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On Matrix Sequences of modified Tribonacci-Lucas Numbers

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ABSTRACT

In this paper, we define modified Tribonacci-Lucas matrix sequence and investigate its properties.

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1. Introduction and Preliminaries

Tribonacci sequence $\{T_n\}_{n\geq 0}$, Tribonacci-Lucas sequence $\{K_n\}_{n\geq 0}$ and modified Tribonacci-Lucas sequence $\{G_n\}_{n\geq 0}$ are defined, respectively, by the third-order recurrence relations

$$T_{n+3} = T_{n+2} + T_{n+1} + T_n, \qquad T_0 = 0, T_1 = 1, T_2 = 1, \tag{1.1}$$
$$K_{n+3} = K_{n+2} + K_{n+1} + K_n, \qquad K_0 = 3, K_1 = 1, K_2 = 3, \tag{1.2}$$
$$G_{n+3} = G_{n+2} + G_{n+1} + G_n, \qquad G_0 = 4, G_1 = 4, G_2 = 10. \tag{1.3}$$

The sequences $\{T_n\}_{n\geq 0}$, $\{K_n\}_{n\geq 0}$ and $\{G_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$T_{-n} = -T_{-(n-1)} - T_{-(n-2)} + T_{-(n-3)}, \tag{1.4}$$
$$K_{-n} = -K_{-(n-1)} - K_{-(n-2)} + K_{-(n-3)}, \tag{1.5}$$
$$G_{-n} = -G_{-(n-1)} - G_{-(n-2)} + G_{-(n-3)}, \tag{1.6}$$

for $n = 1, 2, 3, \ldots$ respectively. Therefore, recurrences (1.1)-(1.3) hold for all integers n. Basic properties of Tribonacci, Tribonacci-Lucas and modified Tribonacci-Lucas sequences are given in [8].

Next, we present the first few values of the Tribonacci, Tribonacci-Lucas and modified Tribonacci-Lucas numbers with positive and negative subscripts:

Table 1. The first few values of the special third-order numbers with positive and negative subscripts.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13
T_n	0	1	1	2	4	7	13	24	44	81	149	274	504	927
T_{-n}		0	1	−1	0	2	−3	1	4	−8	5	7	−20	18
K_n	3	1	3	7	11	21	39	71	131	241	443	815	1499	2757
K_{-n}		−1	−1	5	−5	−1	11	−15	3	23	−41	21	43	−105
G_n	4	4	10	18	32	60	110	202	372	684	1258	2314	4256	7828
G_{-n}		2	−2	4	0	−6	10	−4	−12	26	−18	−20	64	−62

For all integers n, Tribonacci, Tribonacci-Lucas and modified Tribonacci-Lucas numbers can be expressed using Binet's formulas as

$$T_n = \frac{\alpha^{n+1}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\beta^{n+1}}{(\beta-\alpha)(\beta-\gamma)} + \frac{\gamma^{n+1}}{(\gamma-\alpha)(\gamma-\beta)}, \tag{1.7}$$

$$K_n = \alpha^n + \beta^n + \gamma^n, \tag{1.8}$$

$$G_n = (\alpha+1)\alpha^n + (\beta+1)\beta^n + (\gamma+1)\gamma^n \tag{1.9}$$

respectively. Here, α, β and γ are the roots of the cubic equation

$$x^3 - x^2 - x - 1 = 0.$$

Moreover,

$$\alpha = \frac{1+\sqrt[3]{19+3\sqrt{33}}+\sqrt[3]{19-3\sqrt{33}}}{3},$$

$$\beta = \frac{1+\omega\sqrt[3]{19+3\sqrt{33}}+\omega^2\sqrt[3]{19-3\sqrt{33}}}{3},$$

$$\gamma = \frac{1+\omega^2\sqrt[3]{19+3\sqrt{33}}+\omega\sqrt[3]{19-3\sqrt{33}}}{3}$$

where

$$\omega = \frac{-1+i\sqrt{3}}{2} = \exp(2\pi i/3).$$

Note that

$$\alpha+\beta+\gamma = 1,$$
$$\alpha\beta+\alpha\gamma+\beta\gamma = -1,$$
$$\alpha\beta\gamma = 1.$$

Note that the Binet form of a sequence satisfying (1.7) and (1.9) for non-negative integers is valid for all integers n. The generating functions for the modified Tribonacci-Lucas sequence $\{G_n\}_{n\geq 0}$ is

$$\sum_{n=0}^{\infty} G_n x^n = \frac{4+2x^2}{1-x-x^2-x^3}. \tag{1.10}$$

Recently, there have been so many studies of the sequences of numbers in the literature that concern about subsequences of the Horadam (Generalized Fibonacci) numbers and generalized Tribonacci numbers such as Fibonacci, Lucas, Pell and Jacobsthal numbers; third-order Pell, third-order Pell-Lucas, Padovan, Perrin, Padovan-Perrin, Narayana, third order Jacobsthal and third order Jacobsthal-Lucas numbers. The sequences of numbers were widely used in many research areas, such as physics, engineering, architecture, nature and art. On the other hand, the matrix sequences have taken so much interest for different type of numbers. We present some works on matrix sequences of the numbers in the following Table 2.

Table 2. A few special study on the matrix sequences of the numbers.

