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by

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On The Lie Group of Umbrella Matrices

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ABSTRACT

In this paper, Umbrella matrices are defined which are selected from GL (n, $\mathbb{R}$) and it is shown that Umbrella matrices form a matrix group with respect to the matrix product.

Next, a characterization of Umbrella matrices is given; among these it is proved that the group of Umbrella matices which are selected from O (n) and the group of Doubly Umbrella matrices are subgroup of Umbrella matrices which are selected from GL (n, $\mathbb{R}$). Later it is shown that this matrix group is a Lie subgroup of GL (n, $\mathbb{R}$) and a characteristic property is given about this Lie group.

Finally it is shown that, the Lie group under consideration is not connected and noncompact are shown. At the end of this work, the Maurer-Cartan forms are investigated and using the principal I-forms the dimension of the Lie group under consideration is computed.

I. INTRODUCTION

Umbrella Matrices are first defined in [1] and the related Umbrella Motions are studied in [7]. Later in [13], another definition of Umbrella Matrices is given and the Lie group structures are investigated. In this paper, we define the Umbrella Matrices in GL (n, $\mathbb{R}$) differently from [13]. Next, we show that the Lie group of [13] and the Lie group of Doubly Umbrella Matrices which is investigated in [9] are Lie subgroup of the Lie group of this work.

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II. THE GROUP OF UMBRELLA MATRICES AND A CHARACTERIZATION

We will define Umbrella Matrices as follows;

DEFINITION II. 1: A matrix $A \in GL(n, \mathbb{R})$ is an Umbrella Matrix, if $AS = S$, where $S = [1\ 1\ \ldots\ 1]^t \in \mathbb{R}^n_1$ and $[\ ]^t$ denotes the transpose of a matrix and $GL(n, \mathbb{R})$ is the set of all nxn, nonsingular matrices. The set of Umbrella Matrices will be denoted by $\mathbb{H}(n)$.

Notice that this definition is different from the definitions given in the literature by [1] and [13].

THEOREM II. 1: $\mathbb{H}(n)$ is a subgroup of $GL(n, \mathbb{R})$.

Proof: We will prove this theorem in three steps. First, we will show.

i) $(AB)S = S, \quad \forall A,B \in \mathbb{H}(n)$.

If $A,B \in \mathbb{H}(n)$, then from definition we get $AS = S$, $BS = S$ and $(AB)S = A(BS)$. Hence we get $(AB)S = AS$. Then $(AB)S = S$. Therefore $AB \in \mathbb{H}(n)$, for all $A,B \in \mathbb{H}(n)$. Secondly, we will show

ii) $A^{-1} \in \mathbb{H}(n)$, $\forall A \in \mathbb{H}(n)$ where A^{-1} is the inverse element of A in $\mathbb{H}(n)$. Let $A \in GL(n, \mathbb{R})$. Then $A^{-1} \in GL(n, \mathbb{R})$ and so that $(A^{-1}A)S = A^{-1}(AS) = S$. If $A \in \mathbb{H}(n)$, we can write $(A^{-1}A)S = S$. Hence $A^{-1} \in \mathbb{H}(n)$, for all $A \in \mathbb{H}(n)$. Thirdly, we will show

iii) $AB^{-1} \in \mathbb{H}(n)$, for all $A,B \in \mathbb{H}(n)$.

Let $A,B \in \mathbb{H}(n)$. Then from (i) and (ii), $AB \in \mathbb{H}(n)$ and $B^{-1} \in \mathbb{H}(n)$, for all $A,B \in \mathbb{H}(n)$. Therefore $AB^{-1} \in \mathbb{H}(n)$, for all $A,B \in \mathbb{H}(n)$. Hence $\mathbb{H}(n)$ is a subgroup of $GL(n, \mathbb{R})$.

DEFINITION II. 2: The above group of matrices will be called Umbrella Matrix group. Now, we will prove the following theorem on the characteristic of the Umbrella Matrices.

THEOREM II. 2:

$$A = [a_{ij}] \Leftrightarrow \sum_{j=1}^{n} a_{ij} = 1,\ \det A \neq 0,\ 1 \leq i, j \leq n; \text{ for all } A \in \mathbb{H}(n).$$

Proof: ($\Rightarrow$): Let $A \in \mathbb{H}(n)$. Then $AS = S$. Therefore, we can write

$$\begin{bmatrix} a_{11} a_{12} & \dots & a_{1n} \\ a_{21} a_{22} & \dots & a_{2n} \\ \vdots & & \vdots \\ a_{n1} a_{n2} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \sum_{j=1}^{n} a_{1j} \\ \sum_{j=1}^{n} a_{2j} \\ \vdots \\ \sum_{j=1}^{n} a_{nj} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \Rightarrow \sum_{j=1}^{n} a_{ij} = 1, \quad 1 \leq i \leq n.$$

($\Rightarrow$): Let $A = [a_{ij}] \in \mathbb{R}^n_n$ and $\det A = 0$. If $\sum_{j=1}^{n} a_{ij} = 1, 1 \leq i \leq n$,

we can write

$$\begin{bmatrix} \sum_{j=1}^{n} a_{1j} \\ \sum_{j=1}^{n} a_{2j} \\ \vdots \\ \sum_{j=1}^{n} a_{nj} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} a_{11} a_{12} & \dots & a_{1n} \\ a_{21} a_{22} & \dots & a_{2n} \\ \vdots & & \vdots \\ a_{n1} a_{n2} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

$\Rightarrow AS = S$.

