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ON THE SHEAF OF THE FUNDAMENTAL GROUPS.

by

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ON THE SHEAF OF THE FUNDAMENTAL GROUPS.

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SUMMARY

In a recent paper [1] we have constructed the sheaf of the fundamental groups of a topological space and gave some characterizations. In this paper, we first give some characterizations which are the converses of the characterizations in paper [1]. Finally, we obtain some results.

1. INTRODUCTION

Let X be a locally arcwise connected topological space and π_1 (X, x) be the fundamental group at x for any point $x \in X$. Then the disjoint union $H = \bigvee_{x \in X} \pi_1(X, x)$ is a set over X with natural projection $\varphi: H \to X$ mapping each $\sigma_x = [\alpha]_x$ onto the base point x.

We introduced on H a natural topology as follows [4, 5]: Let $x \in X$ be an arbitrary fixed point. Then there exists an arcwise connected open neighborhood $U = U(x)$. If $[\alpha]_x \in \pi_1(X, x)$ is an arbitrary fixed element and $y \in W$ is any point, then there exists an element $[\beta]_y \in \pi_1(X, y)$ which uniquely corresponds to $[\alpha]_x$, since $\pi_1(X, x) \cong \pi_1(X, y)$. Therefore we can define a mapping $s: U \to H$ with $s(y) = [\beta]_y$ for any $y \in W$ such that $\varphi \circ s = 1_U$ and $s(x) = [\alpha]_x$ $\in s(U) \subset H$.

For each $x \in X$ all such sets s(U) form a system of neighborhoods of $[\alpha]_x \in H$ which induces a topology in H.

In this topology s is continuous and φ is a locally topological mapping. s is called a section over U and the totality of section over U is denoted by $\Gamma(U, H)$. For the definition of a section over any open set $W \subset X$ see [1].

In this paper, we assume that topological spaces are locally arc-wise connected.

2. CHARACTERIZATIONS.

Now, we give the following theorem.

Theorem 1. Let (H, φ) be a sheaf over X, $W \subset X$ an open set and $s \in \Gamma(W, H)$. Then φ: $s(W) \rightarrow W$ is a topological mapping and $s = (\pi \mid s(W))^{-1}$.

Proof. If we consider the statement $\varphi os = 1_W$, then $(so(\pi \mid s(W) os)(x) = so(\pi os)(x) = s(x)$, for each $x \in W$.

Definitions 1. Let X_1, X_2 be topological spaces and H_1, H_2 be corresponding sheaves. Then a mapping f^*: $H_1 \rightarrow H_2$ is called stalk preserving, if for each stalk $(H_1)_{x_1} \subset H_1$ there exists a stalk $(H_2)_{x_2}$ such that $f^*((H_1)_{x_1}) \subset (H_2)_{x_2}$ [2].

2. Let f^*: $H_1 \rightarrow H_2$ be a stalk preserving and continuous mapping. Then the mapping f^* is called a sheaf morphism between the sheaves H_1 and H_2.

3. Let f^*: $H_1 \rightarrow H_2$ be a sheaf morphism. If f^* is a homomorphism on each stalk, then it is called a sheaf homomorphism between the sheaves H_1 and H_2.

4. Let f^*: $H_1 \rightarrow H_2$ be a sheaf homomorphism. If f^* is a homemorphism, then it is called a sheaf isomorphism between the sheaves H_1 and H_2.

For the definitions which are not giving in this paper see [1]. We can now give the following theorems.

Theorem 2. Let the parirs (X_1,H_1) and (X_2,H_2) be given. If the mapping f^*: $H_1 \rightarrow H_2$ is given as a sheaf homomorphism, then there exists a unique continuous mapping f: $X_1 \rightarrow X_2$ such that the pair (f, f^*) is a homomorphism between the pairs (X_1, H_1) and (X_2, H_2).

Proof. To prove this theorem we must first find a mapping f from X_1 into X_2. However, for each $(H_1)_{x_1} \subset H_1$ there exists a stalk $(H_2)_{x_2} \subset H_2 \ni f^*((H_1)_{x_1}) \subset (H_2)_{x_2}$, since f^* is stalk preserving. Therefore, to any point $x_1 \in X_1$ there uniquely corresponds a point $x_2 \in X_2$. If we denote this correspondence by $f(x_1) = x_2$, then we obtain a mapping f from X_1 into X_2.

