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Continuous Operators on Paranormed Spaces and Matrix Transformations of Strong Cesaro Summable Sequences

By

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SUMMARY

In this paper, the concept of a paranormed β_j-space is defined, where $j \geq 1$ is an integer, and two Banach-Steinhaus type theorems are proved for the sequences of continuous linear functionals on such space. For example, the necessary and sufficient conditions are given for a sequence $(A_n(x))$ of continuous linear functionals to be in the space of generalized entire sequences, for each x belonging to a paranormed β_j-speace. These results are then used to characterize the matrix transformations between the strong Cesaro summble sequence space $w(p)$ and the space of generalized entire sequences.

1. *Introduction*

In §. 2 we prove the theorems which are the generalizetions of some results given in [10]. These theorems and some results given in [10] are applied to characterize the classes $(w(p), l_\infty(q))$, $(w(p), c_0(q))$, $(w(p), c(q))$, $(w(p), w_\infty(q))$ and $(w(p), w_0 q))$ for $0 < \inf p_k \leq p_k \leq 1$ and bounded $q = (q_k)$, $q_k > 0$, in § 3. N, R, C will denote the set of natural numbers, real numbers and complex numbers, respectively. By (x_k) we mean the sequence $(x_1, x_2, \ldots)$ and by $\Sigma_k x_k$ we mean $\sum_{k=1}^{\infty} x_k$.

X will denote a nontrivial complex linear space with zero element θ and a paranorm g for which the following conditions are satisfied:

$g(\theta)=0$, $g(x)=g(-x)$, $g(x+y)\leq g(x)+g(y)$ for every $x, y \in X$ and $\lambda\to\lambda_o$, $g(x-x_o)\to 0$ imply $g(\lambda x-\lambda_o x_o)\to 0$, where $\lambda_o \in C$ and $x_o \in X$.

Let Y be a subset of X, we denote the closure of Y in X by $\bar{Y}$. Y will be called everywhere dense in X if and only if $\bar{Y} = X$ and nowhere dense in X if and only if $\bar{Y}$ contains no neighbourhood.

Extending the definitions of Sargent [1] and Maddox-Willey [10], we can define a paranormed β_j-space as follows: Let (X_n) be a sequence of subsets of X and let j denote a positive integer. If $\theta \in X_k$, for $k=1,\ldots,j$; and $x, y \in X_n$ implies $x \mp y \in X_{n+j}$ for every $n \in N$, then we say that the sequence (X_n) is an α_j-sequence. If the sequence (X_n) is an α_j-sequence in X such that $X = \bigcup_{n=1}^{\infty} X_n$, where each X_n is nowhere dense in X then X will be called an α_j-space, otherwise β_j-space.

Taking $j=1$ in this definition we can obtain α-and β-spaces which are given by Sargent in [1] and are generalized by Maddox-Willey in [10]. By this definition, if X is an α-space or a β- space then it is also an α_j-space or a β_j-space, respectively, but the converse of this does not hold for $j\neq 1$. It is also easily seen that every α_j- space is of the first category and any complete paranormed space is a β_j-space for $j\geq 1$.

For any $a \in X$ and $\delta>0$, we write

$$S(a, \delta) \equiv \{x : x \in X,\ g(x-a)<\delta\}.$$

Let $G\subset X$, if l.hull (G), the set of all finite linear combinations of elements of G, is dense in X then G is called a fundamental set in X. A sequence (b_k) of elements of X is said to be a basis in X if for each $x \in X$ there exists a unique complex sequence (λ_k) such that $g(x- \sum_{k=1}^{n} \lambda_k b_k) \to 0$ $(n \to \infty)$. Hence any basis in X is also a fundamental set in X.

X^* will denote the set of continuous linear functionals on X. A linear functional A on X is an element of X^* if and only if

$$\|A\|_M \equiv \sup \{|A(x)| : g(x) \leq 1/M\} < \infty \text{ for some } M > 1.$$

If X is a space of complex sequences $x = (x_k)$, then the generalized Köthe-Toeplitz dual of X is denoted by X^+, i. e.,

$$X^+ = \{(\alpha_k) : \Sigma_k \alpha_k x_k \text{ converges for every } x \in X\}.$$

We now list some sets of complex sequences which are considered in this paper ([2], [3]); if (p_k) is a sequence of strictly positive real numbers, then

$$l_\infty(p) = \{(x_k) : \sup_k |x_k|^{p_k} < \infty\},$$

$$c_o(p) = \{(x_k) : \lim_k |x_k|^{p_k} = 0\},$$

$$c(p) = \{(x_k) : \lim_k |x_k - t|^{p_k} = 0, \text{ for some } t \in C\},$$

$$w_\infty(p) = \{(x_k) : \sup_n (n^{-1} \sum_{k=1}^{n} |x_k|^{p_k}) < \infty\},$$

$$w_o(p) = \{(x_k) : \lim_n (n^{-1} \sum_{k=1}^{n} |x_k|^{p_k}) = 0\}.$$

$$w(p) = \{(x_k) : \lim_n (n^{-1} \sum_{k=1}^{n} |x_k - t|^{p_k}) = 0, \text{ for some } t \in C\}.$$

We write $e_i = (0, 0, \ldots, 0, 1, 0, 0, \ldots)$, where 1 is in the i th place and there are zeros in the other places, and $e = (1, 1, 1, \ldots)$, and we write $l_\infty(e) = l_\infty$, $c_o(e) = c_o$, $c(e) = c$, $w_\infty(e) = w_\infty$, $w_o(e) = w_o$ and $w(e) = w$.

We now give some known results which will be useful in what follows.

Lemma 1. $w(p)$ is a linear space if and only if $\sup_k p_k < \infty$, where $p_k > 0$ [3].

Lemma 2. Let $H = \max(1, \sup_k p_k)$, and let $0 < \inf_k p_k \leq p_k \leq \sup_k p_k < \infty$. Then $w(p)$ is a paranormed space with the paranorm g which is defined by

$$(1) \qquad g(x) = \sup_n (n^{-1} \sum_{k=1}^{n} |x_k|^{p_k})^{1/H} \quad [4].$$

$w(p)$ is also a complete space under the norm (1) [6].

$w_o(p)$ and $w_\infty(p)$ are also paranormed spaces under the norm (1). It is easily showed that the sequence $(e, e_1. e_2, \ldots)$ is a basis for the space w(p).

If we write

(2) $$h(x) = \sup_r (2^{-r} \Sigma_r |x_k|^{p_k})^{1/H} \text{ for all } x \in w(p),$$

we have

(3) $$2^{-1} g(x) \leq h(x) \leq 2 g(x),$$

where Σ_r is the sum over $2^r \leq k < 2^{r+1}$ and g is defined by (1).

Lemma 3. Let $0 < p_k \leq 1$ for every $k \in N$ and

$$\mathcal{M} = \{(\alpha_k): \sum_{r=1}^{\infty} \max_r \{(2^r B^{-1})^{1/p_k} |\alpha_k|\} < \infty, \text{ for some } B > 1\},$$

where $\max_r$ is the maximum taken over the range $2^r \leq k < 2^{r+1}$. Then $w(p)^+ = \mathcal{M}$ [8].