Name of sequence	work on the matrix sequences of the numbers
Generalized Fibonacci	[2],[3],[4],[9],[10],[11],[12],[13],[16]
Generalized Tribonacci	[1],[6],[7],[14],[15]
Generalized Tetranacci	[5]

In this paper, the matrix sequences of modified Tribonacci-Lucas numbers will be defined. Then, by giving the generating functions, the Binet formulas, and summation formulas over this new matrix sequence, we will obtain some fundamental properties on modified Tribonacci-Lucas numbers. Also, we will present the relationship between matrix sequences of Tribonacci, Tribonacci-Lucas and modified Tribonacci-Lucas numbers.

Tribonacci and Tribonacci-Lucas matrix sequences are defined as follows (see [6]).

Definition 1.1 For any integer $n \geq 0$, the Tribonacci matrix $(\mathcal{T}_n)$ and Tribonacci-Lucas matrix $(\mathcal{K}_n)$ are defined by

$$\mathcal{T}_n = \mathcal{T}_{n-1} + \mathcal{T}_{n-2} + \mathcal{T}_{n-3}, \qquad (1.11)$$

$$\mathcal{K}_n = \mathcal{K}_{n-1} + \mathcal{K}_{n-2} + \mathcal{K}_{n-3}, \qquad (1.12)$$

respectively, with initial conditions

$$\mathcal{T}_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ \mathcal{T}_1 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \ \mathcal{T}_2 = \begin{pmatrix} 2 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

and

$$\mathcal{K}_0 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & -2 & -1 \\ -1 & 4 & -1 \end{pmatrix}, \ \mathcal{K}_1 = \begin{pmatrix} 3 & 4 & 1 \\ 1 & 2 & 3 \\ 3 & -2 & -1 \end{pmatrix}, \ \mathcal{K}_2 = \begin{pmatrix} 7 & 4 & 3 \\ 3 & 4 & 1 \\ 1 & 2 & 3 \end{pmatrix}.$$

The sequences $\{\mathcal{T}_n\}_{n\geq 0}$ and $\{\mathcal{K}_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\mathcal{T}_{-n} = -\mathcal{T}_{-(n-1)} - \mathcal{T}_{-(n-2)} + \mathcal{T}_{-(n-3)}$$

and

$$\mathcal{K}_{-n} = -\mathcal{K}_{-(n-1)} - \mathcal{K}_{-(n-2)} + \mathcal{K}_{-(n-3)}$$

for $n = 1, 2, 3, ...$ respectively. Therefore, recurrences (1.11) and (1.12) hold for all integers n.

The following theorem gives the nth general terms of the Tribonacci and Tribonacci-Lucas matrix sequences.

Theorem 1.2 For any integer $n \geq 0$, we have the following formulas of the matrix sequences:

$$\mathcal{T}_n = \begin{pmatrix} T_{n+1} & T_n + T_{n-1} & T_n \\ T_n & T_{n-1} + T_{n-2} & T_{n-1} \\ T_{n-1} & T_{n-2} + T_{n-3} & T_{n-2} \end{pmatrix}, \qquad (1.13)$$

$$\mathcal{K}_n = \begin{pmatrix} K_{n+1} & K_n + K_{n-1} & K_n \\ K_n & K_{n-1} + K_{n-2} & K_{n-1} \\ K_{n-1} & K_{n-2} + K_{n-3} & K_{n-2} \end{pmatrix}. \qquad (1.14)$$

Proof. It is given in [6]. □

We now give the Binet formulas for the Tribonacci and Tribonacci-Lucas matrix sequences.

Theorem 1.3 For every integer n, the Binet formulas of the Tribonacci and Tribonacci-Lucas matrix sequences are given by

$$\mathcal{T}_n = A_1\alpha^n + B_1\beta^n + C_1\gamma^n, \qquad (1.15)$$

$$\mathcal{K}_n = A_2\alpha^n + B_2\beta^n + C_2\gamma^n \qquad (1.16)$$

where

$$A_1 = \frac{\alpha\mathcal{T}_2 + \alpha(\alpha-1)\mathcal{T}_1 + \mathcal{T}_0}{\alpha(\alpha-\gamma)(\alpha-\beta)}, B_1 = \frac{\beta\mathcal{T}_2 + \beta(\beta-1)\mathcal{T}_1 + \mathcal{T}_0}{\beta(\beta-\gamma)(\beta-\alpha)}, C_1 = \frac{\gamma\mathcal{T}_2 + \gamma(\gamma-1)\mathcal{T}_1 + \mathcal{T}_0}{\gamma(\gamma-\beta)(\gamma-\alpha)},$$

$$A_2 = \frac{\alpha\mathcal{K}_2 + \alpha(\alpha-1)\mathcal{K}_1 + \mathcal{K}_0}{\alpha(\alpha-\gamma)(\alpha-\beta)}, B_2 = \frac{\beta\mathcal{K}_2 + \beta(\beta-1)\mathcal{K}_1 + \mathcal{K}_0}{\beta(\beta-\gamma)(\beta-\alpha)}, C_2 = \frac{\gamma\mathcal{K}_2 + \gamma(\gamma-1)\mathcal{K}_1 + \mathcal{K}_0}{\gamma(\gamma-\beta)(\gamma-\alpha)}.$$

Proof. It is given in [6]. □

2. The Matrix Sequences of modified Tribonacci-Lucas Numbers

In this section, we define modified Tribonacci-Lucas matrix sequence and investgate its properties.

