COMMUNICATIONS

DE LA FACULTÉ DES SCIENCES
DE L'UNIVERSITÉ D'ANKARA

Série A: Mathématiques, Physique et Astronomie

TOME 23 A ANNÉE 1974

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by

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Faculté des Sciences de l'Université d'Ankara
Ankara, Turquie

Communications de la Faculté des Sciences de l'Université d'Ankara

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Characterizations of Spherical Curves in Euclidean n-Space

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(Received Dec. 12, 1974)

ABSTRACT:

We give the characterizations for the regular curves each of which lies on the (n–1) –sphere S^{n-1} of n dimensional Euclidean Space E^n. We express these characterizations in the higher curvatures of the curves.

I. Basic Concepts.

Theorem I.1 : If X is parametrized curve in u dimensional Euclidean space E^n then X can always be peparametrized by an arc length parameter [1].

Theorem I.1 says that, in general, we can have the arc-length parametrized curve X (s) with arc-length parameter s as a parametrized curve in E^n.

Let I be an open interval in the real line $\mathbb{R}$. We shall interpret this liberally to include not only the usual type of open interval $a < s < b$ (a,b real numbers), but also the types of $a < s$ (a half line to $+\infty$), $s < b$ (a half-line to $-\infty$), and also the whole real line. Henceforth we denote an arc-length parametrized curve of E^n by a map $X:I \to E^n$ which is a C^∞ parametrization by arc-length.

We assume that at each point X (s), of the curve $X:I \to E^n$, the derived vectors

$$\{X', X'', \ldots, X^{(r)}\}$$

are linearly independent, where

$$X' = \frac{dX}{ds}(s),\ X'' = \frac{d^2X}{ds^2}(s),\ \ldots,\ X^{(r)} = \frac{d^rX}{ds^r}(s).$$

Therefore there exists an algorithm, called the Gramm-Schmidt process, for converting X', X'' ..., $X^{(r)}$ into an orthonormal basis

$$\{V_1, V_2, \ldots, V_r\}$$

of the tangent space $T_{E^n}(X(s))$ of E^n at the point $X(s) \varepsilon E^n$. This system is called the Frenet r-handed (or r-frame) of the curve X at the point X (s) [2].

If we denote the inner product (dot product) $E^n x\ E^n \to \mathbb{R}$ over E^n by $< , >$ we have

$$<V_i, V_j> = \delta_{ij}$$

and then the derivatives of the frame vectors satisfy the following Frenet equations.

$$\text{(I.1)} \qquad \begin{cases} V'_i = -k_{i-1} V_{i-1} + k_i V_{i+1}, & 2 \leq i \leq r\text{-}1 \\ V'_1 = k_1 V_2, & \\ V'_r = -k_{r-1} V_{r-1}, & \end{cases}$$

where k_i, $1 \leq i \leq r-1$, is the curvature, with order i, of the curve X at its point X (s) [2]. These formulae (I.1) are called the Frenet Formulae which give us the derived vectors V'_i, $1 \leq i \leq r$. Then we mention the following theorem which is importent in Chapter III.

Theorem I.2 : Let $X: I \to E^n$ *be a regular curve. At the point* X (s) *of it if Frenet n-frame is*

$$\{V_1, V_2, \ldots, V_n\}$$

then we have

$$X^{(p)} = \sum_{j=1}^{p} a_j V_j, \quad 1 \leq p \leq n,$$

where $a_j \varepsilon \mathbb{R}$.

Proof : We use the induction process:

(i) Since the curve is given by its arc-length parameter s, $X' = V_1$. Therefore if $p = 1$ the theorem is trivial.

(ii) Let us suppose that the theorem is proved for the cases

$1 \leq r < n$. Then we prove that the theorem is also valid for the case $p = r+1$.

Since we can write

$$X^{(r)} = \sum_{j=1}^{r} a_j V_j$$

differentiating this, with respect to s, we obtain

$$X^{(r+1)} = \sum_{j=1}^{r} a_j' V_j + \sum_{j=1}^{r} a_j V_j'.$$

Using Equations (I.1) this gives us

$$X^{(r+1)} = \sum_{j=1}^{r} [a'_j V_j + a_j (-k_{j-1} V_{j-1} + k_j V_{j+1})]$$

where if we write that

$$\begin{aligned} b_1 &= a_1' - k_1 a_2, \\ b_j &= a'_j + a_{j-1} k_{j-1} - a_{j+1} k_j, \quad 2 \leq j \leq r-1, \\ b_r &= a_r' + a_{j-1} k_{r-1}, \\ b_{r+1} &= a_r k_r, \end{aligned}$$

then we obtain

$$X^{(r+1)} = \sum_{j=1}^{r+1} b_j V_j$$

which completes the theorem.