Lemma 4. Let $0 < \inf_k p_k \leq p_k \leq 1$ for every $k \in N$ and let denote by $\mathcal{M}(p)$ the set of all sequences $\alpha = (\alpha_k)$ such that $\sum_{r=o}^{\infty} \max_r |2^{r/p_k} |\alpha_k| < \infty$, where $\max_r$ is over $[2^r, 2^{r+1})$. Then, for an arbitrary a, $\alpha \in \mathcal{M}(p)$ and $x \in w(p)$ (with $x_k \longrightarrow s\ (w(p))$,

(4) $$A(x) = a.s + \Sigma_k \alpha_k x_k$$

defines an element A of $w(p)^*$ and conversly every element of $w(p)^*$ can be represented in this form [3].

The following Banach-Steinhaus type theorems were given in [10].

Lemma 5. Let X be a paranormed space and let (A_n) be a sequence of elements of X^*, and suppose $q = (q_k)$ is bounded. Then, for some $M > 1$, $\sup_n (\|A\|_M)^{q_n} < \infty$ implies $(A_n(x)) \in l_\infty(q)$ for every $x \in X$. If X is a β-space then the converse is true.

Lemma 6. Let X be a paranormed space and let (A_n) be a sequence of elements of X^*.

(İ) If X has fundamental set G and if q is bounded then the following propositions

(5) $(A_n(b)) \in c_o\ (q)$ for every $b \in G$

and

(6) $\lim_M \limsup_n (\|A_n\|_M)^{q_n} = 0$

together imply

(7) $(A_n(x)) \in c_o\ (q)$ for every $x \in X$.

(ii) If $q \in c_o$ then (6) implies (7).

(iii) Let X be a β-space; then (7) implies (6) even if q is unbounded.

Lemma 7. Let X be a paranormed space and let (A_n) be a sequence of elements of X^* and suppose q is bounded.

(i) If X has fundamental set G, and if there is an $s \in X^*$ such that $(A_n(b) - s(b)) \in c_o\ (q)$ for all $b \in G$ and

(8) $\lim_M \limsup_n (\|A_n - s\|_M)^{q_n} = 0$

then

(9) $(A_n(x)) \in c\ (q)$ on X.

(ii) If $q \in c_o$ and if there is an $s \in X^*$ such that (8) holds then (9) is true.

(iii) If X is a β-space and if (9) is true, then there is an $s \in X^*$ such that (8) holds.

Let X and Y be sets of sequence and let $A = (a_{nk})$ denote an infinite matrix of complex numbers. We say that $A \in (X, Y)$ if and only if $\Sigma_k a_{nk} x_k$ converges for every $x = (x_k) \in X$ and $n \in N$, and $(\Sigma_k a_{nk} x_k) \in Y$ for every $x \in X$.

Lemma 8. Let $0 < p_k \leq 1$. Then $A \in (w\ (p), c)$ if and only if

(i) There exists an integer $B > 1$ such that

(10) $$C = \sup_n \left(\sum_{r=o}^{\infty} \max_r \{(2^r B^{-1})^{1/p_k} |a_{nk}|\}\right) < \infty,$$

(ii) $\lim_n a_{nk} = \alpha_k$ exists for every fixed k,

(iii) $\lim_n \Sigma_k a_{nk} = \alpha$ exists.

Using the same method of proof of Lemma 8 we can get the following

Lemma 9. Let $0<p_k\leq 1$. Then $A \in (w(p), l_\infty)$ if and only if there exists an integer $B>1$ such that (10) holds.

We shall frequently use the following inequalities.

Take a, b $\in$ C, if $0<p\leq 1$ then

$$|a|^p-|b|^p\leq|a+b|^p\leq|a|^p+|b|^p,$$

and we can write the following inequality which is known as Minkowski's inequality, if $p\geq 1$ then

$$(\Sigma_k|a_k+b_k|^p)^{1/p}\leq(\Sigma_k|a_k|^p)^{1/p}+(\Sigma_k|b_k|^p)^{1/p}.$$

2. *Theorems of Banach-Steinhaus Type*

In the next two sections $q=(q_k)$ will denote a sequence of strictly postive real numbers.

Theorem 1. *Let X be a paranormed space and let (A_n) be a sequence of elements of X^* and suppose that q is bounded. If there exists an integer $M>1$ such that*

$$\text{(11)} \qquad \sup_n \left(n^{-1}\sum_{k=1}^{n}(\|A_k\|_M)^{q_k}\right)<\infty,$$

then

$$\text{(12)} \qquad (A_n(x)) \in w_\infty(q) \text{ for every } x\in X.$$

If X is a β_j-space then the converse is also true.

Prof. Suppose that there exists an integar $M>1$ such that (11) holds, and choose any $x\in X$. By the continuity of scalar multiplication in a paranormed space, there exists a number $K\geq 1$ such that $g(K^{-1}x)\leq 1/M$, where the number M is that of (11). Hence we have

$$n^{-1}\sum_{k=1}^{n}|A_k(x)|^{q_k}=n^{-1}\sum_{k=1}^{n}K^{q_k}|A_k(K^{-1}x)|^{q_k}$$

$$\leq K^Q n^{-1}\sum_{k=1}^{n}|A_k(K^{-1}x)|^{q_k}$$

$$\leq K^Q n^{-1}\sum_{k=1}^{n}(\|A_k\|_M)^{q_k}$$

for any $n\in N$, where $Q=\sup_k q_k$, so that (12) holds.

Now assume that X is a β_j-space and (12) holds, and let $Q = \sup_k q_k$. We define the number j as Q if Q is an integer, and otherwise the first integer which is greater than Q. Let us define

$$X_m = \{(x_k) : x \in X \text{ and } n^{-1} \sum_{k=1}^{n} | A_k(x) |^{q_k} \leq 2^m \text{ for all } n \in N\}$$

for any $m \in N$. Clearly $\theta \in X_m$ for $m = 1, \ldots, j$. By the definition $j \geq 1$, so using the Minkowski's inequalitiy, we have

$$(n^{-1} \sum_{k=1}^{n} | A_k(x \mp y) |^{q_k})^{1/j} \leq (n^{-1} \sum_{k=1}^{n} | A_k(x) |^{q_k})^{1/j}$$

$$+ (n^{-1} \sum_{k=1}^{n} | A_k(y) |^{q_k})^{1/j}$$

by the linearity of each A_n. Now, if $x, y \in X_m$ for any $m \in N$, then by this inequality we find

$$n^{-1} \sum_{k=1}^{n} | A_k(x \mp y) |^{q_k} \leq (2^{m/j} + 2^{m/j})^{1/j} = 2^{m+j},$$

so $x \mp y \in X_{m+j}$. Thus (X_m) is an α_j-sequence. Also $X = \bigcup_{m=1}^{\infty} X_m$.