Definition 2.1 For any integer $n \geq 0$, the modified Tribonacci-Lucas matrix $(\mathcal{G}_n)$ is defined by

$$\mathcal{G}_n = \mathcal{G}_{n-1} + \mathcal{G}_{n-2} + \mathcal{G}_{n-3}, \tag{2.1}$$

with initial conditions

$$\mathcal{G}_0 = \begin{pmatrix} 4 & 6 & 4 \\ 4 & 0 & 2 \\ 2 & 2 & -2 \end{pmatrix}, \mathcal{G}_1 = \begin{pmatrix} 10 & 8 & 4 \\ 4 & 6 & 4 \\ 4 & 0 & 2 \end{pmatrix}, \mathcal{G}_2 = \begin{pmatrix} 18 & 14 & 10 \\ 10 & 8 & 4 \\ 4 & 6 & 4 \end{pmatrix}.$$

The sequence $\{\mathcal{G}_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\mathcal{G}_{-n} = -\mathcal{G}_{-(n-1)} - \mathcal{G}_{-(n-2)} + \mathcal{G}_{-(n-3)}$$

for $n = 1, 2, 3, \ldots$. Therefore, recurrences (2.1) holds for all integers n.

The following theorem gives the nth general terms of the modified Tribonacci-Lucas matrix sequence.

Theorem 2.2 For any integer $n \geq 0$, we have the following formula of the matrix sequence:

$$\mathcal{G}_n = \begin{pmatrix} G_{n+1} & G_n + G_{n-1} & G_n \\ G_n & G_{n-1} + G_{n-2} & G_{n-1} \\ G_{n-1} & G_{n-2} + G_{n-3} & G_{n-2} \end{pmatrix}. \tag{2.2}$$

Proof. We prove (2.2) by strong mathematical induction on n. If $n = 0$ then, since $G_{-3} = 4$, $G_{-2} = -2$, $G_{-1} = 2$, $G_0 = 4, G_1 = 4, G_2 = 10$, we have

$$\mathcal{G}_0 = \begin{pmatrix} G_1 & G_0 + G_{-1} & G_0 \\ G_0 & G_{-1} + G_{-2} & G_{-1} \\ G_{-1} & G_{-2} + G_{-3} & G_{-2} \end{pmatrix} = \begin{pmatrix} 4 & 6 & 4 \\ 4 & 0 & 2 \\ 2 & 2 & -2 \end{pmatrix}$$

which is true and

$$\mathcal{G}_1 = \begin{pmatrix} G_2 & G_1 + G_0 & G_1 \\ G_1 & G_0 + G_{-1} & G_0 \\ G_0 & G_{-1} + G_{-2} & G_{-1} \end{pmatrix} = \begin{pmatrix} 10 & 8 & 4 \\ 4 & 6 & 4 \\ 4 & 0 & 2 \end{pmatrix}$$

which is true. Assume that the equality holds for $n \leq k$. For $n = k + 1$, we have

$$\begin{aligned}
\mathcal{G}_{k+1} &= \mathcal{G}_k + \mathcal{G}_{k-1} + \mathcal{G}_{k-2} \\
&= \begin{pmatrix} G_{k+1} & G_k + G_{k-1} & G_k \\ G_k & G_{k-1} + G_{k-2} & G_{k-1} \\ G_{k-1} & G_{k-2} + G_{k-3} & G_{k-2} \end{pmatrix} + \begin{pmatrix} G_k & G_{k-1} + G_{k-2} & G_{k-1} \\ G_{k-1} & G_{k-2} + G_{k-3} & G_{k-2} \\ G_{k-2} & G_{k-3} + G_{k-4} & G_{k-3} \end{pmatrix} \\
&\quad + \begin{pmatrix} G_{k-1} & G_{k-2} + G_{k-3} & G_{k-2} \\ G_{k-2} & G_{k-3} + G_{k-4} & G_{k-3} \\ G_{k-3} & G_{k-4} + G_{k-5} & G_{k-4} \end{pmatrix} \\
&= \begin{pmatrix} G_k + G_{k-1} + G_{k+1} & G_k + G_{k-1} + G_{k-1} + G_{k-2} + G_{k-2} + G_{k-3} & G_k + G_{k-1} + G_{k-2} \\ G_k + G_{k-1} + G_{k-2} & G_{k-1} + G_{k-2} + G_{k-2} + G_{k-3} + G_{k-3} + G_{k-4} & G_{k-1} + G_{k-2} + G_{k-3} \\ G_{k-1} + G_{k-2} + G_{k-3} & G_{k-2} + G_{k-3} + G_{k-3} + G_{k-4} + G_{k-4} + G_{k-5} & G_{k-2} + G_{k-3} + G_{k-4} \end{pmatrix} \\
&= \begin{pmatrix} G_{k+2} & G_k + G_{k+1} & G_{k+1} \\ G_{k+1} & G_k + G_{k-1} & G_k \\ G_k & G_{k-1} + G_{k-2} & G_{k-1} \end{pmatrix}.
\end{aligned}$$

Thus, by strong induction on n, this proves (2.2). □

We now give the Binet formula for the modified Tribonacci-Lucas matrix sequence.