II. Osculating p–Spheres S^p and The Curvature Lines.

Definition II.1 : The p*–sphere* S^p *in* E^n *which passe sthrough* X (s) *and is in contact with the curve having* p+2 *points in common at the point of contact* X (s) *is called the osculating* p*–sphere to the curve at* X (s).

At any given point, the curve has exactly p-th order contact with its osculating p-sphere for $p = r$.

We suppose that at the point X (s) of the curve $k_{n-1} \neq 0$ and then we will educate the osculating sphere S^{n-2}. In order to do this we will need the following theorem.

Theorem II.1 : *Let* $k_{n-1} \neq 0$ *at every point* X (s) *of a curve* $X: I \to E^n$. *Then at the point* X (s) *of the curve the center of* (n–2)*–osculating sphere* S^{n-2} is

$$a = X - \sum_{i=2}^{n-2} m_i V_i + \lambda V_n , \ a \varepsilon E^n,$$

where $\lambda \varepsilon \mathbb{R}$ and $m_1 = 0$, $m_2 = -1/k_1$ and

$$m_i = \{m'_{i-1} + m_{i-2} k_{i-2}\} \frac{1}{k_{i-1}}, \ 2 < i < n.$$

Proof: Suppose that at the point X (s) there is at least one (n-2)-osculating sphere with the center $a \varepsilon E^n$ and radius $r \varepsilon \mathbb{R}$. In this case let define the function $f: I \to \mathbb{R}$ as

$$\text{(II.1)} \qquad f(s) = < X(s)-a, X(s)-a >-r^2.$$

Since $X(s) \varepsilon S^{n-2}$, we have

$$f(s) = 0.$$

On the other hand, if X (s) is a second order contact point of the curve and the osculating sphere S^{n-2} for the case $\forall s_j \to s$ then we have

$$f(s_1) = 0$$
$$f(s_2) = 0,$$

where $s_1, s_2 \varepsilon I$. Applying the mean value theorem to these equations we obtain f (s) = 0 and f′(s) = 0. Similarly, if X (s) is a n-th order point of the curve and the osculating sphere S^{n-2} for the case

$$\forall s_j \to s$$

we have

$$f(s_j) = 0, \ 1 \leq j \leq n; \ s_j, s \varepsilon I$$
$$f(s) = 0,$$

and from the mean value theorem

$$\text{(II.2)} \qquad f(s) = 0, f'(s) = 0, \ldots, f^{(n-1)}(s) = 0.$$

Since $k_{n-1} \neq 0$ we can imagine that Frenet n-frame is exist at the point X (s) of the curve. Hence, $\{V_1, V_2, \ldots, V_n\}$ is a basis of tangent space $T_{E^n}(X(s))$ and $(X(s)-a) \varepsilon T_{E^n}(X(s))$ can be expressed, in a unique way, as

$$X(s)-a = \sum_{i=1}^{n} m_i V_i, \forall m_i \varepsilon R.$$

Replacing (II.1) in (II.2) we obtain

$$f'(s) = 2 < X'(s), X(s)-a > = 0.$$

On the other hand s is arc-length parameter and so $V_1 = X'$. Hence

$$< V_1, X(s)-a > = 0.$$

and then $m_1 = 0$ so

$$\text{(II.3)} \qquad X(s)-a = \sum_{i=2}^{n} m_i V_i.$$

In Equation (II.1) since $f''(s) = 0$ and using the Frenet formulae we obtain that

$$\frac{1}{2} f''(s) = \frac{d(< V_1, X(s)-a >)}{ds} = 0$$

and then

$$< k_1 V_2, X(s)-a > + < V_1, V_1 > = 0$$

$$k_1 < V_2, X(s)-a > + 1 = 0$$

$$m_2 = -1/k_1.$$

Hence the theorem is proved for the coefficients m_i.