Consequently, $A \in \mathbb{H}(n)$.

III. THE LIE GROUP OF UMBRELLA MATRICES AND A CHARACTERIZATION

In this section, we will show that the group of Umbrella Matrices is a matrix Lie group and later, we will give a characterization on this Lie group.

THEOREM III. 1 : The group of Umbrella Matrices is a Lie subgroup of GL $(n, \mathbb{R})$

Proof: Let $\{x_{11}, x_{21}, \ldots, x_{n1}, \ldots, x_{nn}\}$ be a local coordinate system on $\mathbb{R}^{n^2}$ and S_n is the set of permutations which has n! elements. Now, let us define the following differentiable functions for all $y \in \mathbb{R}^{n^2}$.

$$f_i: \mathbb{R}^{n^2} \to \mathbb{R},\ f_i(y) = \sum_{j=1}^{n} x_{ij}(y) - 1,\ 1 \leq i \leq n \tag{1}$$

and

$$f_{ij}: \mathbb{R}^{n^2} \to \mathbb{R}, f_{ij}(y) = \sum_{\sigma \in S_n} (\mathrm{sgn}\sigma)\, x_{\sigma(1)1}(y) \ldots x_{\sigma(n)n}(y) = \det[x_{ij}(y)], \tag{2}$$

$\det [x_{ij}(y)] \neq 0$, $1 \leq i,j \leq n$ where $(\mathrm{sgn}\sigma)$ denotes signature of $\sigma \in S_n$. Then

$$M = \{y \in \mathbb{R}^{n^2} \mid f_i(y) = 0,\ f_{ij}(y) = 0,\ 1 \leq i,j \leq n\}$$

is a differentiable manifold in $\mathbb{R}^{n^2}$ [3].

However, we will prove this theorem in two steps: First, we will show

i) $\mathbb{H}(n)$ is a differentiable manifold.

Let us define a function Ψ as $\Psi: \mathbb{H}(n) \to M$, where M is a submanifold of $\mathbb{R}^{n^2}$ as above such that let

$\Psi([a_{ij}]) = (b_1, b_2, \ldots, b_{n^2})$, with $a_{ij} = b_{i+(j-1)n}$, $A = [a_{ij}]$

$1 \leq i,j \leq n$. Hence we can find coordinate functions of $\mathbb{H}(n)$ as follows:

$$\begin{array}{ccc} \mathbb{H}(n) & \longrightarrow & \mathbb{R}^{n^2} \\ & \searrow & \downarrow \quad x_{ij},\ 1 \leq i,j \leq n. \\ \Psi_{ij} = x_{ij} \circ \Psi & & \mathbb{R} \end{array} \tag{3}$$

Then from (1) and (3) we get

$$f_i(\Psi([a_{ij}])) = \sum_{j=1}^{n} x_{ij}(\Psi([a_{ij}])) - 1,\ 1 \leq i \leq n$$

$$= \sum_{j=1}^{n} (x_{ij} o \Psi)([a_{ij}]) - 1$$

$$= \sum_{j=1}^{n} \Psi_{ij}([a_{ij}]) - 1. \tag{4}$$

So that we can write $\sum_{j=1}^{n} \Psi_{ij}(a_{ij}) = 1,\ 1 \leq i,j \leq n,$ since $A = [a_{ij}] \in \mathbb{H}(n)$ and Ψ_{ij} are coordinate functions on $\mathbb{H}(n)$. From (4) we get $f_i(\Psi([a_{ij}])) = 0,\ 1 \leq i,j \leq n$. Similarly, from (2) we get $f_{ij}(\Psi([a_{ij}])) = 0,\ 1 \leq i,j \leq n$. Then $\forall\ \Psi([a_{ij}]) \in \mathbb{R}^{n^2} \Rightarrow \Psi([a_{ij}]) \in M$. So $\Psi(\mathbb{H}(n))$ is a subset of M since $\forall\ \Psi([a_{ij}]) \in \Psi(\mathbb{H}(n))$; that is

$$\Psi(\mathbb{H}(n)) \subset M.$$

Also, for all $y \in M$

$$f_i(y) = 0 \Rightarrow \sum_{j=1}^{n} x_{ij}(y) = 1$$

and

$$f_{ij}(y) = 0 \Rightarrow \sum_{\sigma \in S_n} (\operatorname{sgn}\sigma)\, x_{\sigma(1)1}(y) \ldots x_{\sigma(n)n}(y) = \det [x_{ij}(y)].$$

