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Algebraic Structure Of The Continuum And The Factorial Representation

by

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6

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INTRODUCTION

In § 1 of this paper the Method of Factorial Representation is reviewed. § 2 deals with the concept of the Abelean Cluster A_n and the Sequence of Abelean Clusters:

$A_1 \{0\}, A_2 \{1/2\}, A_3 \{1/6, 1/3, 2/3, 5/6\}, \ldots\ldots$

The element

$$x = \sum_{k=1}^{n} a_k e_k$$

of A_n is characterized by the restrictions (1.6) and (1.7) of which the latter leads to

$$A_m \cap A_n = \varnothing, \text{ for } m \neq n.$$

The Sequence of Clusters represents a different version of the Cantor ordering of the rational numbers. § 3 contains the real and complex universe. § 4 deals with the minimal and maximal representatives of the Abelean Clusters and the Univers. § 5 is reserved for the approximation of the irrational numbers. § 6 contains the definition of the extended universe and a criterion for transcendence related to the matrix $T\infty = (a_m)_n$ of infinite rank.

In § 7 certain Diophantine Systems are formulated, which are of special interest in connection with the factorial representation. In § 8, Alef and C are symbolically described as additive infinity: Alef $= \Sigma\, N\,(l)$ and multiplicative infinity $C = \infty!$. In § 9 the factorial representation is being brought in relation with Dedekind Cut.

§ 1

This paragraph contains certain results, which were previously analized in a recent paper of the author[1]).

Throughout this article:

$$e_\lambda = \frac{1}{\lambda!} .$$

A. Every complex number $z = x + iy$ satisfying

(1.1) $$0 \leq | z | \leq 1$$

has a unique factorial representation of the form

(1.2) $$z = x + iy = \sum_{\lambda=1}^{\infty} (a_\lambda + ib_\lambda) e_\lambda$$

whereby

a_λ 's as well as b_λ 's

are integers subject to

(1.3) $$0 \leq a_\lambda \leq \lambda - 1 \text{ and } 0 \leq b_\lambda \leq \lambda - 1 .$$

B. Every rational number

$$x = \frac{p}{q} \qquad \text{with} \qquad [p, q] = 1$$

and subject to

(1.4) $$0 \leq x < 1$$

has a unique finite representation of the form

(1.5) $$x = \frac{p}{q} = \sum_{\lambda=1}^{l} a_\lambda e_\lambda$$

where $l = l(q)$ is obtained by determining the least factorial multiple $l!$ of q and a_λ 's are integers satisfying

(1.6) $$0 \leq a_\lambda \leq \lambda - 1 \qquad \text{for} \qquad 1 \leq \lambda \leq l - 1$$

and

(1.7) $$0 < a_l \leq l - 1 \qquad \text{for} \qquad \lambda = l .$$

C. Any finite representation of the form (1.5) corresponds to a rational number x subject to (1.4).

D. The above defined function

$$l = l\ (q)$$

has the following elementary properties

(1.8) $$l = l\,(q) = q$$

only for q prime or q = 4.

(1.9) $$l\ (q)\ \leq\ \max\ l\ (p_{\mu}^{\alpha_{\mu}})\ \leq\ q$$

$$\text{for}\quad q\ =\ \prod_{\mu=1}^{m}\ p_{\mu}^{\alpha_{\mu}}\ .$$

E. The inverse l^{-1} (q) of (q) — with the exception of the cases stated in (1.8) — is multiple-valued.

We denote the number of integers q corresponding to the same l, in other words the degree of relative multiplicity of l^{-1} (q) with

(1.10) $$I\ =\ I\ (l).$$

This concept plays an important role for the measurement of the density of the prime distribution.

§ 2

ABELEAN CLUSTERS

We call the set A_l consisting of elements of the form

$$x\ =\ \sum_{\lambda=1}^{l}\ a_{\lambda}\ e_{\lambda}$$

as defined by (1.5) and subject to (1.6) and (1.7) an Abelean cluster or Simplexoid of dimension l.