Let us now show that the mapping f is continuous. Let $W \subset f(X_1)$ be an open set. We may be prove that the set $f^{-1}(W)$ is an open in X_1. Since W is an open in X_2, then there exists the arcwise connected open sets W_i in X_2, $i \in I$, such that $W = \bigcup_{i \in I} W_i$. Thus $s^2(W) = \bigcup_{i \in I} s_i^2(W_i)$ is an open set in H_2, for a section $s^2 \in \Gamma(W,H_2)$. However, $f^{*-1}(s^2(W)) = \bigcup_{i \in I} f^{*-1}(s_i^2(W_i))$ is an open set in H_1, since f^* is continuous. Thus there exists the arcwise connected open sets V_i in X_1, $i \in I$, such that $f^{*-1}(s^2(W)) = \bigcup_{i \in I} s_i^1(V_i)$, where s_i^1's are sections over V_i, for each $i \in I$. Hence $\varphi_1(f^{*-1}(s^2(W))) = \bigcup_{i \in I} V_i$ is an open set in X_1. Let us now show that $f^{-1}(W) = \bigcup_{i \in I} V_i$.

1. Let $x_1 \in f^{-1}(W)$. Then there exists only one point $x_2 \in X_2 \ni f(x_1) = x_2$. Hence $s^2(x_2) = \sigma_{x_2} \in s(W) = \bigcup_{i \in I} s_i^2(W_i)$ and there is an element $\sigma_{x_1} \in f^{*-1}(s^2(W)) \in f^*(\sigma_{x_1}) = \sigma_{x_2}$. $\sigma_{x_1} \in s_i^1(V_i)$, for an $i \in I$, since $f^{*-1}(s^2(W)) = \bigcup_{i \in I} s_i^1(V_i)$. Hence $\varphi_1(\sigma_{x_1}) = x_1 \in V_i$. Therefore $f^{-1}(W) \subset \bigcup_{i \in I} V_i$.

2. Let $x_1 \in \bigcup_{i \in I} V_i$. Then $x_1 \in V_i$ and $s_i^1(x_1) \in (H_1)_{x_1}$, for an $i \in I$. From here $f^*(s_i^1(x_1))_{x_2} \in s(W)$ and $\varphi_2(f^*(s_i^1(x_2))_{x_2} = x_2 \in W$. From the definition of f, $f(x_1) = x_2$. Therefore $x_1 \in f^{-1}(W)$. Thus $\bigcup_{i \in I} V_i \subset f^{-1}(W)$.

From (1) and (2) it is obtained that $f^{-1}(W) = \bigcup_{i \in I} V_i$

Thus the mapping $f: X_1 \to X_2$ is continuous. On the other hand, it can be shown that the pair (f, f^*) is a homomorphism between the pairs (X_1, H_1) and (X_2, H_2) [1], and f is unique, since $f o \varphi_1 = \varphi_2 o f^*$.

Now, we can give the following theorem.

Theorem 3. Let the pairs (X_1, H_1), (X_2, H_2) and (X_3, H_3) be given. If the mappings $f_1^*: H_1 \to H_2$ and $f_2^*: H_2 \to H_3$ are sheaf homomrophisms, then there exist a homomorphism between the pairs (X_1, H_1) and (X_3, H_3) such that $f = f_2 \circ f_1$, $f^* = f_2^* \circ f_1^*$.