Now let us consider the sets

$$X_{m,n} = \{ x \in X : n^{-1} \sum_{k=1}^{n} | A_k(x) |^{q_k} \leq 2^m\}.$$

Since each A_k is continuous by the hypothesis, f_n is also continuous for each $n \in N$, where

$$f_n(x) = n^{-1} \sum_{k=1}^{n} | A_k(x) |^{q_k}$$

Thus, for any $m \in N$ and every $n \in N$, the sets $X_{m,n}$ are closed, so that, for any $m \in N$, $X_m = \bigcap_{n \in N} X_{m,n}$ is also closed and so $\bar{X}_m = X_m$ for every $m \in N$. Since X is a β_j-space there exists a $B \in N$ such that X_B is not nowhere dense, whence there is a shpere $S(a, \delta) \subset \bar{X}_B = X_B$. Hence if $g(x\text{-}a) < \delta$ we have

$$n^{-1} \sum_{k=1}^{n} | A_k (x) |^{q_k} \leq 2^B \text{ for every } n \in N,$$

so if $g(x) < \delta$, by the Minkowski's inequality, we have

$$\left(n^{-1} \sum_{k=1}^{n} | A_k (x) |^{q_k}\right)^{1/j} \leq \left(n^{-1} \sum_{k=1}^{n} | A_k (x+a) |^{q_k}\right)^{1/j}$$

$$+ \left(n^{-1} \sum_{k=1}^{n} | A_k (a) |^{q_k}\right)^{1/j}$$

$$\leq 2^{B/j} + 2^{B/j} = 2.2^{B/j}$$

for every $n \in N$ and we obtain

$$n^{-1} \sum_{k=1}^{n} | A_k (x) |^{q_k} \leq 2^{B+j}$$

for all $n \in N$. Taking $M > \delta^{-1}$ we find (11).

Remark. Now let us consider the condition (11) of Theorem 1. If it holds then we find

$$\text{(13)} \qquad \sup_n \left[n^{-1} (\|A_n\|_M)^{q_n}\right] < \infty$$

from the following inequality

$$n^{-1} (\|A_n\|_M)^{q_n} \leq n^{-1} \sum_{k=1}^{n} (\|A_k\|_M)^{q_k} + (n-1)^{-1} \sum_{k=1}^{n-1} (\|A_k\|_M)^{q_k}$$

But it is clear that the converse of this result does not hold in general. Thus the condotion (11) is sufficient for Theorem 1 and (13) is the necessary condition for Theorem 1.

Since every β-space is also a β_j-space, Theorem 1 holds when X is a β-space. Furthermore, if $0<q_k\leq 1$ for every $k \in N$, then it is easily seen from the proof of theorem 1 that we can take β-space in place of β_j-space.

Theorem 2. Let X be a paranormed space and let (A_n) be a sequence of elements of X^*, and suppose q is bounded.

(i) If X has fundamental set G then the following propositions

$$\text{(14)} \qquad (A_n(b)) \in w_o\,(q) \text{ for every } b \in G,$$

(15) $\lim_M \limsup_n (n^{-1} \sum_{k=1}^{n} (\|A_k\|_M)^{q_k}) = 0$

together imply

(16) $(A_n(x)) \in w_o(q)$ *for every* $x \in X$.

(ii) If $q \in c_o$ *then (15) implies (16).*

(iii) Let X *be a* β*-space then (16) implies (15).*

Proof. (i) Suppose X has fundamental set G, conditions (14) and (15) hold and let $\sup_k q_k = Q$. Choose any $x \in X$ and any $\epsilon > 0$. By (15), there exists $M > 1$ and n_o such that

$$n^{-1} \sum_{k=1}^{n} (\|A_k\|_M)^{q_k} < \varepsilon/2^Q, \text{ for all } n \geq n_o.$$

Since the l. hull (G) is dense in X, there exist $\lambda_1, \ldots, \lambda_m \in C$ and $b_1, \ldots, b_m \in G$ such that $g(x - \sum_{k=1}^{n} \lambda_k b_k) < 1/M$, and by (14), there is an $n_1 \geq n_o$ such that

$$n^{-1} \sum_{i=1}^{n} |A_i(b_k)|^{q_i} \leq \varepsilon/2^Q L\, m\, R^{m-1} \text{ for } k = 1, \ldots m,$$

for all $n \geq n_1$, where $R = \max(1, 2^{Q-1})$ and

$L = \max(1, |\lambda_1|, |\lambda_2|, \ldots, |\lambda_m|, |\lambda_1|^Q, |\lambda_2|^Q, \ldots, |\lambda_m|^Q)$.

Hence, using the Minkowski's inequality, we get

$$(n^{-1} \sum_{i=1}^{n} |A_i(x)|^{q_i})^{1/Q} \leq (n^{-1} \sum_{i=1}^{n} |A_i(x - \sum_{k=1}^{m} \lambda_k b_k)|^{q_i})^{1/Q}$$

$$+ (n^{-1} \sum_{i=1}^{n} |\sum_{k=1}^{m} \lambda_k A_i(b_k)|^{q_i})^{1/Q}$$

$$\leq (n^{-1} \sum_{i=1}^{n} |A_i(x - \sum_{k=1}^{m} \lambda_k b_k)|^{q_k})^{1/Q}$$

$$+ (n^{-1} \sum_{i=1}^{n} (\sum_{k=1}^{m} |\lambda_k A_i(b_k)|^{q_i})^{1/Q}$$

$$\leq \left(n^{-1} \sum_{i=1}^{n} \left| A_i \left(x - \sum_{k=1}^{m} \lambda_k b_k\right) \right|^{q_i}\right)^{1/Q}$$

$$+ \left(L R^{m-1} n^{-1} \sum_{i=1}^{n} \sum_{k=1}^{m} | A_i (b_k) |^{q_i}\right)^{1/Q}$$

$$\leq \left(n^{-1} \sum_{i=1}^{n} \left| A_i \left(x - \sum_{k=1}^{m} \lambda_k b_k\right) \right|^{q_i}\right)^{1/Q}$$

$$+ \left(L R^{m-1} \sum_{k=1}^{m} n^{-1} \sum_{i=1}^{n} | A_i (b_k) |^{q_i}\right)^{1/Q}$$

$$\leq 2^{-1} \varepsilon^{1/Q} + 2^{-1} \varepsilon^{1/Q} = \varepsilon^{1/Q}$$

for all $n \geq n_1$, so we have

$$n^{-1} \sum_{i=1}^{n} | A_i (x) |^{q_i} < \varepsilon, \text{ for all } n \geq n_1,$$

which implies (16).

(ii) Suppose $q \in c_o$ and (15) holds. Again, choose any $x \in X$ and any $\varepsilon > 0$. Then, by (15), there exists $M > 1$ and n_o such that

$$n^{-1} \sum_{k=1}^{n} (\|A_k\|_M)^{q_k} < \epsilon/2, \text{ for all } n \geq n_o.$$

Since the scalar multiplication in a paranormed space is continuous, then there is a $K \geq 1$ such that $(g (K^{-1}x) \leq 1/M$. Since $q \in c_o$, we can choose $n_1 \geq n_o$ such that $K^{q_n} \leq 2$ for all $n \geq n_1$, so we have

$$n^{-1} \sum_{k=1}^{n} | A_k (x) |^{q_k} = n^{-1} \sum_{k=1}^{n} | K A_k (K^{-1} x) |^{q_k}$$

$$= n^{-1} \sum_{k=1}^{n} K^{q_k} | A_k (K^{-1} x) |^{q_k}$$

$$\leq 2 n^{-1} \sum_{k=1}^{n} | A_k (K^{-1} x) |^{q_k}$$

$$\leq 2\, n^{-1} \sum_{k=1}^{n} (\|A_k\|_M)^{q_k}$$

thus (16) holds.