Theorem 2.3 For every integer n, the Binet formula of the modified Tribonacci-Lucas matrix sequence is given by

$$\mathcal{G}_n = A_5\alpha^n + B_5\beta^n + C_5\gamma^n \tag{2.3}$$

where

$$A_5 = \frac{\alpha\mathcal{G}_2 + \alpha(\alpha-1)\mathcal{G}_1 + \mathcal{G}_0}{\alpha(\alpha-\gamma)(\alpha-\beta)}, B_5 = \frac{\beta\mathcal{G}_2 + \beta(\beta-1)\mathcal{G}_1 + \mathcal{G}_0}{\beta(\beta-\gamma)(\beta-\alpha)}, C_5 = \frac{\gamma\mathcal{G}_2 + \gamma(\gamma-1)\mathcal{G}_1 + \mathcal{G}_0}{\gamma(\gamma-\beta)(\gamma-\alpha)}.$$

Proof. We prove the theorem only for $n \geq 0$. By the assumption, the characteristic equation of (2.1) is $x^3 - x^2 - x - 1 = 0$ and the roots of it are α, β and γ. So, it's general solution is given by

$$\mathcal{G}_n = A_5\alpha^n + B_5\beta^n + C_5\gamma^n.$$

Using initial condition which is given in Definition 2.1, and also applying lineer algebra operations, we obtain the matrices A_5, B_5, C_5 as desired. This gives the formula for $\mathcal{G}_n$. □

The well known Binet formulas for modified Tribonacci-Lucas is given in (1.7). But, we will obtain this function in terms of modified Tribonacci-Lucas matrix sequence as a consequence of Theorems 2.2 and 2.3. To do this, we will give the formulas for these numbers by means of the related matrix sequences. In fact, in the proof of next corollary, we will just compare the linear combination of the 2nd row and 1st column entries of the matrices.

Corollary 2.4 For every integers n, the Binet's formulas for modified Tribonacci-Lucas numbers is given as

$$G_n = (\alpha+1)\alpha^n + (\beta+1)\beta^n + (\gamma+1)\gamma^n.$$

Proof. From Theorem 2.2, we have

$$\begin{aligned}
\mathcal{G}_n &= A_5\alpha^n + B_5\beta^n + C_5\gamma^n \\
&= \frac{\alpha\mathcal{G}_2 + \alpha(\alpha-1)\mathcal{G}_1 + \mathcal{G}_0}{\alpha(\alpha-\gamma)(\alpha-\beta)}\alpha^n + \frac{\beta\mathcal{G}_2 + \beta(\beta-1)\mathcal{G}_1 + \mathcal{G}_0}{\beta(\beta-\gamma)(\beta-\alpha)}\beta^n \\
&\quad + \frac{\gamma\mathcal{G}_2 + \gamma(\gamma-1)\mathcal{G}_1 + \mathcal{G}_0}{\gamma(\gamma-\beta)(\gamma-\alpha)}\gamma^n \\
&= \frac{\alpha^{n-1}}{(\alpha-\gamma)(\alpha-\beta)}\begin{pmatrix} 10\alpha^2+8\alpha+4 & 8\alpha^2+6\alpha+6 & 4\alpha^2+6\alpha+4 \\ 4\alpha^2+6\alpha+4 & 2\alpha(3\alpha+1) & 4\alpha^2+2 \\ 4\alpha^2+2 & 6\alpha+2 & 2\alpha^2+2\alpha-2 \end{pmatrix} \\
&\quad + \frac{\beta^{n-1}}{(\beta-\gamma)(\beta-\alpha)}\begin{pmatrix} 10\beta^2+8\beta+4 & 8\beta^2+6\beta+6 & 4\beta^2+6\beta+4 \\ 4\beta^2+6\beta+4 & 2\beta(3\beta+1) & 4\beta^2+2 \\ 4\beta^2+2 & 6\beta+2 & 2\beta^2+2\beta-2 \end{pmatrix} \\
&\quad + \frac{\gamma^{n-1}}{(\gamma-\beta)(\gamma-\alpha)}\begin{pmatrix} 10\gamma^2+8\gamma+4 & 8\gamma^2+6\gamma+6 & 4\gamma^2+6\gamma+4 \\ 4\gamma^2+6\gamma+4 & 2\gamma(3\gamma+1) & 4\gamma^2+2 \\ 4\gamma^2+2 & 6\gamma+2 & 2\gamma^2+2\gamma-2 \end{pmatrix}.
\end{aligned}$$

By Theorem 2.3, we know that

$$\mathcal{G}_n = \begin{pmatrix} G_{n+1} & G_n + G_{n-1} & G_n \\ G_n & G_{n-1} + G_{n-2} & G_{n-1} \\ G_{n-1} & G_{n-2} + G_{n-3} & G_{n-2} \end{pmatrix}.$$

Now, if we compare the 2nd row and 1st column entries with the matrices in the above two equations, then we obtain

$$\begin{aligned}
G_n &= \frac{\alpha^{n-1}}{(\alpha-\gamma)(\alpha-\beta)}(4\alpha^2+6\alpha+4) + \frac{\beta^{n-1}}{(\beta-\gamma)(\beta-\alpha)}(4\beta^2+6\beta+4) \\
&\quad + \frac{\gamma^{n-1}}{(\gamma-\beta)(\gamma-\alpha)}(4\gamma^2+6\gamma+4) \\
&= (\alpha+1)\alpha^n + (\beta+1)\beta^n + (\gamma+1)\gamma^n. \quad \square
\end{aligned}$$

Now, we present summation formulas for modified Tribonacci-Lucas matrix sequences.

