COMMUNICATIONS

DE LA FACULTÉ DES SCIENCES DE L'UNIVERSITÉ D'ANKARA

Série A: Mathématiques, Physique et Astronomie

TOME 23 A ANNÉE 1974

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by

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Ankara, Turquie

Communications de la Faculté des Sciences de l'Université d'Ankara

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1.1. Let Σa_n be a given infinite series and $\{S_n\}$ the sequence of its n-th partial sums. Let $\{p_n\}$ be a sequence of constants real or complex, and let us write

$$P_n = p_o + p_1 + \dots + p_n .$$

If

$$\sigma_n = \frac{1}{P_n} \sum_{\nu=o}^{n} p_{n-\nu} s_\nu \rightarrow \infty \qquad (P_n \neq 0)$$

as $\sigma_n \rightarrow \infty$, then we say that the series is summable by Nörlund method (N, p_n) to σ. The series Σa_n is said to be absolutely summable by the Nörlund method or summable $|N, p_n|$ if $\{\sigma_n\}$ is of bounded variation, i.e.

$$\Sigma \mid \Delta\sigma_n \mid < \infty .$$

Suppose that $\varnothing(t)$ is an even and integrable function, periodic with period 2π. And let

$$(1.1.1) \quad \varnothing(t) \sim \frac{a_o}{2} + \sum_{n=1}^{\infty} a_n \operatorname{Cos} nt$$

and

$$\Phi(t) = \int_0^t \mid \varnothing(u) \mid du$$

1.2. Very recently Hsiang [1] has proved a theorem for absolute Nörlund summability of (1.1.1). The theorem of Hsiang is as follows:

Theorem. Let $\{P_n\}$ be a sequence of positive constants.

If $\{\Delta p_n\} = \{(p_n - p_{n-1})\}$ is monotonic and bounded and if

$$\text{(i)} \quad \sum_{n=2}^{\infty} \frac{n}{P_n} (\log n)^{-\Delta} < \infty$$

for some $\Delta > 0$, and

$$\text{(ii)} \quad (\log \frac{1}{t})^{\Delta} \quad | \varnothing (t) | = 0 (1), \text{ as } t \to 0,$$

then the series $a_0/2 + \Sigma a_n$ is summable $| N, p_n |$.

The object of this paper is to prove the above mentioned theorem of Hsiang under a weaker condition i.e. to replace condition (ii) by a weaker condition and to give a new short proof.

Hence we prove the following theorem:

Theorem. *Let* $\{p_n\}$ *be a sequence of positive constants such that* $\{\Delta p_n\} = \{(p_n - p_{n-1})\}$ *is monotonic and bounded, and if*

$$\text{(i)} \quad \sum_{n=2}^{\infty} \frac{n}{P_n(\log n)^{\Delta}} < \infty$$

for some $\Delta > 0$, *and*

$$\text{(ii)} \quad \int_0^t |\varnothing(u)| \, du = 0 \left\{ \frac{t}{(\log \frac{1}{t})^{\Delta}} \right\}, \text{ as } t \to o ,$$

then the series $a_0/2 + \Sigma a_n$ *is summable* $| N, p_n |$.

1.3. For the proof of our theorem we require the following lemmas.

Lemma 1. *If* $\{p_n\}$ *is defined as in the theorem, and if the series*

$$\Sigma \left|\frac{t_n}{p_n}\right| < \infty ,$$

where $t_n = \sum_{\nu=o}^{n} S_\nu$, *then* Σa_n *is summable* $| N, p_n |$

Lemma 2 *Let*

$$K_n (t) = \sum_{K=o}^{n} D_k (t) ,$$

where $D_k (t) = \frac{1}{2} + \cos t + \ldots + \cos nt.$

Then

$$| K_n (t) | < 2n^2$$

and

$$| K_n (t) | < \frac{C^*}{t^2} \text{ for } \frac{1}{n} \leq t \leq \pi$$

Lemma 3 *If* (ii) *is satisfied then*

$$\int_{1/n}^{e^{-\Delta/\eta}} \frac{|\varnothing(t)|}{t^2} dt = O\left(\frac{n}{(\log n)^{\Delta}}\right)$$