(iii) Let X be a β-space and suppose (16) holds. Choose any $\varepsilon > 0$, and for any $m \in N$, we define

$$X_m = \{x \in X : n^{-1} \sum_{k=1}^{n} | 2^{-m} A_k (x) |^{q_k} \leq \varepsilon/2, \text{ for all } n \geq m \}.$$

Clearly $\theta \in X_1$. Let x, $y \in X_m$ for any $m \in N$, then we write

$$| 2^{-(m+1)} A_k (x \mp y) |^{q_k} = | 2^{-(m+1)} A_k (x) \mp 2^{-(m+1)} A_k(y) |^{q_k}$$

$$\leq (| 2^{-(m+1)} A_k (x) | + | 2^{-(m+1)} A_k (y) |)^{q_k}$$

$$\leq [2 \max (|2^{-(m+1)} A_k (x) | , | 2^{-(m+1)} A_k (y) |)]^{q_k}$$

$$= [\max(| 2^{-m} A_k (x) | , | 2^{-m} A_k (y) |)]^{q_k}$$

$$= \max (| 2^{-m} A_k (x) |^{q_k}, | 2^{-m} A_k (y) |^{q_k}),$$

so we have

$$n^{-1} \sum_{k=1}^{n} | 2^{-(m+1)} A_k (x \mp y) |^{q_k}$$

$$\leq n^{-1} \sum_{k=1}^{n} \max (| 2^{-m} A_k (x) |^{q_k}, | 2^{-m} A_k (y) |^{q_k}) \leq \varepsilon/2$$

for all $n \geq m+1$, thus $x \mp y \in X_{m+1}$ so that (X_m) is an α-sequence. Also $X = \bigcup_{m=1}^{\infty} X_m$. Since each A_n is a continuous linear functional, we see that $\bar{X}_m = X_m$, for all $m \in N$, in a similar way of the proof of Theorem 1. Since X is a β-space, there exists a $B \in N$ such that X_B is nowhere dense in X, that is, there is a sphere $S(a, \delta) \subset \bar{X}_B = X_B$.

Hence, if $g(x-a) < \delta$, then we have $n^{-1} \sum_{k=1}^{n} | 2^{-B} A_k (x) |^{q_k} \leq \varepsilon/2$ for all $n \geq B$, and

$$n^{-1} \sum_{k=1}^{n} | 2^{-(B+1)} A_k (x) |^{q_k}$$

$$\leq n^{-1} \sum_{k=1}^{n} \max (| 2^{-B} A_k (x+a) |^{q_k}, \quad | 2^{-B} A_k (a) |^{q_k})$$
$$\leq \varepsilon /2 \text{ for all } n \geq B+1.$$

Now let us write $\rho = 2^{-(B+1)} \delta$ and choose $M > \rho^{-1}$. Then if $g(x) < \rho$, by the subadditivity of g, we have $g(2^{B+1} x) < \delta$. Thus, if $g(x) \leq 1/M$, then we find

$$n^{-1} \sum_{k=1}^{n} | A_k (x) |^{q_k} = n^{-1} \sum_{k=1}^{n} | 2^{-(B+1)} A_k (2^{B+1} x) |^{q_k}$$
$$\leq \varepsilon/2$$

for all $n \geq B+1$.

Since $\varepsilon > 0$ was arbitrary we obtain (15). This completes the proof of Theorem 2.

It is obvious that ve can give a remark here like as at the end of Theorem 1. If (15) holds then, in a similar way to the remark of Theorem 1, we find

(17) $$\lim_M \lim \sup_n [n^{-1} (\|A_n\|_M)^{q_n}] = 0.$$

Thus, in the case q is bounded, (14) and (15) are sufficient conditions for (16) If X is a β-space, 16) implies (15) so that (17). But (17) may not be a sufficient condition for (16) in general.

3. *Matrix Transformations*

We now apply the above theorems and the lemmata 5–7 to characterize the classes $(w(p), l_\infty (q))$, $(w(p), c_o (q))$, $(w(p), c (q))$, $(w (p), w_\infty (q))$ and $(w(p), w_o (q))$.

Let $p = (p_k)$ and $q = (q_k)$ be bounded sequences of strictly positive real numbers. If $Q = \max (1, \sup_k q_k)$ then it follows by lemma 1 of [3], $c_o (q) = c_o (Q^{-1} q)$, $l_\infty (q) = l_\infty (Q^{-1} q)$ and $c(q) = c\, Q^{-1} q)$. For this reason without loss of generality we may assume $q_n \leq 1$ for all $n \in N$ for the following first three theorems.

Theorem 3. *Let* $0 < m = \inf_k p_k \leq p_k \leq 1$ *and* $0 < q_k \leq 1$ *for all* $k \in N$. *The necessary and sufficient condition for* $A \in (w(p), l_\infty (q))$

(18) $$T(B) = \sup_n (\sum_{r=0}^{\infty} \max_r \{ (2^r B^{-1})^{1/p_k} | a_{nk} | \})^{q_n} < \infty,$$

for some $B > 1$.

Proof. We define, for each $n \in N$,

(19) $A_n(x) = \Sigma_k a_{nk} x_k$ on $w(p)$.

First let us show that (18) is necessary. Suppose $A \in (w(p), l_\infty(q))$ then the series in (19) is convergent for every $n \in N$. Hence, by the definition of Köthe-Toeplitz duals of sequence spaces, $(a_{n1}, \ldots) \in w(p)^+$ for each $n \in N$. Now we define

$$f_{r,n}(x) = \Sigma_r a_{nk} x_k \text{ on } w(p),$$

where the sum is over the range $2^r \leq k < 2^{r+1}$. Clearly each $f_{r,n}$ is linear on $w(p)$. Since $w(p)$ is a paranormed space by lemma 2, there is a $K \geq 1$ such that $g(K^{-1}x) \leq 1/B$, by the continuity of scalar multiplication where B satisfies the condition of lemma 3. Then, for each $n \in N$ and for every $r \geq 0$, we have

$$\begin{aligned} | f_{r,n}(x) | &= | K f_{r,n}(K^{-1}x) | = K | f_{r,n}(K^{-1}x) | \\ &= K | \Sigma_r a_{nk}(K^{-1}x_k) | \leq K \Sigma_r | a_{nk}(K^{-1}x_k) | \\ &= K \Sigma_r (2^r B^{-1})^{1/p_k} | a_{nk} | (2^{-r}B)^{1/p_k} | K^{-1}x_k | \\ &\leq K \max_r \{(2^r B^{-1})^{1/p_k} |a_{nk}|\} \Sigma_r (2^{-r}B |K^{-1}x_k|^{p_k})^{1/p_k} \\ &\leq K \max_r \{(2^r B^{-1})^{1/p_k} | a_{nk} | \} B 2^{-r} \Sigma_r | K^{-1}x_k |^{p_k} \\ &\leq K \max_r \{(2^r B^{-1})^{1/p_k} | a_{nk} | \} B g(K^{-1}x). \end{aligned}$$