Theorem 2.5 For all integers m and j, we have

$$\sum_{k=0}^{n} \mathcal{G}_{mk+j} = \frac{\mathcal{G}_{mn+m+j} + \mathcal{G}_{mn-m+j} + (1-K_{-m})\mathcal{G}_{mn+j} - \mathcal{G}_{m+j} - \mathcal{G}_{j-m} + (K_m - 1)\mathcal{G}_j}{K_m - K_{-m}}. \tag{2.4}$$

Proof. Note that

$$\begin{aligned}\sum_{k=0}^{n}\mathcal{G}_{mk+j} &= \mathcal{G}_{mn+j}+\sum_{k=0}^{n-1}\mathcal{G}_{mk+j}=\mathcal{G}_{mn+j}+\sum_{k=0}^{n-1}(A_5\alpha^{mk+j}+B_5\beta^{mk+j}+C_5\gamma^{mk+j})\\ &= \mathcal{G}_{mn+j}+A_5\alpha^{j}\left(\frac{\alpha^{mn}-1}{\alpha^{m}-1}\right)+B_5\beta^{j}\left(\frac{\beta^{mn}-1}{\beta^{m}-1}\right)+C_5\gamma^{j}\left(\frac{\gamma^{mn}-1}{\gamma^{m}-1}\right).\end{aligned}$$

Simplifying the last equalities in the last expression imply (2.4) as required. □

As in Corollary 2.4, in the proof of next Corollary, we just compare the linear combination of the 2nd row and 1st column entries of the relevant matrices.

Corollary 2.6 For all integers m and j, we have

$$\sum_{k=0}^{n}G_{mk+j}=\frac{G_{mn+m+j}+G_{mn-m+j}+(1-K_{-m})G_{mn+j}-G_{m+j}-G_{j-m}+(K_m-1)G_j}{K_m-K_{-m}}. \tag{2.5}$$

Note that using the above Corollary we obtain the following well known formulas (taking $m=1, j=0$ and $m=-1, j=0$ respectively):

$$\sum_{k=0}^{n}G_k=\frac{1}{2}\left(G_{n+1}+2G_n+G_{n-1}-6\right)$$

and

$$\sum_{k=0}^{n}G_{-k}=\frac{1}{2}(-G_{-n+1}-G_{-n-1}+14).$$

or

$$\sum_{k=1}^{n}G_{-k}=\frac{1}{2}(-G_{-n+1}-G_{-n-1}+6).$$

Note that the last Corollary can be written in the following form:

$$\sum_{k=1}^{n}G_{mk+j}=\frac{G_{mn+m+j}+G_{mn-m+j}+(1-K_{-m})G_{mn+j}-G_{m+j}-G_{j-m}+(K_{-m}-1)G_j}{K_m-K_{-m}}.$$

We now give generating functions of $\mathcal{G}$.

Theorem 2.7 The generating function for the modified Tribonacci-Lucas matrix sequence is given as

$$\sum_{n=0}^{\infty}\mathcal{G}_nx^n=\frac{1}{1-x-x^2-x^3}\begin{pmatrix}4x^2+6x+4 & 2x+6 & 2x^2+4\\ 2x^2+4 & 2x^2+6x & -2x^2+2x+2\\ -2x^2+2x+2 & 4x^2-2x+2 & 4x^2+4x-2\end{pmatrix}.$$

Proof. Suppose that $g(x)=\sum_{n=0}^{\infty}\mathcal{G}_nx^n$ is the generating function for the sequence $\{\mathcal{G}_n\}_{n\geq 0}$. Then, using Definition 2.1, we obtain

$$\begin{aligned}g(x) &= \sum_{n=0}^{\infty}\mathcal{G}_nx^n=\mathcal{G}_0+\mathcal{G}_1x+\mathcal{G}_2x^2+\sum_{n=3}^{\infty}\mathcal{G}_nx^n\\ &= \mathcal{G}_0+\mathcal{G}_1x+\mathcal{G}_2x^2+\sum_{n=3}^{\infty}(\mathcal{G}_{n-1}+\mathcal{G}_{n-2}+\mathcal{G}_{n-3})x^n\\ &= \mathcal{G}_0+\mathcal{G}_1x+\mathcal{G}_2x^2+\sum_{n=3}^{\infty}\mathcal{G}_{n-1}x^n+\sum_{n=3}^{\infty}\mathcal{G}_{n-2}x^n+\sum_{n=3}^{\infty}\mathcal{G}_{n-3}x^n\\ &= \mathcal{G}_0+\mathcal{G}_1x+\mathcal{G}_2x^2-\mathcal{G}_0x-\mathcal{G}_1x^2-\mathcal{G}_0x^2+x\sum_{n=0}^{\infty}\mathcal{G}_nx^n+x^2\sum_{n=0}^{\infty}\mathcal{G}_nx^n+x^3\sum_{n=0}^{\infty}\mathcal{G}_nx^n\\ &= \mathcal{G}_0+\mathcal{G}_1x+\mathcal{G}_2x^2-\mathcal{G}_0x-\mathcal{G}_1x^2-\mathcal{G}_0x^2+xg(x)+x^2g(x)+x^3g(x).\end{aligned}$$

Rearranging the above equation, we get

$$g(x)=\frac{\mathcal{G}_0+(\mathcal{G}_1-\mathcal{G}_0)x+(\mathcal{G}_2-\mathcal{G}_1-\mathcal{G}_0)x^2}{1-x-x^2-x^3}$$

which equals the $\sum_{n=0}^{\infty}\mathcal{G}_n x^n$ in the Theorem. This completes the proof. □

The well known generating functions for modified Tribonacci-Lucas numbers is as in (1.10). However, we will obtain these functions in terms of modified Tribonacci-Lucas matrix sequence as a consequence of Theorem 2.7. To do this, we will again compare the the 2nd row and 1st column entries with the matrices in Theorem 2.7. Thus, we have the following corollary.