for $0 < \eta < 1.$

Proof. Integrating by parts we have

$$\int_{1/n}^{-\Delta/\eta} \frac{|\Phi(t)|}{t^2} dt = \left[\frac{\Phi(t)}{t^2}\right]_{1/n}^{e^{-\Delta/\eta}} + 2 \int_{\frac{1}{n}}^{e^{-\Delta/\eta}} \frac{\Phi(t)}{t^3} dt$$

$$= O\left[\frac{1}{t(\log \frac{1}{t})^{\Delta}}\right]_{1/n}^{e^{-\Delta/\eta}} + O\left[\int_{1/n}^{e^{-\Delta/\eta}} \frac{dt}{t^{\eta} (\log \frac{1}{t})^{\Delta} . t^{2-\eta}}\right]$$

* where C is any positive finite constant.

$$= O\left(\frac{n}{(\log n)^{\Delta}}\right) + O\left[\int_{1/n}^{e^{-\Delta/\eta}} \frac{dt}{t^{\eta}\,(\log \frac{1}{t})^{\Delta}\,.\,t^{2-\eta}}\right]$$

But $\dfrac{1}{t^{\eta}(\log \frac{1}{t})^{\Delta}}$ is monotonic decreasing in $\left(\frac{1}{\eta}, e^{-\Delta/\eta}\right)$

therefore

$$\int_{1/n}^{e^{-\Delta/\eta}} \frac{|\varnothing(t)|}{t^2}\,dt = O\left(\frac{n}{(\log n)^{\Delta}}\right)$$

$$+ O\left[\frac{n^{\eta}}{(\log n)^{\Delta}}\int_{1/n}^{e^{-\Delta/\eta}} t^{-2+\eta}\,dt\right]$$

$$= O\left(\frac{n}{(\log n)^{\Delta}}\right) + O\left(\frac{n}{(\log n)^{\Delta}}\right)$$

$$= O\left(\frac{n}{(\log n)^{\Delta}}\right)$$

Hence the lemma is proved.

Lemma 4. *If* (ii) *is satisfied, then*

$$t_n = O\left(\frac{n}{(\log n)^{\Delta}}\right),$$

as $\eta \to \infty$.

Proof.

$$\pi t_n = \int_0^{\pi} \varnothing\;(t) \sum_{\nu=0}^{\eta} D_{\nu}^{*}\,(t)\;dt\;.$$

$$= \int_0^{1/n} + \int_{1/n}^{e^{-\Delta/\eta}} + \int_{e^{-\Delta/\eta}}^{\pi}$$

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* $D_{\nu}(t)$ is the same as defined in lemma 2.

where $0 < \eta < 1$,

$$= M_1 + M_2 + M_3 \text{ , say}$$

By Riemann – Lebesgue theorem $M_3 = O(1)$ because

$$\sum_{\upsilon=0}^{\eta} D_\upsilon(t) = \frac{1}{2}\left\{ \frac{\sin(\eta+1)\, t/2}{\sin t/2} \right\}^2 .$$

Now by lemma 2

$$| M_1 | \leq 2 \int_0^{1/n} |\varnothing(t)|\, n^2\, dt = O\left(\frac{n}{(\log n)^\Delta}\right)$$

Lastly

$$| M_2 | \leq \int_{1/n}^{e^{-\Delta/\eta}} \frac{|\varnothing(t)|}{t^2}\, dt = O\left(\frac{n}{(\log n)^\Delta}\right),$$

by lemma 3. Hence finally $t_n = O\left(\frac{n}{(\log n)^\Delta}\right)$.

1.4. The proof of our theorem is a direct consequence of lemmas 1 and 4.

ACKNOWLEDGEMENT

I would like to express my indebtedness to Dr. S. M. Mazhar for his kind help and constant encouragement during the preparation of this paper.

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[1] Hsiang, F. C. On the Nörlund Summability of a Fourier Series. J. Australian Math. Soc. 7 (1967), 252-256.

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Prix de l'abonnement annuel

Turquie : 15 TL; Étranger: 30 TL.

Prix de ce numéro : 5 TL (pour la vente en Turquie).

Prière de s'adresser pour l'abonnement à : Fen Fakültesi Dekanlığı Ankara, Turquie.

Ankara Üniversitesi Basımevi. Ankara – 1975