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On Almost-Continuity And Almost-A Continuity Of Real Functions

by

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On Almost-Continuity And Almost-A Continuity Of Real Functions

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SUMMARY

The purpose of this note is to give some new concepts of continuity for real functions and to investigate the relations between concepts of continuity.

1. INTRODUCTION

Let $A = (a_{nk})$ be an infinite matrix of real numbers and $x = (x_k)$ be a sequence of real numbers. The sequence $((Ax)_n)$ defined by

$$(Ax)_n = \sum_{k=1}^{\infty} a_{nk} x_k \qquad (1)$$

is called the A-transform of x whenever the above series converges for $n = 1, 2, \ldots$ .The sequence x is said to be A-summable to x_0 if the sequence $((Ax)_n)$ converges to x_0. A is called conservative if $x \in c$ implies $((Ax)_n \in c$, where c is the linear space of convergent sequences. A is called regular if it is conservative and preserves the limit of each convergent sequence. A is called strongly regular if A is regular and

$$\lim_n \sum_{k=1}^{\infty} | a_{nk} - a_{n,k+1} | = o \qquad (2)$$

[3]. Throughout this study R stands for real numbers and N denotes the set of positive integras.

2. Definitions.

Let m denote the linear space of bounded sequences.

A sequence $x \in m$ is said to be almost convergent and s is called its generalized limit if each Banach limit of x is s [3]. The class F of almost convergent sequences was characterized by G. G. Lorentz [3], who proved that a sequence $x = (x_k)$ is almost convergent if and only if

$$\lim_p \frac{x_n + x_{n+1} + \dots + x_{n+p-1}}{p} = s \tag{3}$$

uniformly in n. We shall write F-lim $x = s$ or Lim $x = s$, shortly. We denote by Lx the following sequence

$$\left(\frac{1}{p} \sum_{j=n}^{n+p-1} x_j\right).$$

If the method A sums all almost convergent sequences then A is called strongly regular [3]. It is clear that a convergent sequence is almost convergent and its limit and generalized limit are identical.

We shall now speak of some basic concepts. Let X, Y be topological spaces. Then $f: X \to Y$ is called continuous on X if and only if the inverse image of every open set in Y is open in X and f is called sequentially continuous at a point $x_0 \in X$ if and only if for every sequence $x_n \to x_0$ (in X) we have $f(x_n) \to f(x_0)$ (in Y). It is known that if $f: X \to Y$ is continuous on X, then f is sequentially continuous on X, but not conversely in general. Furthermore, if X, Y are metric spaces, then the sequentially continuity on X implies continuity on X [4]. Thus the concepts of sequential continuity and continuity coincide for R, since R is a metric space with the usual modulus metric.

A function $f: R \to R$ is called c-continuous at the point $x_0 \in R$ if $(c, 1) - \lim f(x_n) = f(x_0)$ whenever $(c, 1) - \lim x_n = x_0$ [6], where (c,1) is the first Cesàro mean and $(c, 1)$ - lim $x_n = x_0$ means that

$$\frac{x_1 + x_2 + \dots + x_n}{n} \to x_0 \ (n \to \infty) \tag{4}$$

Similarly, A-continuity of f was defined by Jozef Antoni-Tibor Salat [1].

We shal give some new additional definitions:

Definition (2.1). Let $x = (x_n)$ be a sequence in R. We shall say that a function $f: R \to R$ is almost continuous at the point $x_0 \in R$ if F-lim $(f(x)) = f(x_0)$ whenever F-lim $x = x_0$.

Definition (2.2). Let $A = (a_{nk})$ be a regular matrix of real numbers and $x = (x_n)$ be a sequence in R. We shall say that a function $f: R \mapsto R$ is A-almost continuous at $x_0 \in R$ if A-lim $(Lf(x)) = f(x_0)$ whenever A-lim $(Lx) = x_0$.

Definition (2.3). Let the matrix $A = (a_{nk})$ and the sequence $x = (x_n)$ be as the definition (2.2). We shall say that a function $f: R \mapsto R$ is almost A-continuous at $x_0 \in R$ if F-lim $(A(f(x))) = f(x_0)$ whenever F-lim $(Ax) = x_0$.

In the case of A is a unit matrix the definitions (2.2) and (2.3) are equivalent.

3. Relations between the concepts of continuity.

Theorem (3.1). If a function $f: R \mapsto R$ is A-almost continuous at $x_0 \in R$ then f is almost continuous at the same point.

Proof. Let $x = (x_n)$ be a sequence in R such that Lx converges to x_0. Since f is A-almost continuous at $x_0 \in R$

A-lim $(Lx) = x_0$ implies A-lim $(Lf(x)) = f(x_0)$,
and so,

Lim $x = x_0$ implies A-lim $(Lx) = x_0$ implies A-lim $(Lf(x)) = f(x_0)$.
Hence,

Lim $x = x_0$ implies A-lim $(Lf(x)) = f(x_0)$,

that is, we have A-lim $(Lf(x)) = f(x_0)$ for very sequence Lx converging to x_0. On the other hand, every subsequence of Lx converges to x_0 since Lx converges to x_0. It is easy to see that to each subsequence of Lf (x) there corresponds a subsequence of Lx which is convergent to x_0. Therefore, A sums every subsequence of Lf(x). Hence the sequence Lf(x) is convergent [2]. Moreover the sequence Lf(x) must converge to $f(x_0)$ since A is regular and A-lim $(Lf(x)) = f(x_0)$ This completes the proof.