From the Equations (II.2) one can write

$$f'''(s) = 2 < X'''(s), X(s)-a > = 0.$$

On the other hand differentiating the equation

$$X' = V_1$$

according to s and using the Frenet Formulae we obtain that

$$X'' = k_1 V_2$$

$$X''' = -k_1^2 V_1 + k_1' V_2 + k_1 k_2 V_3.$$

Replacing the last equation in $f'''(s) = 0$ and using (II.3) we have

$$< -k_1^2 V_1 + k'_1 V_2 + k_1 k_2 V_3, \sum_{i=2}^{n} m_i V_i > = 0$$

or

$$k_1' m_2 + k_1 k_2 m_3 = 0.$$

Since $m_2 = -1/k_1$ for m_3 we can have

$$m_3 = m_2' / k_2$$

and the theorem is also proved for the case $i = 3$.

Suppose that the theorem is valid for the cases for p such that $2 < p < n-1$ and then we will prove it for the case $p+1$.

Let define a function $\Psi_p(s)$ by the equation

$$\text{(II.3)} \qquad f^{(p)}(s) = < X^{(p)}(s), X(s)-a > + \Psi_p(s).$$

From the derivative of

$$f'(s) = 2 < X'(s), X(s)-a >$$

we can see that in the expression of $f^{(p)}(s)$ the derivatives higher than $X^{(p)}(s)$ do not exist. From the Theorem (I.2) we can write

$$\text{(II.4)} \qquad X^{(p)}(s) = \sum_{j=1}^{p} a_j V_j, \ \forall a_j \in \mathbb{R}.$$

Replacing (II.3) and (II.4) in the Equation $f^{(p)}(s) = 0$ we have

$$< \sum_{j=1}^{p} a_j V_j, \sum_{j=2}^{n} m_j V_j > + \Psi_p(s) = 0$$

or

$$\sum_{j=2}^{p} m_j a_j + \Psi_p(s) = 0.$$

Therefore we have

$$\text{(II.5)} \qquad m_p = -\frac{1}{a_p} \left[\sum_{j=2}^{p-1} m_j a_j + \Psi_p(s) \right].$$

Differentiating the Equations (II.3) and (II.4) we have, respectively,

$$f^{(p+1)}(s) = \langle X^{(p+1)}(s), X(s)-a \rangle + \langle X^{(p)}(s), V_1 \rangle + \Psi'_p(s) = 0$$

and

$$X^{(p+1)}(s) = (a'_1 - k_1 a_2) V_1 + \sum_{j=2}^{p-1} (a'_j + a_{j-1} k_{j-1} - a_{j+1} k_j) V_j + (a'_p + a_{p-1} k_{p-1}) V_p + a_p k_p V_{p+1}.$$

Hence replacing the values of $X^{(p+1)}(s)$ and $X(s)-a$ in $f^{(p+1)}(s) = 0$ we have

$$\sum_{j=2}^{p-1} (a'_j + a_{j-1} k_{j-1} - a_{j+1} k_j) m_j + (a'_p + a_{p-1} k_{p-1}) m_p + a_p k_p m_{p+1} + a_1 + \Psi'_p(s) = 0$$

From the last equation, calculation gives us that the value of m_{p+1} is

$$m_{p+1} = -\frac{1}{a_p k_p} \{ \sum_{j=2}^{p-1} (a'_j + a_{j-1} k_{j-1} - a_{j+1} k_j) m_j + (a'_p + a_{p-1} k_{p-1}) m_p + a_1 + \Psi_p'(s) \}.$$

And from the Equation (II.5) differentiation gives us that

$$m'_p = -\frac{1}{a_p} [\sum_{j=2}^{p-1} (a'_j + a_{j-1} k_{j-1} + a_{j+1} k_j) m_j + (a'_{p-1} + a_{p-2} k_{p-2}) m_{p-1} + (a'_p + a_{p-1} k_{p-1}) m_p + a_1 + \Psi'_p(s)].$$

Hence we can write

$$\text{(II.6)} \qquad m_{p+1} = \{ m'_p + m_{p-1} k_{p-1} \} \frac{1}{k_p}$$

and since we obtain m_{p+1} from the equation $f^{(p+1)}(s) = 0$ the coefficients $m_2, m_3, \ldots, m_{n-1}$ are determinet in an equation like (II.6). $m_n = \lambda \varepsilon \mathbb{R}$ is a parameter. ■

Corollary I : At any point $X(s)$ *of a regular curve* $X: I \to E^n$ *if* $k_{n-1} \neq 0$ *then all the centers of* (n-2) *-osculating spheres* S^{n-2} *are collinear.*