Corollary 2.8 The generating functions for the modified Tribonacci-Lucas sequence $\{G_n\}_{n\geq 0}$ is given as

$$\sum_{n=0}^{\infty}G_n x^n=\frac{2x^2+4}{1-x-x^2-x^3}.$$

3. Relation Between Tribonacci, Tribonacci-Lucas and modified Tribonacci-Lucas Matrix Sequences

The following theorem shows that there always exist interrelation between Tribonacci, Tribonacci-Lucas and modified Tribonacci-Lucas matrix sequences.

Theorem 3.1 For the matrix sequences $\{\mathcal{T}_n\}$ and $\{\mathcal{G}_n\}$, we have the following identities.

(a) $22\mathcal{T}_n=7\mathcal{G}_{n+4}-13\mathcal{G}_{n+3}+\mathcal{G}_{n+2}$,

(b) $22\mathcal{T}_n=-6\mathcal{G}_{n+3}+8\mathcal{G}_{n+2}+7\mathcal{G}_{n+1}$,

(c) $22\mathcal{T}_n=2\mathcal{G}_{n+2}+\mathcal{G}_{n+1}-6\mathcal{G}_n$,

(d) $22\mathcal{T}_n=3\mathcal{G}_{n+1}-4\mathcal{G}_n+2\mathcal{G}_{n-1}$,

(e) $22\mathcal{T}_n=-\mathcal{G}_n+5\mathcal{G}_{n-1}+3\mathcal{G}_{n-2}$,

(f) $\mathcal{G}_n=4\mathcal{T}_{n+4}-6\mathcal{T}_{n+3}$,

(g) $\mathcal{G}_n=-2\mathcal{T}_{n+3}+4\mathcal{T}_{n+2}+4\mathcal{T}_{n+1}$,

(h) $\mathcal{G}_n=2\mathcal{T}_{n+2}+2\mathcal{T}_{n+1}-2\mathcal{T}_n$,

(i) $\mathcal{G}_n=4\mathcal{T}_{n+1}+2\mathcal{T}_{n-1}$,

(j) $\mathcal{G}_n=4\mathcal{T}_n+6\mathcal{T}_{n-1}+4\mathcal{T}_{n-2}$.

Proof. The proof follows from the following equalities:

$$\begin{aligned}
22T_n &= 7G_{n+4}-13G_{n+3}+G_{n+2},\\
22T_n &= -6G_{n+3}+8G_{n+2}+7G_{n+1},\\
22T_n &= 2G_{n+2}+G_{n+1}-6G_n,\\
22T_n &= 3G_{n+1}-4G_n+2G_{n-1},\\
22T_n &= -G_n+5G_{n-1}+3G_{n-2}
\end{aligned}$$

and

$$G_n = 4T_{n+4}-6T_{n+3}, \tag{3.1}$$
$$G_n = -2T_{n+3}+4T_{n+2}+4T_{n+1}, \tag{3.2}$$
$$G_n = 2T_{n+2}+2T_{n+1}-2T_n, \tag{3.3}$$
$$G_n = 4T_{n+1}+2T_{n-1}, \tag{3.4}$$
$$G_n = 4T_n+6T_{n-1}+4T_{n-2}. \tag{3.5}$$

□

Theorem 3.2 For the matrix sequences $\{\mathcal{K}_n\}$ and $\{\mathcal{G}_n\}$, we have the following identities.

(a) $2\mathcal{K}_n = -3\mathcal{G}_{n+4} + 4\mathcal{G}_{n+3} + 3\mathcal{G}_{n+2}$,

(b) $2\mathcal{K}_n = \mathcal{G}_{n+3} - 3\mathcal{G}_{n+1}$,

(c) $2\mathcal{K}_n = \mathcal{G}_{n+2} - 2\mathcal{G}_{n+1} + \mathcal{G}_n$,

(d) $2\mathcal{K}_n = -\mathcal{G}_{n+1} + 2\mathcal{G}_n + \mathcal{G}_{n-1}$,

(e) $2\mathcal{K}_n = \mathcal{G}_n - \mathcal{G}_{n-2}$,

(f) $\mathcal{G}_n = \mathcal{K}_{n+3} - \mathcal{K}_{n+2}$,

(g) $\mathcal{G}_n = \mathcal{K}_{n+1} + \mathcal{K}_n$,

(h) $\mathcal{G}_n = 2\mathcal{K}_n + \mathcal{K}_{n-1} + \mathcal{K}_{n-2}$.

Proof. The proof follows from the following equalities:

$$\begin{aligned} 2K_n &= -3G_{n+4} + 4G_{n+3} + 3G_{n+2}, \\ 2K_n &= G_{n+3} - 3G_{n+1}, \\ 2K_n &= G_{n+2} - 2G_{n+1} + G_n, \\ 2K_n &= -G_{n+1} + 2G_n + G_{n-1}, \\ 2K_n &= G_n - G_{n-2} \end{aligned}$$

and

$$\begin{aligned} G_n &= K_{n+3} - K_{n+2}, \\ G_n &= K_{n+1} + K_n, \\ G_n &= 2K_n + K_{n-1} + K_{n-2}. \end{aligned}$$

□

Lemma 3.3 For all non-negative integers m and n, we have the following identities.

(a) $\mathcal{G}_0\mathcal{T}_n = \mathcal{T}_n\mathcal{G}_0 = \mathcal{G}_n$,

(b) $\mathcal{T}_0\mathcal{G}_n = \mathcal{G}_n\mathcal{T}_0 = \mathcal{G}_n$.