Proof. Since the mappings f_1^*, f_2^* are continuous, the mappings $f_2^* \circ f_1^*$ is also continuous. By Theorem 2, there is a continuous mapping f from X_1 into X_2. Clearly, $f_2^* \circ f_1^*$ preserves the stalks with respect to f and $f_2^* \circ f_1^*$ is a homomorphism on each stalk. Hence the pair $(f, f_2^* \circ f_1^*)$ is a homomorphism between the pairs (X_1, H_1) and (X_3, H_3). Now, let us show that $f = f_2 \circ f_1$. Since ,for any stalk $(H_1)_{x_1} \subset H_1$, there is a stalk $(H_2)_{x_2} \subset H_2 \ni f_1^* ((H_1)_{x_1} \subset (H_2)_{x_2}$ and for any stalk $(H_2)_{x_2}$, there is a stalk $(H_3)_{x_3} \ni f_2^* ((H_2)_{x_2}) \subset (H_3)_{x_3}$, $(f_2^* \circ f_1^*)((H_1)_{x_1}) = f_2^*(f_1^*((H_1)_{x_1})) \subset f_2^*((H_2)_{x_2}) \subset (H_3)_{x_3}$ and $f(x_1) = x_3$. On the other hand $f_1(x_1) = x_2$ and $f_2(x_2) = x_3$, since $f_1^*((H_1)_{x_1}) \subset (H_2)_{x_2}$ and $f_2^*((H_2)_{x_2}) \subset (H_3)_{x_3}$. So, $(f_2 \circ f_1)(x_1) = f_2(f_1(x_1)) = x_3$. Therefore $f_2 \circ f_1 = f$.

Now, let *C* be the category of the sheaves of the fundamental groups and sheaf homomorphisms and *D* be the category of the topological spaces and continuous mappings. Then, we can define a mapping $T: C \to D$ as follows:

For any sheaf H and every morphism $f^*: H_1 \to H_2$, let $T(H) = X$ and $T(f^*) = f: X_1 \to X_2$.

It is easily shown that,

1. If $H_1 = H_2$ and $f^* = 1_{H_1}$, then $T(1_{H_1}) = 1_{X_1} = T(H_1)$.

2. If $f_1^*: H_1 \to H_2$ and $f_2^*: H_3 \to H_2$ are two sheaf homomorphisms, then $T(f_2^* \circ f_1^*) = T(f_2^*) \circ T(f_1^*)$.

Thus, the mapping: $T: C \to D$ is a covarient functor. Now, we can state the following theorem.

Theorem 4. There is a covarient functor from the category of the sheaves of the fundamental groups and sheaf homomorphisms to the topological spaces and continuous mappings.

Now, we can give the following theorem.

Theorem 5. Let the pairs (X_1, H_1) and (X_2, H_2) be given. If the mapping $f^*: H_1 \to H_2$ is a sheaf isomorphism, then there exists an isomorphism between the pairs (X_1, H_1) and (X_2, H_2).

Proof. It follows from the Theorem 2 that, there exists a continuous mapping f from X_1 into X_2. Let us now show that f is a bijection. In fact, for any two elements $x_1, y_1 \in X_1$, if $f(x_1) = f(y_1) = x_2$, then there is a stalk $(H_2)_{x_2}$, $x_2 \in X_2$, $\ni f^*((H_1)_{x_1}) = f^*((H_1)_{y_1}) = (H_2)_{x_2}$. However, this is impossible, since f^* is one-to-one. Therefore $x_1 = y_1$. On the other hand, for each stalk $(H_2)_{x_2}$, there exists a stalk $(H_1)_{x_1} \ni f^*((H_1)_{x_1}) = (H_2)_{x_2}$ since f^* is onto. It follows from this reason that, for each $x_2 \in X_2$, there exists an element $x_1 \in X_1 \ni f(x_1) = x_2$. Hence f is a bijection. By Theorem 2, there is a continuous mapping $g: X_2 \to X_1$, since f^{*-1} is continuous. It is similarly shown that g is a bijection. On the other hand, it can be shown that $g = f^{-1}$. Therefore f is a homeomorphism.

Clearly, f^{*-1} preserves the stalks with respect to f. Thus, the pair (f, f^*) is an isomorphism.