A. Every A_l contains

(2.1) $$N\ (l)\ =\ [\ (l\ -\ 1)\ !\ ]\ .\ (l - 1)$$

elements.

The statement A. follows from (1.5), (1.6), (1.7) immediately.

B. $$A_m \cap A_n\ =\ \varnothing \qquad \text{for} \qquad m\ \neq\ n$$

A result which follows from (1.7).

C. Every rational number

$$x = \frac{p}{q}, \qquad [p, q] = 1, \qquad 0 \leq x < 1$$

is element of only one A_l with $l = l(q)$.

The statement C. is a simple conclusion drawn from B.

We shall refer to an abelean cluster A_l whose elements are ordered according to their magnitude as an ordered abelean cluster A_l^*.

In connection of B. and C. and with respect to this last definition we can now express:

D. The Sequence

$$(2.2) \qquad S : \qquad A_1^*, A_2^*, \ldots, A_l^*, \ldots$$

represents a different version of the Cantor ordering of the rational elements of the unit interval.

§ 3

The Real And Complex Universe.

We call the set of elements of the form

$$x = \sum_{\lambda=1}^{\infty} a_\lambda e_\lambda$$

where a_λ's are integers subject to

$$0 \leq a_\lambda \leq \lambda - 1$$

the Real Universe $A\infty$ and consequently the set of elements of *a* the form (1.2) and subjest to (1.3) the complex Universe $K\infty$. Obviously

$$A\infty \subset K\infty$$

It is easy to make the following observations concerning A_l and $A\infty$.

A. $$A_l \subseteq A\infty .$$

B. $$S = \bigcup_{\lambda=1}^{l} A_\lambda$$

is an abelean group under addition mod. 1.

C. $$A\infty = \bigcup_{\lambda=1}^{\infty} A_\lambda \quad \text{and} \quad A\infty = \bar{A}\infty .$$

D. $A\infty$ is an abelean group addition mod. 1.

E. $A\infty$ is isomorphic to the closed unit interval.

These remarks can organisally be extended to K_l and $K\infty$.

§ 4

The Minimal and the Maximal axis of A_l and $A\infty$.

We call

$$m_l = \sum_{\lambda=1}^{l} e_\lambda \tag{4.1}$$

the minimal axis

$$M_l = \sum_{\lambda=1}^{l} (\lambda - 1) e_\lambda \tag{4.2}$$

the maximal axis of A_l, and

$$m_\infty = \sum_{\lambda=1}^{\infty} e_\lambda = (e - 2) \tag{4.3}$$

minimal axis and

$$M_\infty = \sum_{\lambda=1}^{\infty} (\lambda - 1) e_\lambda = 1 \tag{4.4}$$

the maximal axis of the real universe $A\infty$ respectively.

We call R_l defined by

$$M\infty - M_l = R_l = \sum_{\lambda=l+1}^{\infty} (\lambda - 1) e_\lambda \tag{4.5}$$

the residua complement of M.

Due to (4.4) and the identity

$$M_l + e_l = 1 \tag{4.6}$$

we observe elementarily the following property:

A.

$$(4.7) \qquad e_l = R_l \quad .$$

From the geometric point of view this result means that e_l is a maximal element of the residual space R_l .

§ 5

Approximation of the Irrational Numbers

We call $x \in A$ with infinitely many non-zero components a proper element of A.

A. According to A., B., C., of § 1 with the exclusion of $x = 1$ all the other proper elements of A are irrational bumbers, satisfying $0 \leq x \leq 1$.

B. For every proper element x of A the following inequality holds:

$$(5.1) \quad x = \sum_{\lambda=1}^{n} a_\lambda e_\lambda + \sum_{\lambda=n+1}^{\infty} a_\lambda e_\lambda \leq \sum_{\lambda=1}^{n} a_\lambda e_\lambda + R_n = x_n + R_n .$$

Hence:

C.

$$(5.2) \qquad x - x_n \leq R_n$$

or according to (4.7)

$$(5.3) \qquad x - x_n \leq e_n .$$

We call

$$x_n = \sum_{\lambda=1}^{n} a_\lambda e_\lambda$$

of (5.1) the proper n-th representative of the irrational number x.

