the logarithm is that branch that vanishes at $z=0$.

Writing this formula as

$$\log(f(z)/z)=-2c\int_{-\pi}^{\pi}\log(1-ze^{-it})d\mu(t),$$

where the logarithm on the left of the equation is that branch that is analytic in K and vanishes at $z=0$, for this class of functions we have

Theorem 4. For a given entire function F(w) and a given point $0\neq z\in K$, the maximum for either of the functionals

$$\mathrm{Re}\ \{F(\log(f(z)/z))\},\ |F(\log(f(z)/z))|$$

in the class $S_\alpha\ (\gamma)$ is attained only for a function of the form

$$f(z)=z/(1-e^{it}z)^{2c}.$$

Here we exclude from consideration the case in which for the extremal function $F'(\log(f(z)/z)) = 0$.

Using appropriate $F(w)$, bounds for $|f(z)|$ or others may be obtained [3].

Proof. By theorem 2, $\mu(t)$ is a step function with jump points satisfying the equation

$$\mathrm{Re}\,\{-2c\ F'(\log(f(z)/z))ie^{-it}z\,(1-e^{it}\bar{z})\} = 0$$

This equation has for $z \neq 0$ and $F'(\log(f(z)/z)) \neq 0$ at most two roots with respect to e^{it}. Consequently, by the known criterion the maximum number of jump points is half of the number of roots and the theorem follows.

If $\alpha = 0$, $\gamma = 0$ we obtain results pertaining to the class of starlike functions. If only $\gamma = 0$ we obtain results pertaining to α–starlike functions.

5. Class S_0 of Convex Functions. For a function $f(z)$ analytic in K, with $f(0) = 0$, $f'(0) = 1$ to be convex it is necessary and sufficient that

$$\mathrm{Re}\,\{1 + zf''(z)/f'(z)\} > 0\,,\ z \in K\,.$$

A representation for the logarithm of the derivative is

$$\log f'(z) = -2\int_{-\pi}^{\pi} \log(1-e^{-it}z)\ d\mu(t)\ .$$

The logarithm on the left is that branch that is 0 for $z = 0$. We then have for the class S_0

Theorem 5. For a given entire function $F(w)$ and a given point $0 \neq z \in K$, the maximum for either of the functionals

$$\mathrm{Re}\,\{F(\log f'(z))\}\ ,\ |F(\log f'(z))|$$

in the class S_0 is attained only for a function of the form

$$f(z) = z/(1-e^{it}z).$$

Here we exclude from consideration the case in which for the extremal function $F'(\log f'(z)) = 0$.

Let us consider a more general class. Namely,

6. Class $S_0(\alpha)$ of α–convex Functions. These functions are analytic in K, with $f(0) = 0$, $f'(0) = 1$, $f(z)\,f'(z) \neq 0$ in $0 < |z| < 1$ and satisfy the condition

$$\mathrm{Re}\ \{ (1-\alpha)zf'(z)/f(z) + \alpha(1+ zf''(z)/f'(z)) \} > 0,\ z\in K,\ \alpha \text{ real.}$$

This condition implies at once the integral representation

$$(1-\alpha)\ zf'(z)/f(z) + \alpha\ (1+ zf''(z)/f'(z))$$
$$= \int_{-\pi}^{\pi} (e^{it} + z)/(e^{it} - z)\ d\mu(t)$$

Using the condition $\mu(\pi) - \mu(-\pi) = 1$, we can write

$$(1-\alpha)\ zf'(z) / f(z) + \alpha\ (1 + zf''(z) / f'(z)) -1$$
$$= \int_{-\pi}^{\pi} 2z/(e^{it}-z)d\mu(t)$$

or, dividing by z and arranging

$$(1 - \alpha)\ f'(z)/f(z) + \alpha\ f''(z)/f'(z) - (1-\alpha)/z$$
$$= \int_{-\pi}^{\pi} 2/(e^{it} -z)\ d\mu(t)\ .$$

Integrating on z from 0 to z we obtain

$$\log(\alpha z^{1-1/\alpha}\ df(z)^{1/\alpha}/dz)$$
$$= \int_{-\pi}^{\pi} \log\ (1- e^{-it}z)^{-2/\alpha}\ d\mu(t),\ \alpha > 0.$$