Proof: From the Theorem (II.1) the center of S^{n-2} at X (s) is

$$a = X(s) - \sum_{i=2}^{n-1} m_i V_i - \lambda V_n, \quad \lambda \varepsilon |R,$$

where,

$$m_2 = -1/k_1, \ m_i = \{m'_{i-1} + m_{i-2} k_{i-2}\} \frac{1}{k_{i-1}}.$$

For $\forall s \varepsilon I$, m_i, V_i, X(s) are constant. Hence a, the center of (n–2)–osculating sphere S^{n-2}, lies on the straight line which passes throught the point $X(s) - \sum_{i=2}^{n-1} m_i V_i$ and parallel to the vector V_n. ■

Definition (II.2) : Let X: I → E^n *be a given ragular curve. The locus of the centers of* (n–2)*–osculating skteres* S^{n-2} *is called the curvature line of the curve, at the point* X (s).

Colollary II. At any point X (s) *of a regular curve* X: I → E^n *if* $k_{n-1} \neq 0$ *then the center of* (n–1) – *osculating sphere* S^{n-1} *is*

$$a = X(s) - \sum_{i=2}^{n} m_i V_i$$

where, $m_1 = 0$, $m_2 = -1/k_1$ and

$$m_i = \{m'_{i-1} + m_{i-2} k_{i-2}\} \frac{1}{k_{i-1}}, \ 2 < i \leq n.$$

Proof: According to Definition (II.1) the point X (s) is a (n+1)–th order contact point of the curve X and its (n–1)–osculating sphere. Therefore we have the expressions (II.2) and also $f^{(n)}(s) = 0$. Hence we can repeat here the same proof of Theorem (II.1). ■

Corollary III. If $\forall k_{n-1} \neq 0$, s ε I, *for the curve* X: I → E^n *then at the point* X (s) *the osculating sphere* S^{n-1} *is unique and its radius is* $r = (\sum_{i=2}^{n} m_i^2)^{1/2}$.

Proof: According to Corollary II the center of (n–1)–osculating sphere S^{n-1} is unique so S^{n-1} is unique.

At the point X (s) the radius r of osculating sphere S^{n-1} is

$$r = \| a - X(s) \|.$$

From the corollary II we have

$$r = \| a - X(s) \| = \| \sum_{i=2}^{n} m_i V_i \|$$

$$= (\sum_{i=1}^{n} m_i^2)^{1/2}. \blacksquare$$

III. Spherical Curve of E^n *and Its Characterization.*

In this paragraph we will give a necessary and suffiecent condition for a curve of E^n to be a spherical curve.

Definition III. 1 : Let $X: I \to E^n$ *be a curve and* $S^p \subset E^n$ *be a p-sphere. If* $X \subset S^p$ *then* X *is called a spherical curve in* E^n.

The case p=n-1 is supposed in this paragraph. Because of $S^p = S^{n-1} \cap H_{n(p+1)}$ every curve X of S^p in E^n lies in a (p+1)–hyperplane $H_{n(p+1)}$ [3] so this case is not a special case. Since a (p+1) – hyperplane $H_{n(p+1)}$ is isomorph to Euclidean (p+1)–space E^{p+1}, a curve of S^p can be taken as another curve of another sphere $S_0^p \subset E^{p+1}$. Hence $X \subset S^p \subset E^n$ so we can see that $k_{p+1} \neq 0$. Therefore we can only suppose the curves of S^{n-1} whose curvature $k_{n-1} \neq 0$.

Theorem III. 1 : Let $X: I \to E^n$ *be a regular curve such that*

$$k_{n-1} \neq 0, \forall s \in I, m_1 = 0, m_2 = -\frac{1}{k_1}$$

$$m_i = \{ m'_{i-1} + m_{i-2} k_{i-2} \} \frac{1}{k_{i-1}}, \quad 2 < i \leq n,$$

and $X \subset S_0^{n-1}$; *where* S_0^{n-1} *is an* (n–1)*–sphere with the center* O. *Then*

$$< X(s), V_i > = m_i$$

where $\{ V_1, V_2, ..., V_n \}$ *is the Frenet n-frame at the point* X (s) *of hec urve.*

Proof: We apply the induction process:

If $i = 2$ and the radius of S_0^{n-1} is r we can write

$$< X(s), X(s) > = r^2$$

and then from this by differentiation, with respect to s,

$$< X(s), X'(s) > = 0$$

or

$$< X''(s), X(s) > + < X'(s), X'(s) > = 0$$

or

$$< X''(s), X(s) > + 1 = 0.$$

On the other hand since we know that $V_2 = X''(s) / \| X''(s) \|$, $\| X''(s) \| = k_1$ [2] we can have

$$k_1 < V_2, X(s) > = -1$$

or

$$< V_2, X(s) > = -1 / k_1 = m_2$$

which proves the theorem in the case $i=2$.