D. Due to (1.6), x_n is the largest element of A_n satisfying

$$(5.4) \qquad 0 < x_n < x .$$

The concept proper n—th representative can also be applied for

$$(5.5) \qquad y = \sum_{\lambda=1}^{l} a_\lambda e_\lambda \ , \ y \in A_l \ .$$

For $n < l$ we call

$$(5.6) \qquad y_n = \sum_{\lambda=1}^{n} a_\lambda e_\lambda$$

correspondingly the proper n-th representative of y. (5.3) remains also valid for this interpretation.

§ 6

The extended Universe and The Criterion of Transcendence

We call the sequence

$$(6.1) \qquad E: e_1, e_2, \ldots, e_n, \ldots$$

the fundamental base of the Universe and the sequence

$$(6.2) \qquad Y: y_1, y_2, \ldots, y_n, \ldots$$

introduced by

$$(6.3) \qquad y_n = \sum_{\lambda=1}^{\infty} b_{n\lambda} \ e_\lambda \qquad \text{for } n = 1, 2, \ldots$$

the **regular** base of the Extenden Universe U_∞, provided the infinite matrix of the transformation

$$(6.4) \qquad T_\infty = (b_{n\lambda}), \ (n, \lambda) = 1, 2, \ldots$$

is of infinite rank and the components are finite complex numbers. Obviously:

A. $$A_\infty \subset K_\infty \subset U_\infty \ .$$

We refer to x as a transcendental element of U_∞, whenever the power sequence

$$(6.5) \qquad P: \quad x, x^2, \ldots, x^n, \ldots$$

is a regular base of U_∞.

Examble 1. the power sequence

$$(6.6) \qquad E_1 : e^m = \sum_{\lambda=0}^{\infty} m^\lambda e_\lambda ,$$

with the infinite matrix :

$$(6.7) \qquad T_\infty = (m^\lambda)$$

$$m = 1, 2, \ldots ; \lambda = 0, 1, 2, \ldots$$

is a regular base.

Examble 2. The power sequence

$$(6.8) \qquad E : (e - 2), (e - 2)^2, \ldots , (e - 2)^n, \ldots$$

is a regular base.

Remark.

$$(6.9) \qquad (e - 2)^n = \sum_{\lambda=0}^{n} (\lambda^n) (- 2)^{n-\lambda} . e^\lambda$$

is a linear combination of

$$(6.10) \qquad 1, \quad e^2 , \ldots , e^n .$$

B. If x is transcendental with repect to U_∞ then $x - [x]$ is trascendental with respect to A. [x] is the integral part of x.

We call $(c_\lambda z^\lambda)$ — c and z complex — local contraction and c_λ the λ — th contractor.

C. If $\quad f(z) = \sum_{\lambda=0}^{\infty} c_\lambda z^\lambda e_\lambda$

is analytic over $\qquad | z | < R \leq \infty$

then $\qquad f(z) \in U_\infty$.

§ 7

Certain Diophantine Systems

A close study of linear independence of the fundamental and regular bases leads to certain mathematical questions, which are attractive for their own sake.

The following Diophantine System is a typical example of this kind:

$$(7.1)\qquad s.\,(\lambda - 1) = \sum_{k=1}^{m} t_k \; a_{k\lambda}$$

$$\lambda = 1, 2, \ldots, n, \ldots$$

where t_k' s are integers and $a_{k\lambda}$'s are integers subject to

$$(7.2)\qquad 0 \leq a_{k\lambda} \leq \lambda - 1$$

$$(7.3)\qquad m \leq \infty .$$

Examble 1. $x = \sum_{\lambda=1}^{\infty} a_{1\lambda} e_\lambda \in A_\infty$, $\lambda = \sum_{\lambda=1}^{\infty} a_{2\lambda} e_\lambda \in A_\infty$ and $x + y = 1$

$s=1$, $t_1 = t_2 = 1$, $m = 2$ and $a_{1\lambda} + a_{2\lambda} = \lambda - 1$.