where $\max_r$ is the maximum taken over $2^r \leq k < 2^{r+1}$ and $g(x) = \sup_r (2^{-r} \Sigma_r | x_k |^{p_k})$, for $0 < p_k \leq 1$, $k = 1, \ldots,$ and since $g(K^{-1}x) \leq 1/B$, it is easily seen that $2^{-r}B | K^{-1}x_k |^{p_k} \leq 1$. Thus, for each $n \in N$ and every integer $r \geq 0$, $f_{r,n}$ is bounded, so that continuous functionals, i. e., for each $n \in N$ and every integer $r \geq 0$ $f_{r,n} \in w(p)^*$. We also have

$$\lim_s \sum_{r=0}^{s} f_{r,n}(x) = \lim_s \sum_{r=0}^{s} \Sigma_r a_{nk} x_k = \sum_{k=1}^{\infty} a_{nk} x_k = A_n(x)$$

so that, for each $n \in N$, A_n is continuous, i. e., for all $n \in N$, $A_n \in w(p)^*$ by the corollary of Theorem 11 in ([7], p. 114). Hence, for each $n \in N$, we can write

(20) $\quad | A_n (x) | \leq \| A_n \|_B \, g(x)$ on $w(p)$.

Now we show that, for each $n \in N$,

(21) $$\| A_n \|_B = C(n, B) = \sum_{r=o}^{\infty} \max_r \{ (2^r B^{-1})^{1/p_k} \, | a_{nk} | \}.$$

where $B > 1$ for which $\| A_n \|_B$ is defined.

Choose any $n \in N$. First suppose that B is a number such that

$$\max_r \{ (2^r B^{-1})^{1/p_k} \, | a_{nk} | \} \geq (r+3)^{-1}, \text{ for all } r \geq 0.$$

If we define a sequence $x = (x_k)$ as follows:

$x_{k(r)} = [\log (3+r)]^{-1} (2^r B^{-1})^{1/p_{k(r)}} \operatorname{sgn} a_{nk(r)}$, $x_k = 0$ for $k \neq k(r)$, where $k(r)$ denotes an integer k in the interval $2^r \leq k < 2^{r+1}$ such that

(22) $\max_r \{ (2^r B^{-1})^{1/p_k} \, | a_{nk} | \} = (2^r B^{-1})^{1/p_{k(r)}} \, | a_{nk(r)} |$,

then, for this sequence, we have

$$2^{-r} \Sigma_r | x_k |^{p_k} = B^{-1} [\log (3+r)]^{-p_{k(r)}} \longrightarrow 0 \ (r \longrightarrow \infty).$$

since $0 < m = \inf_k p_k \leq p_k \leq 1$, so that $x \in w(p)$, and also we find

$$\Sigma_r a_{nk} x_k = \max_r \{ (2^r B^{-1})^{1/p_k} \, | a_{nk} | \} [\log (3+r)]^{-1}$$
$$\geq [(3+r) \log (3+r)]^{-1},$$

so that the series in (19) is divergent. This is a contradiction. Thus for each $n \in N$,

$$C(n,B) = \sum_{r=o}^{\infty} \max_r \{ 2(^r B^{-1})^{1/p_k} \, | a_{nk} | \} < \infty \text{ for some } B > 1,$$

since $(a_{n1}, \ldots) \in w(p)^+$ for all $n \in N$.

If $g(x) = \sup_r (2^{-r} \Sigma_r | x_k |^{p_k}) \leq 1/B$, then $(2^{-r} B \, | x_k |^{p_k} \leq 1$ for all $k \in N$ and since $\sup_k p_k \leq 1$, we find

$$| A_n (x) | = | \Sigma_k a_{nk} x_k | \leq \Sigma_k | a_{nk} x_k | = \sum_{r=o}^{\infty} \Sigma_r | a_{nk} x_k |$$

$$\leq \sum_{r=o}^{\infty} \max_r \{(2^r B^{-1})^{1/p_k} \mid a_{nk} \mid \} \Sigma_r (2^{-r} B \mid x_k \mid^{p_k})^{1/p_k}$$

$$\leq [C(n,B)] B g(x)$$

so we hawe

(23) $\| A_n \|_B \leq C(n,B) < \infty$.

Now for each n we take any integer $s>0$ and define $x \in w(p)$ by $x_k = 0$ for $k \geq 2^{s+1}$, $x_{k(r)} = (2^r B^{-1})^{1/p_{k(r)}} \operatorname{sgn} a_{nk(r)}$, and $x_k = 0$ $(k \neq k_r)$ for $0 \leq r \leq s$, where k(r) is a number which satisfies the condition (22). Since $g(x) = \sup_r (2^{-r} \Sigma_r \mid x_k \mid^{p_k}) = 1/B$, for this sequence, we get

$$\sum_{r=o}^{s} \max_r \{(2^r B^{-1})^{1/p_k} \mid a_{nk} \mid \} \leq \| A_n \|_B$$

by (20), so that we find

(24) $C(n,B) \leq \| A_n \|_B$.

Hence, wo obtain $\| A_n \|_B = C(n,B)$ by (23) and (24).

By lemma 2, we know that w (p) is a complete paranormed space so it is a β-space. Thus, by lemma 5, it is seen that (18) holds.

Now, for the sufficiency, suppose that (18) holds for some integer $B > 1$. Then there exists an integer $M > 1$ such that

$$C(n,M) = \sum_{r=o}^{\infty} \max_r \{(2^r M^{-1})^{1/p_k} \mid a_{nk} \mid \} < \infty$$

for each $n \in N$.

Since w (p) is a paranormed space, there is a $K \geq 1$ such that $g(K^{-1} x) \leq 1/M$ by the continuity of scalar multiplication on the paranormed space. Hence,

$$g(K^{-1} x) = \sup_r (2^{-r} \Sigma_r \mid K^{-1} x_k \mid^{p_k}) \leq 1/M,$$

and for this reason we can take $2^{-1} M \mid K^{-1} x_k \mid^{p_k} \leq 1$ for every $k \in N$. Thus, we have

$$\Sigma_k | a_{nk} x_k | = \sum_{r=o}^{\infty} \Sigma_r | a_{nk} x_k |$$

$$\leq \sum_{r=o}^{\infty} \max_r \{ (2^r M^{-1})^{1/p_k} | a_{nk} | \} K \Sigma_r (2^{-r} M | K^{-1} x_k |^{p_k})^{1/p_k}$$

$$\leq K \sum_{r=o}^{\infty} \max_r \{ (2^r M^{-1})^{1/p_k} | a_{nk} \} M 2^{-r} \Sigma_r | K^{-1} x_k |^{p_k}$$

$$\leq K [C (n,m)] M g (K^{-1} x),$$

which implies that the series in (19) is absolute convergent for each $n \in N$, that is, $(a_{n1}, \ldots) \in w(p)^{+}$ for each $n \in N$. Now we can show, by a similar way of the proof of the necessity, that $A_n \in w(p)^*$ for each $n \in N$, where $\| A_n \|_M = C(n, M)$. Then $(A_n(x)) \in l_\infty(q)$ by lemma 5, so it is seen that $A \in (w(p), l_\infty(q))$. This completes the pıoof of the Theorem.