Suppose that the theorem is proved in the cases $p < n$. Then we can write

$$< X(s), V_p > = m_p$$

which gives us, by differentiation, with respect to s,

$$< V_1, V_p > + < X(s), V'_p > = m'_p$$

in this last equation, replacing the Frenet Formulae (I,1) we have

$$< X(s), V_{p+1} > = \{ m'_p + m_{p-1} k_{p-1} \} \frac{1}{k_p}$$

$$< X(s), V_{p+1} > = m_{p+1}$$

which completes the theorem. ■

Theorem III. 2 : Let $X: I \to E^n$ *be a given regular curve such that* $k_{n-1} \neq 0$, $\forall s \varepsilon I$. *If* $X \subset S_0^{n-1}$ *then all the* (n–1)–*osculating spheres of the curve* X *coincide with* S_0^{n-1}.

Proof: Suppose that the center of (n–1)–osculating sphere at the point X (s) of X is a. From the Corollary II of Theorem II.1 we have

$$a = X(s) - \sum_{j=2}^{n} m_j V_j$$

where $m_1 = 0$, $m_2 = -1/k_1$, $m_i = \{m'_{i-1} + m_{i-2}k_{i-2}\} \dfrac{1}{k_i}$, $2 < i \leq n$ and $\{V_1, V_2, \dots, V_n\}$ is the Frenet n-frame at X(s) of X. According to Theorem III.1 the expression of a can be write as

$$a = X(s) - \sum_{j=1}^{n} < X(s), V_j > V_j.$$

Since $\{V_1, V_2, \dots, V_n\}$ is a basis of the tangent space $T_{E^n}(O)$ we can have

$$X(s) = \sum_{j=1}^{n} < X(s), V_j > V_j$$

and then

$$a = X(s) - X(s)$$

or

$$a = 0$$

which shows that the centers of S_0^{n-1} and (n–1)–osculating sphere at X (s) of X coincide. On the other hand since d (X (s), O) = r we see that the theorem is completed. ■

Corollary I. If $S_b^{n-1} \subset E^n$ *is an* (n–1)*–sphere and the curve* X *is* $X \subset S_b^{n-1}$ *then* (n–1) *– osculating sphere at the point* X (s) *of* X *is* S_b^{n-1}.

The proof of this corollary can be given in the light of the fact that "The spheres with the same radius are isometric".

The radius of an (n–1)–osculating sphere of a curve X depends on the center X (s) of the sphere. The following theorem makes clear this dependence.

Theorem III.3 : Let $X: I \to E^n$ *be a given regular curve whose* $k_{n-1} \neq 0$ *for* $\forall s \in I$ *and let* $m_n \neq 0$ *(see the Corollary II of* Theorem II.1). *The radii of* (n–1)*–osculating spheres at* X (s) *of* X *are constant for* $\forall s \in I \Leftrightarrow$ *The centers of the* (n–1) *– osculating spheres are the same point.*

Proof: We need the following calculation:

As we know, let a (s) and r (s) be, respectively, the center and the radius of (n–1)–osculating sphere at X (s) of the curve X. Since X (s) is a point of the (n–1)–osculating sphere we have

$$< X(s)-a(s), X(s)-a(s) > = (r(s))^2$$

which gives us, by differentiation with respect to s,

$$\text{(III.1)} \quad < V_1, X(s)-a(s) > - < \frac{da}{ds}(s), X(s)-a(s) >$$

$$= r(s). \frac{dr}{ds}(s).$$

According to Corollary II of Theorem II.1 we have

$$< V_1, X(s)-a(s) > = 0$$

and replacing this in (III.1) we obtain

$$\text{(III.2)} \quad < \frac{da}{ds}(s), X(s)-a(s) > = - r(s). \frac{dr}{ds}(s).$$

Now we can give the proof:

(Necessity): Suppose that at every point X (s) of the curve X the radii r (s) are constant. According to Corollary II of Theorem II.1 the radius of (n-1)–osculating sphere at X (s) is

$$r(s) = \left(\sum_{i=2}^{n} m^2_i \right)^{1/2}$$

$$r(s) = \text{constant} \Rightarrow dr/ds = 0$$

and

$$\text{(III.3)} \qquad \sum_{i=2}^{n} m'_i m_i = 0.$$

From the corollary II of Theorem II.1 since we have

$$m'_i = - k_{i-1} m_{1-i} + k_i m_{i+1}, \quad 2 < i \leq n-1 .$$

Equation (III.3) reduces to

$$\text{(III.4)} \quad m_2 m'_2 + \sum_{i=3}^{n-1} m_i [- k_{i-1} m_{i-1} + k_i m_{i+1}] + m_n m'_n = 0$$

or replacing $m'_2 = m_3 k_2$ in (III.4)

$$m_2m_3k_2 - \sum_{i=3}^{n-1} k_{i-1}m_{i-1}m_i + \sum_{i=3}^{n-1} k_im_im_{i+1} + m_nm'_n = 0$$

or after some cancellations

(III.5) $\quad m_n\,(m'_n + k_{n-1}m_{n-1}) = 0.$

Then acording to Theorem II.1 we have

$$\frac{da}{ds}(s) = V_1 - \sum_{i=2}^{n} m'_i\, V_i - \sum_{i=2}^{n} m_iV'_i$$

where replacing (I.1) and (III.4) we obtain

(III.6) $\quad \dfrac{da}{ds} = (\,m'_n + k_{n-1}m_{n-1})\,V_n.$

From the Equations (III.5) and (III.6)

$$\frac{da}{ds} = 0,\ \forall s \in I$$

and so

$$a\,(s) = \text{constant}.$$

(Sufficiency): Suppose that $a\,(s) = \text{constant}$, $\forall s \in I$. Then $\dfrac{da}{ds} = 0$ and according to (III.2)

$$< \frac{da}{ds}\,,\ X\,(s) - a\,(s) > \; = -\,r\,(s)\,\frac{dr}{ds} = 0$$

or

$$r\,(s)\,\frac{dr}{ds}\,(s) = 0.$$

In the last equation if $r\,(s) = 0$ then from Corollary III of Theorem II.1 we have

$$\sum_{i=2}^{n} m^2_i = 0$$

which gives us $m_i = 0\,,\ 2 \leq i \leq n$.

On the other hand

$$m_2 = -\,1/k_1.$$

Since $k_1 = \|\,X''(s)\,\| / \|\,X'\,(s)\,\|$ [2] the case $m_2 = 0$ implies that $\|\,X'\,(s)\,\| = 0$ or $\|\,X''\,(s)\,\| \to \infty$. In the case $\|\,X'\,(s)\,\| = 0$

the curve is not regular. Then we must have $\| X'(s) \| \neq 0$ and in E^n $\| X''(s) \|$ can not be infinitive. Therefore we must have

$$\frac{dr}{ds}(s) = 0$$

and so $r(s) = \text{constant}$. ■

A characterization of a curve in E^n to be an (n–1)–sphere can be given by the following theorem.

Theorem III.4 : Let $X: I \rightarrow E^n$ *be a regular curve such that* $k_{n-1} \neq 0, \forall s \varepsilon I$ *and* $m_n(s) \neq 0$. *The curve* X *lies on a* (n–1)–*sp- here* $\Leftrightarrow$ *The centers of* (n–1)–*osculating spheres of the curve* X *are all the same point.*

Proof: (Necessity): Suppose that X lies on S_b^{n-1}. Then according to Corollary I of Theorem III.2, for $\forall s \varepsilon I$ at the poînts X (s) of X, (n–1) – osculating sphere is S_b^{n-1} whose center is a fixed point.

(Sufficiency): Suppose that, at the point X (s), the center of (n–1)–osculating sphere of X is a fixed point b. Then Theorem III.3 says that, at every point X (s) of X, the radii of (n–1)–osculating spheres are also equal. Hence for $\forall s \varepsilon I$ at every point X (s),

$$d(X(s), b) = r = \text{constant}$$

which means that X is a spherical curve. For $\forall s \varepsilon I$ the curvature $k_{n-1} \neq 0$ implies that this sphere is S_b^{n-1}. ■

Another characterization of a curve in E^n to be on a (n–1)–sphere can be given in terms of its curvatures by the following theorem.