Examble 2. For $e - 2 = x_n + r_n$, $n = 1, 2, \ldots$. the residual sequence

$$r_1 , r_2 , \ldots , r_n , \ldots$$

is a regular base and

$$t_k = 1 \text{ for } k = 1, 2, \ldots , n , \ldots$$

$$a_{k\lambda} = 1 \text{ for } \lambda \geq k \text{ and } a_{k\lambda} = 0 \text{ for } \lambda < k .$$

§ 8

Symbolic Interpretetion of Alef and C

A. *The power of the countable set*

$$(8.1)\ \text{Alef} = \sum_{\lambda=1}^{\infty} N(\lambda) \text{ (additive infinity)}$$

and in connection with (1.2) and (1.3).

B. *The power of the Continuum*

$$(8.2)\qquad C = 1.2.3.\ldots.n.\ldots = \infty \quad \text{(mutiplicative infinity).}$$

§ 9

The Dedekind Cut and Factorial Representation

For every $x \in A_{\infty,}$ we weall the finite or infinite sequence

(9.1) $Ak_1 , Ak_2 , \ldots , Ak_n , \ldots$

which is obtained from the sequence

(9.2) $A_1 , A_2 , \ldots , A_n , \ldots$

by the omission of those clusters A_n in which there is no-representative of x, The Reduced Sequence of the Representatives of x. Obviously (9.1) might as well be identical with (9.2).

A. Every $x \in A_{\infty}$ is a Dedekind Cut represented by the finite or infinite sequence

(9.3) $x_{k_1} , x_{k_2} , \ldots , x_{k_n} , \ldots$

$x_{k_n} \in A_{\infty}$ and maximal relative to x.

B. $x_{k_n} = \varnothing_x (A_{k_n})$ is the relative choice function of Zermelo.

C. Every subset of (9.3) has a maximal element, x is a maximal element (Zorin ' s Lemma).

REFERENCES

z1e *Orhan Hamdi Alisbah*, M.E.T.U. Jour. of Pure And Appl. Sc. Vol. 1, No 2, 1968, p. 71–77.

z2e On the suggestion of the author *Uluğ Çapar* of M.E.T.U has made an interesting survey concerning the nature of l (q) in 1967 (unpublished). His observetions are not included.

z3e On the suggsetion of the auther *Ellen Fenwick* of *Rutgers* (with the help of *Rabert Turkelson* of *Rutgers*) has conducted a programed survey concerning the related distribution of l^{-1} (g), in 1968. The method they used is different in nature from the one applied in the previous paper of the author, (unpublished).

ÖZET

Continuum'un Cebirsel Yapısı ve Faktöryel Temsil

Bu makale aslında bundan önce Orta Doğu Teknik Üniversitesi Temel ve Uygulamalı Bilimler Dergisi Cilt 1, Sayı 2, yıl 1968 Sayfa 71 de yayımlanan bir araştırmanın devamıdır. § 1 de temel konuya değinilmiştir. § 2 de Abel Kümeleri tanımlanmıştır. Düzenlenmiş Abel Kümeleri yardımıyle rasyonel sayıların sıralanabilirliği Cantor metodundan farklı bir şekilde gösterilmiştir. § 3 de reel ve komplex universe tanımlanmış, § 4 te Abel Kümelerinin ve universe'in minimal ve maximal eksenleri ifade olunmuştur. § 5 irrasyonel sayıların yaklaşık değerlerinin hesaplanması ve hata tahminine dairdir. § 6 da genelleştirilmiş universe tanımlanmakta ve bir transcendence kriteri verilmektedir. Aynı paragrafta e ve (e–2) örnekleri incelenmektedir. § 7 de faktöryel temsil ile bazı Diophant denklem sistemleri ve o tipten iki örnek irdelenmektedir. § 8 de sembolik olarak Alef $= \Sigma N(l)$, $C = \infty!$ eşitlikleri ileri sürülmektedir. § 9 da faktöryel temsil Dedekind kesiti ile karşılaştırılmaktadır.

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