Remark 1. If $0 < \inf_k q_k \leq q_k \leq \sup_k q_k = Q < \infty$, then we can take,

$$\text{(25)} \quad \sup_n \sum_{r=o}^{\infty} \max_r \{ (2^r B^{-1/q_n})^{1/p_k} | a_{nk} | \} < \infty \text{ for some } B > 1,$$

instead of (18).

Similarly we can show that this condition is also necessary and sufficient for $A \in (w(p), l_\infty(q))$, and also

$$\| A_n \|_B = \sum_{r=o}^{\infty} \max_r \{ (2^r B^{-1/q_n})^{1/p_k} | a_{nk} | \},$$

where B is a number for which $\| A_n \|_B$ is defined. So in this case we prove the sufficiency without using the result of lemma 5, as follows: Since w (p) is a paranormed space and the scalar multiplication in a paranormed space is continuous then there exists a $K \geq 1$ such that $g(K^{-1}x) \leq B^{-1/v} < B^{-1}$, where $v = \inf q_k$. Then for any $n \in N$, we have

$$| A_n (x) |^{q_n} = | \Sigma_k a_{nk} x_k |^{q_n} \leq (\Sigma_k | a_{nk} x_k |)^{q_n}$$

$$= (\sum_{r=o}^{\infty} \Sigma_r |a_{nk} x_k |)^{q_n}$$

$$\leq K^Q \left(\sum_{r=o}^{\infty} \Sigma_r \mid a_{nk} \mid \mid x_k \; K^{-1} \mid \right)^{q_n}$$

$$\leq K^Q \left[\sum_{r=o}^{\infty} \max_r \{ (2^r B^{-1/q_n})^{1/p_k} |a_{nk}| \} \Sigma_r (2^{-r} B^{1/V} |K^{-1} x_k|^{p_k})^{1/p_k} \right]^{q_n}$$

$$\leq K^Q \left[\sum_{r=o}^{\infty} \max_r \{ (2^r B^{-1/q_n})^{1/p_k} \mid a_{nk} \mid \} B^{1/V} g (K^{-1} x) \right]^{q_n}$$

$$\leq K^Q \left[\sum_{r=o}^{\infty} \max_r \{ (2^r B^{-1/q_n})^{1/p_k} \mid a_{nk} \mid \} \right]^{q_n}$$

$$\leq K^Q \left[1 + \left(\sum_{r=o}^{\infty} \max_r \{ (2^r B^{-1/q_n})^{1/p_k} \mid a_{nk} \mid \} \right)^Q \right] \text{ on } w(p),$$

so $(A_n(x)) \in l_\infty(q)$ for all $x \in w(p)$.

In fact, the condition (18) implies (25) without the restriction $\inf_k q_k \neq 0$. Suppose that (18) holds. Then there is a constant $H > 1$ such that

$$\left(\sum_{r=o}^{\infty} \max_r \{ (2^r B^{-1})^{1/p_k} \mid a_{nk} \mid \} \right)^{q_n} \leq H \text{ for some } B > 1,$$

for all $n \in N$. Hence we have

$$\left(\sum_{r=o}^{\infty} \max_r \{ (2^r B^{-1})^{1/p_k} H^{-1/q_n} \mid a_{nk} \mid \} \right)^{q_n} \leq 1$$

or

$$\left(\sum_{r=o}^{\infty} \max_r \{ (2^r B^{-1})^{1/p_k} H^{-1/q_n} \mid a_{nk} \mid \} \right)^{q_n/Q} \leq 1$$

for all $n \in N$, where $Q = \sup_k q_k$. If we take $M = B^Q H$, then we have

$$M^{1/q_n p_k} = B^{Q/q_n p_k} H^{1/q_n p_k} \geq B^{1/p_k} H^{1/q_n}$$

or

$$M^{-1/q_n p_k} \leq B^{-1/p_k} H^{-1/q_n} \text{ for all } k, n \in N .$$

Thus, from the above result, we find

$$\sum_{r=o}^{\infty} \max_r \{(2^r M^{-1/q_n})^{1/p_k} |a_{nk}|\} \leq \sum_{r=o}^{\infty} \max_r \{(2^r B^{-1})^{1/p_k} H^{-1/q_n} |a_{nk}|\}$$

$$\leq (\sum_{r=o}^{\infty} \max_r \{ (2^r B^{-1})^{1/p_k} H^{-1/q_n} |a_{nk}|\})^{q_n/Q} \leq 1,$$

for all $n \in N$, so that (25) holds.

If $0 < \inf_k q_k = v \leq q_k \leq \sup_k q_k < \infty$ for every $k \in N$, then we can show that (25) implies (18) by the same method.

Theorem 4. Let $0 < \inf_k p_k \leq p_k \leq 1$ *for every* $k \in N$ *and let* $q = (q_k)$ *be bounded.* $A \in (w(p), c_o(q))$ *if and only if*

(26) $$\lim_n | a_{nk} |^{q_n} = 0, \text{ for } k = 1, \ldots,$$

(27) $$\lim_B \lim\sup_n (\sum_{r=o}^{\infty} \max_r \{(2^r B^{-1})^{1/p_k} | a_{nk} | \})^{q_n} = 0,$$

and

(28) $$\lim_n | \Sigma_k a_{nk} |^{q_n} = 0.$$

Proof. Define A_n by (19) on $w(p)$. Suppose that $A \in (w(p), c_o(q))$ then $A \in (w(p), l_\infty)$, since $c_o(q) \subset l_\infty(q)$. Thus, by lemma 9, there is constant H such that

$$C(n, B) = \sum_{r=o}^{\infty} \max_r \{ (2^r B^{-1})^{1/p_k} | a_{nk} | \} \leq H \text{ for some } B > 1,$$

for all $n \in N$. Now by the same method in the proof of Theorem 3, we can show that $A_n \in w(p)^*$ for every $n \in N$, where $\| A_n \|_B = C(n, B)$ and B is a number for which $\| A_n \|_B$ is defined.

Since w (p) is a complete paranormed space, it is-seen that (27) is necessary by lemma 6 (iii). Considering the sequences e and e_k, it is easily seen that the conditions (28) and (26) are also necessary.

Now suppose that the conditions (26), (27) and (28) hold. We know that the sequence $(e, e_1, e_2, \ldots)$ is a basis of w (p) and it is also a fundamental set of w (p). Hence by (26) and (28), $A_n(b) \in c_o(q)$ for every $b \in (e, e_1, e_2, \ldots)$. Then, by lemma 7 (i), it is enough to show that

$$\lim_B \limsup_n (\| A_n \|_B)^{q_n} = 0.$$

By a similar way of the proof of the theorem 3, it is easily shown that $A_n \in w(p)^*$, where $\| A_n \|_B = C(n, B)$ and B is an integer $B > 1$ such that $\| A_n \|_B$ is defined. Then the condition required holds by (27), so we obtain $(A_n(x)) \in c_o(q)$ for every $x \in w(q)$ by lemma 7 (i), i. e., $A \in (w(p), c_o(q))$.