Let $F = (f, f^*)$ be an isomorphism between the pairs (X_1, H_1) and (X_2, H_2). If $W_1 \subset X_1$ is an open set and $\Gamma(W_1, H_1)$ is the set all of the section over W_1, then $f^* \circ s^1 o f^{-1}: f(W_1) \to H_2$ is a continuous map and $\varphi_2 \circ (f^* o s^1 o f^{-1}) = 1_{f(W_1)}$, for a section $s^1 \in \Gamma(W_1, H_1)$. Hence $f^* o s^1 o f^{-1} \in \Gamma(f(W_1), H_2)$. Therefore we can define a mapping $f_*: \Gamma(W_1, H_1) \to \Gamma(f(W_1), H_2)$ as follows:

If $s^1 \in \Gamma(W_1, H_1)$, then let $f_*(s^1) = f^* o s^1 o f^{-1}$.

f_* is a homomorphism. In fact, if $s^1_1, s^1_2 \in \Gamma(W_1, H_1)$, then

$f_*(s^1_1 . s^1_2) = f^* o(s^1_1 . s^1_2) o f^{-1}$.

On the other hand, for any $x \in f(W_1)$,

$$\begin{aligned}(f^* o (s^1_1 . s^1_2) o f^{-1})(x) &= f^*((s^1_1 . s^1_2)(f^{-1}(x)) \\ &= f^*(s^1_1(f^{-1}(x)) . s^1_2(f^{-1}(x))) \\ &= f^*(s^1_1(f^{-1}(x))) . f^*(s^1_2(f^{-1}(x))) \\ &= ((f^* o s^1_1 o f^{-1}) . (f^* o s^1_2 o f^{-1}))(x).\end{aligned}$$

Therefore $f_*(s^1_1 \cdot s^1_2) = f_*(s^1_1) \cdot f_*(s^1_2)$. Moreover f_* is a bijection. In fact, if $s^1_1, s^1_2 \in \Gamma(W_1, H_1)$ are any two sections and $f_*(s^1_1) = f_*(s^1_2)$, then $f^* o s^1_1 o f^{-1} = f^* o s^1_2 o f^{-1}$ and $s^1_1 = s^1_2$. On the other hand, if $s^2 \in \Gamma(f(W_1), H_2)$, then $f^{*-1} o s^2 o f \in \Gamma(W_1, H_1)$ and $f_*(f^{*-1} o s^2 o f) = s^2$. Therefore f_* is an isomorphism.

Now, we can state the following theorem.

Theorem 6. Let $F = (f, f^*)$ be an isomorphism between the pairs (X_1, H_1), (X_2, H_2) and $W_1 \subset X_1$ be an open set. Then $\Gamma(W_1, H_1)$ is isomorphic to $\Gamma(f(W_1), H_2)$.

Now, let (X, H) be a pair, $W \subset X$ be an open and $F = (1_X, 1_H) = 1_{(X,H)}$. Then $f_* = 1_{\Gamma(W,H)}$. Moreover, if (X_1, H_1), (X_2, H_2) and (X_3, H_3) are any pairs and $(X_1, H_1) \overset{F_1}{\cong} (X_2, H_2)$, $(X_2, H_2) \overset{F_2}{\cong} (X_3, H_3)$, then $(X_1, H_1) \overset{F_2 o F_1}{\cong} (X_3, H_3)$. By Theorem 6, $\Gamma(W_1, H_1) \overset{f_*}{\cong} \Gamma(f_2(f_1(W_1)), H_3)$, for an open $W_1 \subset X_1$. It is shown that $f_* = f_{1*} o f_{2*}$.

Now, let C be the category of pairs and isomorphisms, D be the category of groups and isomorphisms. Then, we can define a map $T: C \to D$ as follows:

If (X_1, H_1) any pair and $F = (f, f^*)$ any isomorphism and $W_1 \subset X_1$ is an open, then let $T((X_1, H_1)) = \Gamma(W_1, H_1)$ and $T(F) = f_*: \Gamma(W_1, H_1) \to \Gamma(f(W_1), H_2)$.

Thus, we can state the following theorem.

Theorem 7. There is a covarient functor from the category of pairs and isomorphisms to the category of groups and isomorphisms.

ÖZET

Bu makalede, daha önceki bir çalışmamızda vermiş olduğumuz bazı karakterizasyon teoremlerinin karşıtları teşkil edilip ispatları verilmiş, daha sonra da bir topolojik uzayın esas gruplarının demetinde kesitler ele alınarak teşkil ettikleri gruba ait bazı özellikler elde edilmiştir.

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