Proof. Identities can be established easily. Note that to show (a) we need to use the relations (3.1)-(3.5). □

We need the following Theorem.

Theorem 3.4 For all non-negative integers m and n, we have the following identities.

$$\begin{aligned} \mathcal{T}_m\mathcal{T}_n &= \mathcal{T}_{m+n} = \mathcal{T}_n\mathcal{T}_m, \\ \mathcal{K}_{m+n} &= \mathcal{T}_m\mathcal{K}_n = \mathcal{K}_n\mathcal{T}_m. \end{aligned}$$

Proof. It is given in [6]. □

The following theorem gives relations between the matrices $\mathcal{T}_n$ and $\mathcal{G}_n$.

Theorem 3.5 For all non-negative integers m and n, we have the following identities.

(a) $\mathcal{T}_m\mathcal{G}_n = \mathcal{G}_n\mathcal{T}_m = \mathcal{G}_{m+n}$,

(b) $\mathcal{G}_m\mathcal{G}_n = \mathcal{G}_n\mathcal{G}_m = 16\mathcal{T}_{m+n+8} - 48\mathcal{T}_{m+n+7} + 36\mathcal{T}_{m+n+6}$,

(c) $\mathcal{G}_m\mathcal{G}_n = \mathcal{G}_n\mathcal{G}_m = 4\mathcal{T}_{m+n+4} + 8\mathcal{T}_{m+n+3} - 4\mathcal{T}_{m+n+2} - 8\mathcal{T}_{m+n+1} + 4\mathcal{T}_{m+n}$,

(d) $\mathcal{G}_m\mathcal{G}_n = \mathcal{G}_n\mathcal{G}_m = 16\mathcal{T}_{m+n+2} + 16\mathcal{T}_{m+n} + 4\mathcal{T}_{m+n-2}$,

(e) $(\mathcal{G}_{m+n+2} - 2\mathcal{G}_{m+n+1} + \mathcal{G}_{m+n}) = \frac{1}{22}(2\mathcal{G}_{m+2} + \mathcal{G}_{m+1} - 6\mathcal{G}_m)(\mathcal{G}_{n+2} - 2\mathcal{G}_{n+1} + \mathcal{G}_n)$.

Proof.

(a) By Lemma 3.3, we have

$$\mathcal{T}_m\mathcal{G}_n = \mathcal{T}_m\mathcal{T}_n\mathcal{G}_0.$$

Now, from Theorem 3.4 and again by Lemma 3.3 we obtain $\mathcal{T}_m\mathcal{G}_n = \mathcal{T}_{m+n}\mathcal{G}_0 = \mathcal{G}_{m+n}$.

Similarly, it can be shown that $\mathcal{G}_n\mathcal{T}_m = \mathcal{G}_{m+n}$.

(b) Using Theorem 3.1 and Theorem 3.4, we obtain

$$\begin{aligned}\mathcal{G}_m\mathcal{G}_n &= (4\mathcal{T}_{m+4} - 6\mathcal{T}_{m+3})(4\mathcal{T}_{n+4} - 6\mathcal{T}_{n+3})\\ &= 36\mathcal{T}_{m+3}\mathcal{T}_{n+3} - 24\mathcal{T}_{m+3}\mathcal{T}_{n+4} - 24\mathcal{T}_{m+4}\mathcal{T}_{n+3} + 16\mathcal{T}_{m+4}\mathcal{T}_{n+4}\\ &= 36\mathcal{T}_{m+n+6} - 24\mathcal{T}_{m+n+7} - 24\mathcal{T}_{m+n+7} + 16\mathcal{T}_{m+n+8}\\ &= 36\mathcal{T}_{m+n+6} - 48\mathcal{T}_{m+n+7} + 16\mathcal{T}_{m+n+8}\\ &= 16\mathcal{T}_{m+n+8} - 48\mathcal{T}_{m+n+7} + 36\mathcal{T}_{m+n+6}.\end{aligned}$$

It can be shown similarly that $\mathcal{G}_n\mathcal{G}_m = 16\mathcal{T}_{m+n+8} - 48\mathcal{T}_{m+n+7} + 36\mathcal{T}_{m+n+6}$.

Similarly, the remaining of identities can be proved by considering again using Theorem 3.1 and Theorem 3.4. □

Comparing matrix entries and using Theorem 1.2 and Theorem 2.2 we have next result.

Corollary 3.6 For Tribonacci and modified Tribonacci-Lucas numbers, we have the following identities:

(a) $G_{m+n} = G_{n+1}T_m + G_n\,(T_{m-1} + T_{m-2}) + G_{n-1}T_{m-1}$,

(b) $G_{n+1}G_m + G_n\,(G_{m-1} + G_{m-2}) + G_{n-1}G_{m-1} = 16T_{m+n+8} - 48T_{m+n+7} + 36T_{m+n+6}$,

(c) $G_{n+1}G_m + G_n\,(G_{m-1} + G_{m-2}) + G_{n-1}G_{m-1} = 4T_{m+n+4} + 8T_{m+n+3} - 4T_{m+n+2} - 8T_{m+n+1} + 4T_{m+n}$,

(d) $G_{n+1}G_m + G_n\,(G_{m-1} + G_{m-2}) + G_{n-1}G_{m-1} = 16T_{m+n+2} + 16T_{m+n} + 4T_{m+n-2}$,

(e) $G_{m+n} - 2G_{m+n+1} + G_{m+n+2} = \frac{1}{22}(G_{n+3}\,(2G_{m+2} + G_{m+1} - 6G_m) + G_{n+2}(-4G_{m+2} + 15G_m - 5G_{m-1} - 6G_{m-2}) + G_{n+1}(2G_{m+2} - G_{m+1} - 11G_m + 4G_{m-1} + 12G_{m-2}) + G_n(-2G_{m+1} + G_m + 7G_{m-1} - 6G_{m-2}) + G_{n-1}(2G_{m+1} + G_m - 6G_{m-1}))$.