Remark 2. If $0 < \inf_k q_k = v \leq q_k \leq \sup_k q_k < \infty$, then we can take the following condition instead of (27) in the Theorem 4:

$$\text{(29)} \qquad \lim_B \limsup_n \left(\sum_{r=o}^{\infty} \max_r \left\{ (2^r B^{-1/q_n})^{1/p_k} | a_{nk} | \right\} \right) = 0$$

By the way of the proof of the Theorem 3 and 4, we can show that (26), (28) and (29) are necessary and sufficient for $A \in (w(p), c_o(q))$. We can also show that (27) implies (29) and if $\inf_k q_k \neq 0$ then (29) implies (27).

Theorem 5. *Let* $0 < \inf_k p_k \leq p_k \leq 1$ *for every* $k \in N$ *and let* $q \in c_o$. $A \in (w(p), c_o(q))$ if and only if (27) holds.

Proof. This follows, by the methods of Theorem 4, from lemma 6 (ii) and (iii).

Theorem 6. *Let* $0 < \inf_k p_k \leq p_k \leq 1$ *for every* $k \in N$ *and let q be bounded.* $A \in (w(p), c(q))$ *if and only if*

$$\text{(30)} \qquad \sup_n \left(| a_{no} | + \sum_{r=o}^{\infty} \max_r \left\{ (2^r B^{-1})^{1/p_k} | a_{nk} | \right\} \right) < \infty,$$

for some $B > 1$,

and there exists a sequence $(\alpha_o, \alpha_1, \ldots)$ *such that*

(31) $lim_n \mid a_{nk} - \alpha_k \mid^{q_n} = 0, \text{ for } k = 0, 1, \ldots$

and

(32) $lim_B \, lim \, sup_n \left(|a_{no} - \alpha_o| + \sum_{r=o}^{\infty} max_r \{(2^r B^{-1})^{1/p_k} |a_{nk} - \alpha_k|\}\right)^{q_n} = 0$

Proof. Define A_n by

(33) $A_n(x) = a_{no} \, t(x) + \Sigma_k \, a_{nk} \, x_k$ on $w(p)$,

where $x_k \to t \; (w(p))$.

First, suppose $A \in (w(p), c(q))$. Then, by the definition of Köthe-Toeplitz dual of a seqeunce space, $(a_{n1}, \ldots) \in w(p)^{+}$ for each $n \in N$. Also by lemma 4, $A_n \in w(p)^*$ for each $n \in N$, where

(34) $\| A_n \|_B = | a_{no} | + \sum_{r=o}^{\infty} max_r \{ (2^r B^{-1})^{1/p_k} | a_{nk} | \},$

and $B > 1$ is an integer for which $\| A_n \|_B$ is defined. Hence there exists a $l \in w(p)^*$ such that $lim_n \mid A_n(b) - l(b) \mid = 0$ for every $b \in (e, e_1, \ldots)$ and

$$lim_B \; lim \; sup_n \left(\| A_n - l \|_B \right)^{q_n} = 0,$$

by lemma 7 (iii). Then, by lemma 4, we can write

$$l(x) = \alpha_o \, t(x) + \Sigma_k \, \alpha_k \, x_k,$$

where $\alpha = (\alpha_k) \in w(p)^{+}$ and α_o is an airbitrary real number and

$$\| l \|_B = | \alpha_o | + \sum_{r=o}^{\infty} max_r \{ (2^r B^{-1})^{1/p_k} | \alpha_k | \}.$$

Thus, we have $(U_n(x)) \in c_o(q)$ for $U_n = (A_n - l) \in w(p)^*$, so that $U \in (w(p), c_o(q))$. Then the necessity of the conditions are seen by the same method of Theorem 4.

Now to prove the sufficiency of the conditions, we must show that the conditions of the theorem imply

$$| \alpha_o | + \sum_{r=o}^{\infty} max_r \{ (2^r B^{-1})^{1/p_k} | \alpha_k | \} < \infty \text{ for some } B > 1.$$

Then, by lemma 4, a function l which is defined by

$$l(x) = \alpha_o\, t(x) + \Sigma_k\, \alpha_k\, x_k$$

defines an element of w (p)*. In fact, if we suppose that the conditions hold, then we have

$$| \alpha_o | + \sum_{r=o}^{\infty} \max_r \{ (2^r B^{-1})^{1/p_k} | \alpha_k | \} = | \alpha_o - a_{no} + a_{no} |$$

$$+ \sum_{r=o}^{\infty} \max_r \{ (2^r B^{-1})^{1/p_k} | \alpha_k - a_{nk} + a_{nk} | \}$$

$$\leq | a_{no} - \alpha_o | + \sum_{r=o}^{\infty} \max_r \{ (2^r B^{-1})^{1/p_k} | a_{nk} - \alpha_k | \}$$

$$+ | a_{no} | + \sum_{r=o}^{\infty} \max_r \{ (2^r B^{-1})^{1/p_k} | a_{nk} | \}$$

$$\leq 1 + \sup_n (| a_{no} | + \sum_{r=o}^{\infty} \max_r \{ (2^r B^{-1})^{1/p_k} | a_{nk} | \} < \infty$$

for sufficiently large B and n. Also the condition (30) implies $(a_{n1}, \ldots) \in w(p)^+$ by lemma 3 and $A_n \in w(p)^*$ by lemma 4, for each $n \in N$, where A_n satisfies the condition (34). Thus, under the given hypothesis, using the result obtained above, we find $U \in (w(p), c_o(q))$ by Theorem 4, where $U_n = (A_n - l)$. Hence $| A_n(x) - l(x) |^{q_n} = | U_n(x) |^{q_n} \to 0$ $(n \to \infty)$ on w (p), that is, $A \in (w(p), c(q))$.

Theorem 7. let $0 < \inf_k p_k \leq p_k \leq 1$ *for every* $k \in N$ *and let q be bounded.* $A \in (w(p)), w_\infty(q))$ *if and only if*

$$\sup_n (n^{-1} \sum_{i=1}^{n} [C(i, B)]^{q_i}) < \infty, \tag{35}$$

where

$$C(i, B) = \sum_{r=o}^{\infty} \max_r \{ (2^r B^{-1})^{1/p_k} | a_{ik} | \} < \infty, \text{ for some } B > 1.$$

Proof. Let A_n be defined by (19) on w (p). Suppose that $A \in (w(p), w_\infty(q))$. Then $A_n \in w(p)^*$ for each $n \in N$, where $\| A_n \|_B = C(n, B)$ and B is an integer for which $\| A_n \|_B$ is defined.

Since w (p) is a complete paranormed space, it is a β-space so that is a β_j-space. Thus, by the convers part of Theorem 1, (35) holds.

