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Homology Group and Generalized Riemann-Roch Theorem

by

Cengiz ULUÇAY

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Homology Group and Generalized Riemann-Roch Theorem

Cengiz ULUÇAY

Dept. of Mathematics, Faculty of Science, Institute of Science, Ankara Üniversity, Ankara

SUMMARY

In this paper we show in particular that if X is a connected complex analytic manifold with fundamental group $F \neq 1$, then the commutator subgroup [F, F] determines complete'y the vector space A(X) of holomorphic functions on X. As a consequence, similarly and generally every normal subgroup D of F such that F/D is commutative determines completely an ideal of A(X).

In this paper we recollect and expand on the results obtained in [1, 2] and deduce thereof the fundamental Theorem. The paper is nevertheless self-contained.

1. **Introduction.** Let X be a connected complex analytic manifold of dimension n with fundamental group $F \neq 1$ and with atlas $\{U_\alpha, z_\alpha)$, $\alpha \in I$, and A(X) the totality of holomorphic functions on X. It is a ring (or C-algebra) or a C-vector space. We denote by A′ (X) a subring of A (X), and by A' the associated restricted subsheaf [1]. It must be understood, however, from the context of the papers [1, 2] that all the restricted sheaves and subsheaves under investigation are analytic, i. e., A-module sheaves, and regular covering spaces of X. Since the base space X is a complex analytic manifold of dimension n with fundamental group $F \neq 1$, and the topology of the regular covering spaces of X is always the induced natural topology of X, it follows that the points of theses spaces may be represented by convergent power series derived from A(X). Accordingly a regular covering space of X corresponding to the normal subgroup $D \subset F$ such that F/D is commutative is just an ideal sheaf A' with the group of cover transformations isomorphic to F/D. Hence

(1) $$A'(X) \cong F/D.$$

In particular the restricted analytic sheaf A determined by the totality of holomorphic functions A(X) on X, is, as regular covering space of X, maximal, while its fundamental group [F, F] is the smallest normal subgroup of F such that F / [F, F] is commutative. Now, F / [F, F] is isomorphic to the group of cover transformations of A which in turn is isomorphic to the group of sections Γ (X, A) and therefore to A(X). In conclusion (1) reads

$$A(X) \cong F/[F, F]. \tag{2}$$

Here F / [F, F] is the homology group. In the recent fundamental paper [2] we have shown among others that if in particular X is a compact Riemann surface with structure restricted sheaf A and genus g, then (2) reads

$$\dim_c A(X) = g. \tag{3}$$

It is then a routine matter to demonstrate the famous duality theorem of Riemann-Roch, say along the lines such as developed in [3], in which however the structure sheaf $\mathcal{O}$ should be replaced by the restricted structure sheaf A introduced by the Author.

It becomes now clear that (1) or (2) may be refered to as the generalized Riemann-Roch Theorem.

In this paper we aim at proving the fundamental Theorem, i. e., the converse of (2).

Namely, given a connected complex analytic manifold X with fundamental group F $\neq$ 1, determine completely the vector space A(X) of holomorphic functions on X.

The converse of (1) or the general theorem that D determines completely an ideal of A(X) is just a consequence of the fundamental Theorem.

2. **Converse Theorem.** We now restate and prove the following

Fundamental Theorem. Let X be a connected complex analytic manifold of dimension n with fundamental group F $\neq$ 1. Then [F, F] determines completely the vector space A(X) of holomorphic functions on X. Furthermore, A(X) and [F, F] are once more connected by (2).

Proof. We consider the commutator subgroup $[F, F]$ of F. It is the smallest normal subgroup of F so that the quotient group $F/[F, F]$ is commutative.