Theorem (3.2). Let $f: R \mapsto R$ be an almost continuous function at $x_0 \in R$. Then f is continuous at x_0 if and only if

$$f(x_{n+1}) - f(x_n) \mapsto o \ (n \mapsto \infty) \tag{5}$$

for each sequence $x = (x_n)$ converging to x_o.

Proof. Necessity. Let f be continuous at $x_o \in R$. Then.

$x_n \mapsto x_o$ $(n \mapsto \infty)$ implies $f(x_n) \mapsto f(x_o)$ $(n \mapsto \infty)$. Hence, for every number $\varepsilon > o$ there exists $n_o(\varepsilon)$ such that

$$| f(x_n) - f(x_o) | < \frac{\varepsilon}{2}$$

for each $n > n_o(\varepsilon)$. Therefore, for $n > n_o(\varepsilon)$ we have

$$| f(x_{n+1}) - f(x_n) | \leq | f(x_{n+1}) - f(x_o) | + | f(x_n) - f(x_o) | < \varepsilon.$$

Sufficiency. Let the sequence $x = (x_n)$ converge to x_o and f be an almost continuous function at $x_o \in R$. Then for any number $\varepsilon > o$, we can choose a number p large enough such that

$$| \frac{1}{p} (f(x_n) + f(x_{n+1}) + \ldots + f(x_{n+p-1})) - f(x_o) | < \frac{\varepsilon}{2} \quad (6)$$

for all $n \in N$.

Let us tak $\varepsilon_1 = \frac{\varepsilon}{p-1}$, $(p > 1)$. By (5), for the number $\varepsilon_1 > o$ we select a number n_o so large that

$$| f(x_n) - f(x_{n+1}) | < \varepsilon_1$$

for all $n > n_o$. Therefore, for $n > n_o$ we get

$$| f(x_n) - f(x_{n+p-1}) | \leq (p-1)\varepsilon_1. \quad (7)$$

Let max $(n_o, p) = M$. By (6) and (7) ,for $n > M$ we have

$$| f(x_n) - f(x_o) | \leq | f(x_n) - \frac{f(x_n) + \ldots + f(x_{n+p-1})}{p} |$$

$$+ | \frac{f(x_n) + \ldots + f(x_{n+p-1})}{p} - f(x_o) |$$

$$\leq \frac{1}{p} | p\, f(x_n) - \sum_{j=n}^{n+p-1} f(x_j) | + \frac{\varepsilon}{2}$$

$$\leq \frac{1}{p} \sum_{j=n}^{n+p-1} | f(x_n) - f(x_j) | + \frac{\varepsilon}{2}$$

$$\leq \frac{1}{p} (1 + 2 + \ldots + (p - 1)) \varepsilon_1 + \frac{\varepsilon}{2} = \varepsilon$$

This completes the proof.

In a recent paper, we have defined the new methods of summability by a suitable rearrangement of the elements on each row of a given matrix summability method [5]. In connection with this we can give the following:

Theorem (3.3). Let $A = (a_{nk})$ be a strongly regular matrix and $f: R \mapsto R$ be a function such that the sequence $f(x)$ is bounded whenever $x = (x_k)$ is bounded. Then the concepts of the A-continuity and the $A\pi$ –continuity corresponding to those permutation functions each of which has a symmetrical mapping (see, definition in [5]) on disjoint blocks of the positive integers, are equivalent.

Proof. We showed in theorem 2.1 [5] that for every bounded sequence $x = (x_k)$ we have

$$\lim_n |(Ax)_n - (A\pi x)_n| = o. \qquad (8)$$

Let $A\pi$ -lim $x_n = x_o$ and f be A-continuous at $x_o \in R$. We shall show that $A\pi$ -lim $f(x_n) = f(x_o)$. Since f is A-continuous at $x_o \in R$ we have

$$\text{A-lim } x_n = x_o \text{ implies A-lim } f(x_n) = f(x_o).$$

By (8) and since A-lim $f(x_n) = f(x_o)$, we get $A\pi$ -lim $f(x_n) = f(x_o)$. Hence, f is A -continuous at $x_o \in R$. In the same way one can prove that f is A-continuous at $x_o \in R$ if the function f is $A\pi$ -continuous at $x_o \in R$. This completes the proof.

ÖZET

Bu makalede, reel funksiyonlar için bazı yeni süreklilik kavramları tarif edilmekte ve bu süreklilik kavramları arasındaki bağıntılar incelenmektedir.

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