Now assume that (35) holds. Similarly $A_n \in w(p)^*$ for each $n \in N$. Hence, by Theprem 1, (35) implies $(A_n(x)) \in w_\infty(q)$ so that $A \in (w(p), w_\infty(q))$.

If we condider Remark 1, we see that (35) is sufficient for $A \in (w(p), w_\infty(q))$. But the following condition can be taken as a necessary condition for $A \in (w(p), w_\infty(q))$:

$$(36) \qquad \sup_n \{ n^{-1} [C(n, B)]^{q_n} \} < \infty.$$

Thus, Theorem 7 can be stated as follows: Let $0 < \inf_k p_k \leq p_k \leq 1$ for every $k \in N$ and let $q = (q_k)$ be bounded. If (35) holds, then $A \in (w(p), w_\infty(q))$, and if $A \in (w(p), w_\infty(q))$ then (36) holds.

Theorem 8. Let $0 < \inf_k p_k \leq p_k \leq 1$ *and let* q *be bounded.* $A \in (w(p), w_o(q))$ *if and only if*

$$(37) \qquad \lim_n (n^{-1} \sum_{i=1}^{n} | a_{ik} |^{q_i}) = 0, \textit{ for } k = 1, 2, \ldots,$$

$$(38) \qquad \lim_n (n^{-1} \sum_{i=1}^{n} | \Sigma_k \, a_{ik} |^{q_i}) = 0,$$

and

$$(39) \qquad \lim_B \lim\sup_n \{ n^{-1} \sum_{i=1}^{n} [C(i, B)]^{q_i} \} = 0,$$

where C (i, B) satisfies the condition of Theorem 7.

Proof. Let us define A_n by (19) on w (p). First suppose that (37), (38) and (39) hold. Tben $A_n \in w(p)^*$ for every $n \in N$, where $\| A_n \|_B = C(n, B)$ and B is an integer for which $\| A_n \|_B$ is defined. Since w (p) is a paranormed space and the sequence $(e, e_1, e_2, \ldots)$ is a basis, (37) and (38) imply $(A_n(b)) \in w_o(q)$ for every $b \in (e, e_1, e_2, \ldots)$ In this case, the above result together with (39) satisfy the hypothesis of Theorem 2. Thus, $(A_n(x)) \in w_o(q)$, so that $A \in (w(p), w_o(q))$, by Theorem 2 (i).

Now suppose $A \in (w(p), w_o(q))$. Similarly, $A_n \in w(p)^*$ for every $n \in N$, where $\| A_n \|_B = C(n, B)$ and B is an integer for which $\| A_n \|_B$ is defined. Since w (p) is a complete paranormed space and therefore it is a β-space, we see that (39) is a necessary condition for $A \in (w(p), w_o(q))$ by Theorem 2 (iii). If we consider the sequences e_k and e, then we also see that (37) and (38) are necessary conditions.

Remark 3. As in Remark 1 and 2, it is easily seen that (37), (38) and (39) are sufficient conditions while the following condiditions are necessary for $A \in (w(p), w_o(q))$:

(40) $$\lim_n n^{-1} | a_{nk} |^{q_n} = 0, \text{ for } k = 1, 2, \ldots,$$

(41) $$\lim_n (n^{-1} | \Sigma_k a_{nk} |^{q_n}) = 0,$$

and

$$\lim_B \limsup_n \{ n^{-1} [C(n, B)]^{q_n} \} = 0.$$

It is obvious that a similar remark to Remarks 1 and 2 for q and $[C(n, B)]^{q_n}$ can be given here.

Theorem 9. *Let* $0 < \inf_k p_k \leq p_k \leq 1$ *for all* $k \in N$, *and let* $q \in c_o$. $A \in (w(p), w_o(q))$ *if and only if* (39) *holds.*

Proof. Using the methods of Theorem 8, this follows from Theorem 2, parts (ii) and (iii).

It is easily showed that (39) is sufficient and (42) is necessary for $A \in w(p), w_o(q))$

ÖZET

X kompleks lineer paranormlu bir uzay ve X in alt cümlelerinin bir dizisi (X_n) olsun. $j \geq 1$ bilinen sabit bir tam sayıyı göstermek üzere, $n = \iota, \ldots, j$ için $\theta \in X_n$ olması $x \pm y \in X_{n+j}$ olmasını gerektiriyorsa (X_n) dizisine α_j-dizisi adını verdik. Eğer (X_n) dizisi X de bir α_j-dizisi ve her bir X_n, X de hiç bir yerde yoğun olmayan bir cümle olmak üzere $X = \bigcup_{n=i}^{\infty} X_n$ yazılabiliyorsa X e bir α_j-uzayı aksi halde bir β_j-uzayı dedik. Bu tip uzaylar üzerinde sürekli lineer fonksiyonellerin oluşturduğu diziler için Banach-Steinhaus tipi iki teorem ispatladık. Örneğin, paranormlu bir β_j-veya β-uzayına ait her x elemanı için sürekli lineer fonksiyonellerin bir $(A_n(x))$ dizisinin kuvvetli Cesaro toplanabilir diziler uzayında *olması için gerek ve yeter koşullar verildi. İspatladığımız bu iki teorem ve diğer Banach-Steinhaus tipi teoremler kullanılarak; kuvvetli Cesaro toplanabilir dizi uzaylarından diğer, bazı tip dizi uzaylarına olan matris dönüşümleri, yani*

$(w(p), l_\infty(q))$, $(w(p), c_0(q))$, $(w(p), c(q,))$ $(w(p), (w_\infty(q))$ $(w(p), (w_0(q))$ *matris sınıfları karakterize edildi.*

REFERENCES

1– W. L.C. Sargent, On some theorems of Hahn, Banach and Steinhaus, J. London Math. Soc. 28 (1953), 438–451.

2– S, Simons, The sequence spaces $l(p_i)$ and $m(p_i)$, Proc. London Math. Soc. (3) 15 (1965), 422–436.

3– I. J. Maddox, Spaces of strongly summable sequences, Quartely J. Math. Oxford, 18 (1967), 345–355.

4– I. J. Maddox, Paranormed sequences spaces generated by infinite matrices, Proc. Camb. Phil. Soc. 64 (1968) 335–340.

5– I. J. Maddox, Continuous and Köthe-Toeplitz duals of certain sequence spaces, Proc. Camb. Phil. Soc. 65 (1969), 431–435.

6– I. J. Maddox, Some properties of paranormed sequence spaces, J. London Math. Soc. (2) 1 (1969). 316–322.

7– I. J. Maddox, Elements of functional analysis, Cam. Unıversty press.

8– C. G. Lascarides and I. J. Maddox, Matrix transformations between some classes of sequenses, Proc. Cam. Phil. Soc. 68 (1970) 99–104.

9– M. A. L. Willey, On sequences of bounded linear functionals with applications to matrix transformations, J. London Math. Soc. (2) 7 (1973), 19–30.

10– I. J. Maddox, and M. A. L. Willey, Continuous operators on paranormed spaces and matrix transformations, Pasific J. Math. 53 (1974), 217–228.

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