The logarithm on the left is that branch that is analytic in K and vanishes at $z = 0$. From this, we may also deduce the representation

$$f(z) = \left\{(1/\alpha)\int_{0}^{z} (\exp\int_{-\pi}^{\pi} \log(z^{1/\alpha-1}\ (1-e^{-it} z)^{-2/\alpha})\ d\mu(t))\ dz\right\}^{\alpha},\ \alpha>0$$

where the logarithm and the power stand for principal values. Applying theorem 2, we have

Theorem 6. For a given entire function F(w) and a given point $z\in K$, the maximum for either of the functionals

$$\mathrm{Re}\ \{F(\log\alpha z^{1-1/\alpha}\ df(z)^{1/\alpha}/dz)\},\ \ |F(\log\alpha z^{1-1/\alpha}\ df(z)^{1/\alpha}/dz)|,$$
$$\alpha > 0$$

in the class $S_o(\alpha)$ is attained only for a function of the form

$$f(z) = \left\{(1/\alpha)\int_0^z z^{1/\alpha-1}(1-e^{it}z)^{-2/\alpha}\,dz\right\}^{\alpha},\quad \alpha > 0\,.$$

The powers in the formula stand for principal values.

Here, we exclude from consideration the case in which $F'(\log \alpha z^{1-1/\alpha}\, df(z)^{1/\alpha}/dz) = 0$.

Using $F(w) = e^w$ we obtain in particular the bound [4]

$$|f(z)| \leqq \left\{(1/\alpha)\int_0^r \rho^{1/\alpha-1}(1-\rho)^{-2/\alpha}\,d\rho\right\}^{\alpha},\quad \alpha > 0.$$

In fact, using theorem 6, we first obtain

$$\alpha\,|z|^{1-1/\alpha}\,|df(z)^{1/\alpha}/dz| \leqq 1/(1-|z|)^{2/\alpha}$$

or,

$$|\,df(z)^{1/\alpha}/dz\,| \leqq (1/\alpha)\,|z|^{1/\alpha-1}/(1-|z|)^{2/\alpha}\,.$$

Integrating both sides of this inequality along the radius joining the points 0 and z, we obtain

$$|f(z)|^{1/\alpha} = \left|\int_0^z \frac{df(z)^{1/\alpha}}{dz}\,dz\right| = \int_0^{|z|}\left|\frac{df(z)^{1/\alpha}}{dz}\right| d|z|$$

$$\leqq (1/\alpha)\int_0^r \rho^{1/\alpha-1}(1-\rho)^{-2/\alpha}\,d\rho$$

from which the assertion follows.

LİTERATURE

[1] C. Uluçay, A General Formulation of an Extremalization Problem..., Communications de la Fac. des Sc. de l'Un. d'Ankara, Tome 19 A, pp. 11–22, (1971).

[2] G.M. Goluzin, On a Variational method in the Theory of Analytic Functions, Leningrad, gosud. Univ. 144, 85-101, 1952, (Russian) Translated by A. W. Goodman, Amer. Math. Soc. Trans., II. Ser. 18, 1–14, (1961).

[3] P. K. Kulshrestha, Distortion of spiral-like mappings, Proc. roy. İrish Acad., Sect. A, 73, 1–5 (1973).

[4] S. S. Miller, Distortion properties of α– starlike functions, Proc. Amer. math. Soc. 38, 311–318 (1973).

Ö Z E T

Bu makalede, bundan önce vermiş olduğumuz bir esas teoremin bir başka şeklini veriyor ve ispat ediyoruz. Son günlerde elde edilen birçok neticeler bu teoremin tabii ve kolay sonuçlarıdır.

Prix de l'abonnement annuel

Turquie: 15 TL; Étranger: 30 TL.
Prix de ce numéro: 5 TL (pour la vente en Turquie).
Prière de s'adresser pour l'abonnement á: Fen Fakültesi Dekanlığı Ankara, Turquie.

Ankara Üniversitesi Basımevi. Ankara–1975