Theorem III.5 : Let $X: I \rightarrow E^n$ *be a regular curve such that for* $\forall s \varepsilon I$, $k_{n-1} \neq 0$, $m_n \neq 0$, $m_1 = 0$, $m_1 = 0$, $m_2 = -1/k_1$ *and for* $2 < i \leq n$,

$$m_i = \{ m'_{i-1} + m_{i-2} k_{i-2} \} \frac{1}{k_{i-1}} .$$

The curve X *lies on a sphere* $S^{n-1} \Leftrightarrow m'_n + m_{n-1} k_{n-1} = 0$.

Proof: (Necessity): According to Corollary II of Theorem II.1 the center of (n–1)–osculating sphere at X (s) is

$$\text{(III.7)} \qquad a(s) = X(s) - \sum_{j=2}^{n} m_j V_j .$$

On the other hand according to Theorem III.4 it is necessary that a (s) is a fixed point for the curve X to lie on a (n–1)–sphere. This implies that

$$da/ds = 0 .$$

Hence from Equation (III.7), by differentiation with respect to s, we have

$$\text{(III.8)} \qquad \frac{da}{ds} = (m'_n + m_{n-1} k_{n-1}) V_n = 0$$

which completes the necessity of the theorem.

(Sufficiency): Suppose that for a curve X we have

$$m'_n + m_{n-1} k_{n-1} = 0.$$

Replacing this in (III.8) we obtain

$$da / ds = 0$$

which implies that

$$a(s) = \text{constant}.$$

Thus jointing this result with Theorem III.4 we see that X is a spherical curve and for $\forall s \in I$, since $k_{n-1} \neq 0$ this lies on a (n–1)–sphere of E^n. ■

IV. Specil Cases.

1. *The Case n=3.*

In the case that n=3 the formulae in Theorem III.5 reduces to

$$\text{(IV.1)} \qquad m'_3 + k_2 m_2 = 0,$$

where replacing $m_3 = m'_2 / k_2$ we have

$$(m'_2 / k_2)' + m_2 k_2 = 0.$$

Since $m_2 = -1/k_1$ the last equation gives us

(IV.2) $$\frac{1}{k_1} k_2 + \left[\left(\frac{1}{k_1}\right)' \frac{1}{k_2}\right]' = 0 ,$$

where replacing

$$\frac{1}{k_1} = \vartheta, \ \frac{1}{k_2} = \sigma \text{ and } k_2 = \tau$$

we obtain

(IV.3) $$\vartheta \tau + (\vartheta' \sigma)' = 0$$

which is well-known, in the books on elementary differential geometry, characterization for spherical curves.

On the other hand in the case n=3 the function f in [4] can be taken as $f = -m_3$. Similarly, taking n=3 in Corollary III of Theorem III.1, the radius of the sphere can be obtained as

$$r = (m_2^2 + m_3^2)^{1/2}$$

or

$$r = \left[\left(\frac{1}{k_1}\right)^2 + f_2\right]^{1/2}$$

which is the same value in [4]. Hence, for n=3, Theorem I.1 in [4], can be obtained as another special case from Theorem III.5. Since $f = -m_3$ are, respectively,

$$m_3 k_2 = m'_2 \quad , \quad m'_3 + k_2 m_2 = 0$$

which can be obtained from Theorem III.5, for n=3.

On the other hand since Theorem 1.2 in [4] is deduced from (IV.2) we can say that it is also another special case of the Theorem III.5 .

REFERENCES

[1] **Auslander. L.** "*Differential Geometry*" A Harper International Edition Jointly Published by Harper & Row. New York., London. 1967.
Library of Congress Catalog Card Nu. 67–10789, pp: 85–86.

[2] **Gluck, H.** "*Higher Curvature of Curves in Euclidean Space*" Amer. Math. Month. 73 (1966). pp: 699–704.

[3] **Borsuk, K.** "*Multidimensional Analytic Geometry*" Copyright C 1969 by Pánstwow Wydawnictwo Naukkowe (PWN-Polish Scientific Publishers) Warszawa (Poland), UI, Miodowo 10. pp: 293–294.

[4] **Yung-Chuw Wong.** "*On an Explicit Characterization of Spherical Curves*" Proc. Amer. Math. Soc. 27 (1972). pp: 239–242.

Özet:

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Ankara Üniversitesi Basımevi. Ankara–1975