Since $[F, F]$ is a normal subgroup then to $[F, F]$ there corresponds a regular covering space $\tilde{A}$ of X. $\tilde{A}$ has the following properties:

1. The fundamental group $\tilde{F}$ of $\tilde{A}$ projects onto and is isomorphic to $[F, F]$.

2. The group $\tilde{T}$ of covering transformations is isomorphic to the factor group $F/[F, F]$.

3. Each covering transformation of the covering space $\tilde{A}$ is a topological mapping of $\tilde{A}$ onto itself such that each point remains above its base point x and only points lying above one and the same base point are interchanged with one another. Namely, if π is the projection map of $\tilde{A}$ into X then $\pi^{-1}(x)$ is preserved under every transformation of $\tilde{T}$. $\pi^{-1}(x)$ is called the fibre over x.

4. If $Y \subset X$ is the subspace with the fundamental group $[F, F]$, $\pi(\tilde{A}) = Y$, with respect to a fixed initial point $O \in X$, then $\pi^{-1}(Y)$ is a disjoint union of open sets s_i in $\tilde{A}$, each of which is mapped homeomorphically onto Y by π . Y is thus evenly covered, and the s_i are called sheets over Y. Since this is true for any fixed initial point, it follows that every open set in X is evenly covered.

5. The group $\tilde{T}$ is additive and interchanges the s_i with one another. We have

$$\tilde{T} \cong \Gamma(Y, \tilde{A}) \cong F/[F, F].$$

Here $\Gamma(Y, \tilde{A})$ is the set of sheets s_i over Y in $\tilde{A}$.

Furtheremore, for each base point $x \in X$, $\pi^{-1}(x) \cong \tilde{T}$. Accordingly all the fibres are isomorphic with each other.

6. In view of property 4. to each point $\sigma \in s_i$ is associated a corresponding local parameter z_α and σ may be represented by a convergent power series in z_α. These power series are however subject to the property stated in 5.

In conclusion, each sheet s_i is obtained by holomorphic extension of the power series σ and so defines a holomorphic function on X as follows: There exists a holomorphic function f say on U_α such that the series σ converges uniformly to f in U_α. Let σ' be another point on s_i. Similarly, there exists a holomorphic function f' say on U_β such that the series σ' converges uniformly to f' in U_β. Now, in view of property 4. since s_i is a homeomorphic mapping, if $U_\alpha \cap U_\beta \neq \varnothing$ then on the intersection $f = f'$, and so s_i defines on Y a holomorphic function f by holomorphic extension. Since O is arbitary it follows that f is holomorphic on every such open set Y, and therefore on every open set in X. Hence f is holomorphic on X. If we consider the totality of power series σ on a fixed fibre, then in view of property 5, this fibre generates $\tilde{A}$ and also A (X). For, conversely if f is holomorphic on X, then f is in particular holomorphic say at the point x and therefore in a neighborhood of x. We may assume without loss of generality that f is holomorphic on a parametric disc U_α and that U_α contains x. Thus $f \mid U_\alpha$ determines a convergent power series with respect to the local parameter z_α. In view of property 4, to $f \mid U_\alpha$ corresponds on the fibre π^{-1} (x) a unique point $\sigma \in \tilde{A}$ which determines at the same time the sheet s_i containing σ. The identity of $\tilde{A}$ and A is thus complete, i.e., the fibres are the stalks, the sheets are the sections, and the power series are the germs of holomorphic functions on X. The fundamental theorem is thus completely proved.

Finally, if D with F/D commutative is given, the same process will yield a regular covering space I with fundamental group isomorphic to D and whose group of covering transformations is isomorphic to F/D. Thus if we choose on the fixed fibre a submodule with a group isomorphic to F/D, then this ideal will generate likewise the ideal sheaf I with fundamental group isomorphic to D.

Of course, the covering spaces are n-dimensional complex analytic manifolds with projection maps holomorphic. Also,

$$I \subset A \Rightarrow \pi^* = \pi \mid I \colon I \to X,\ \pi^* (I) \subset Y = \pi(A).$$

ÖZET

Bu makalede bilhassa gösteriliyor ki, eğer X, Esas grubu F $\neq$ 1 olan irtibatlı n-boyutlu bir kompleks analitik manifold ise bu takdirde [F, F], X üzerinde bütün holomorf fonksiyonları tamamiyle belirtir.

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