Proof.

(a) From Theorem 3.5 (a), we have $\mathcal{T}_m\mathcal{G}_n = \mathcal{G}_n\mathcal{T}_m = \mathcal{G}_{m+n}$. Using Theorem 1.2 and Theorem 2.2, we can write this result as

$$\begin{pmatrix} T_{m+1} & T_m + T_{m-1} & T_m \\ T_m & T_{m-1} + T_{m-2} & T_{m-1} \\ T_{m-1} & T_{m-2} + T_{m-3} & T_{m-2} \end{pmatrix}\begin{pmatrix} G_{n+1} & G_n + G_{n-1} & G_n \\ G_n & G_{n-1} + G_{n-2} & G_{n-1} \\ G_{n-1} & G_{n-2} + G_{n-3} & G_{n-2} \end{pmatrix}$$
$$= \begin{pmatrix} G_{m+n+1} & G_{m+n} + G_{m+n-1} & G_{m+n} \\ G_{m+n} & G_{m+n-1} + G_{m+n-2} & G_{m+n-1} \\ G_{m+n-1} & G_{m+n-2} + G_{m+n-3} & G_{m+n-2} \end{pmatrix}.$$

Now, by multiplying the left-side matrices and then by comparing the 2nd rows and 1st columns entries, we get the required identity in (a).

Similarly, the remaining of identities can be proved by considering again Theorems 3.5, 1.2 and 2.2. □

The next two theorems provide us the convenience to obtain the powers of Tribonacci and modified Tribonacci-Lucas matrix sequences.

Theorem 3.7 For non-negative integers m, n and r with $n \geq r$, the following identities hold:

(a) $\mathcal{T}_n^m = \mathcal{T}_{mn}$,

(b) $\mathcal{T}_{n+1}^m = \mathcal{T}_1^m\mathcal{T}_{mn}$,

(c) $\mathcal{T}_{n-r}\mathcal{T}_{n+r} = \mathcal{T}_n^2 = \mathcal{T}_2^n$.

Proof. The proof is given in [6]. □

To prove the following theorem we need the next lemma.

Lemma 3.8 Let A_5, B_5, C_5 as in Theorem 2.3. Then the following relations hold:

$$A_5B_5 = B_5A_5 = A_5C_5 = C_5A_5 = C_5B_5 = B_5C_5 = (0)\,.$$

Proof. Using $\alpha+\beta+\gamma = 1$, $\alpha\beta+\alpha\gamma+\beta\gamma = -1$ and $\alpha\beta\gamma = 1$, required equalities can be established by matrix calculations. □

We have analogues results for the matrix sequence $\mathcal{G}_n$.

Theorem 3.9 For non-negative integers m, n and r with $n \geq r$, the following identities hold:

(a) $\mathcal{G}_{n-r}\mathcal{G}_{n+r} = \mathcal{G}_n^2$,

(b) $\mathcal{G}_n^m = \mathcal{G}_0^m\mathcal{T}_{mn}$.

Proof.

(a) We use Binet's formula of modified Tribonacci-Lucas matrix sequence which is given in Theorem 2.3. So,

$$\begin{aligned}
& \mathcal{G}_{n-r}\mathcal{G}_{n+r} - \mathcal{G}_n^2 \\
= \; & (A_5\alpha^{n-r} + B_5\beta^{n-r} + C_5\gamma^{n-r})(A_5\alpha^{n+r} + B_5\beta^{n+r} + C_5\gamma^{n+r}) \\
& -(A_5\alpha^n + B_5\beta^n + C_5\gamma^n)^2 \\
= \; & A_5B_5\alpha^{n-r}\beta^{n-r}(\alpha^r - \beta^r)^2 + A_5C_5\alpha^{n-r}\gamma^{n-r}(\alpha^r - \gamma^r)^2 \\
& +B_5C_5\beta^{n-r}\gamma^{n-r}(\beta^r - \gamma^r)^2 \\
= \; & 0
\end{aligned}$$

since $A_5B_5 = A_5C_5 = C_5B_5$ (see Lemma 3.8). Now, we get the result as required.

(b) By Theorem 3.7, we have

$$\mathcal{G}_0^m\mathcal{T}_{mn} = \underbrace{\mathcal{G}_0\mathcal{G}_0\ldots\mathcal{G}_0}_{m \text{ times}}\underbrace{\mathcal{T}_n\mathcal{T}_n\ldots\mathcal{T}_n}_{m \text{ times}}.$$

When we apply Lemma 3.3 (a) iteratively, it follows that

$$\begin{aligned}
\mathcal{G}_0^m\mathcal{T}_{mn} &= (\mathcal{G}_0\mathcal{T}_n)(\mathcal{G}_0\mathcal{T}_n)\ldots(\mathcal{G}_0\mathcal{T}_n) \\
&= \mathcal{G}_n\mathcal{G}_n\ldots\mathcal{G}_n = \mathcal{G}_n^m.
\end{aligned}$$

This completes the proof. □

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