Therefore $\forall\ y \in M \Rightarrow y \in \mathbb{R}^{n^2} \Rightarrow y = (b_1, b_2, \ldots, b_{n^2})$

$$\Rightarrow y = \Psi([a_{ij}]),\ [a_{ij}] \in \mathbb{H}(n)$$

$\Rightarrow y \to (\mathbb{H}(n))$. Then

$$M \subset \Psi(\mathbb{H}(n)). \tag{6}$$

From (5) and (6) we get $M = \Psi(\mathbb{H}(n))$. Now, we can show that Ψ is a diffeomorphism from $\mathbb{H}(n)$ to $M \subset \mathbb{R}^{n^2}$ and therefore $\mathbb{H}(n)$ is isomorphic to M. In fact, for all $A,B \in \mathbb{H}(n)$

$$\Psi(A) = \Psi(B) \Leftrightarrow (a_1, a_2, \ldots, a_{n^2}) = (b_1, b_2, \ldots, b_{n^2})$$

$$\Leftrightarrow a_i = b_i,\ 1 \leq i \leq n^2$$

$$\Leftrightarrow A = B.$$

Hence Ψ is one-to-one. Also $\forall\ (b_1, b_2, \ldots, b_{n^2}) \in M \subset \mathbb{R}^{n^2}\ \exists\ A = [a_{ij}] \in \mathbb{H}(n)$ such that $\Psi(A) = b_1, b_2, \ldots, b_{n^2})$ with $a_{ij} = b_{i+(j-1)n}$, $1 \leq i,j \leq n$. So that Ψ is an isomorphism from $\mathbb{H}(n)$ to M.

On the other hand, $(b_1, b_2, \ldots, b_{n^2})$ are Euclidean coordinate functions on $\mathbb{R}^{n^2}$, they are differentiable. This show that Ψ is a differentiable function, so that $\mathbb{H}(n)$ is differentiable and also diffeomorphic to M. Now, let H_1, H_2 be hyperplanes defined as follows:

$$H_1 = \{y \in \mathbb{R}^{n^2} \mid f_i(y) = 0, \ 1 \leq i \leq n\}$$

and

$$H_2 = \{y \in \mathbb{R}^{n^2} \mid f_{ij}(y) = 0, \ 1 \leq i,j \leq n\}.$$

Then we can write $GL(n, \mathbb{R}) \cap \left(\bigcap_{i=1}^{2} H_i \right) = M$. Also, since $M = \to (\mathbb{H}(n))$ and $\mathbb{H}(n)$ is diffeomorphic to M, we get $GL(n, \mathbb{R}) \cap \left(\bigcap_{i=1}^{2} H_i \right) = \mathbb{H}(n)$.

Hence $\mathbb{H}(n)$ is a differentiable manifold. Secondly, we will show ii) $(A,B) \to AB^{-1}$ is differentiable for all $A,B \in \mathbb{H}(n)$.

Since the group of Umbrella Matrices is a subgroup of $GL(n, \mathbb{R})$ and $GL(n, \mathbb{R})$ is a Lie group, the map $(A,B) \to AB^{-1}$ is differentiable, for all $A,B \in \mathbb{H}(n)$. Consequently, from steps (i) and (ii), $\mathbb{H}(n)$ is a Lie subgroup of $GL(n, \mathbb{R})$.

DEFINITION III. 1: The above Lie group of matrices will be called Umbrella Matrices Lie group and will be denoted by $\mathbb{H}(n)$.

Now, we will give a characterization on the Lie group of Umbrella Matrices.

THEOREM III. 2: Let $\{x_{11}, x_{21}, \ldots, x_{n1}, \ldots, x_{nn}\}$ be a local coordinate system on $GL(n, \mathbb{R})$ and let $X \in \mathfrak{X}(GL(n, \mathbb{R}))$ be the vector field with

$$X = \sum_{i,j=1}^{n} \xi_{ij} \frac{\partial}{\partial x_{ij}}$$ such that $\xi_{ij} \in C^\infty(GL(n, \mathbb{R}), \mathbb{R})$. Then

$$X \in \mathfrak{X}(\mathbb{H}(n)) \Leftrightarrow \sum_{j=1}^{n} \xi_{ij} \Big|_g = 0,$$ for all $g \in \mathbb{H}(n)$ where

$\mathfrak{X}$ (GL (n, $\mathbb{R}$)) denotes the space of vector field on GL (n, $\mathbb{R}$).

Proof: Let $X \in \mathfrak{X}(\mathbb{H}(n))$, then we can find a one-parameter group $\{\Psi_t | t \in \mathbb{R}\}$ which induces a vector field X on $\mathbb{H}(n)$, [10]. So that we can write

$$x_{ij}(\Psi_t(g)) = \Psi_{ij}(t, g). \ 1 \leq i,j \leq n \tag{7}$$

and

$$\frac{d\Psi_{ij}}{dt} \Big|_{(t,g)} = \xi_{ij}(\Psi_{11}, \Psi_{21}, \ldots, \Psi_{nn}) \Big|_{(t,g)}.$$

If $[\Psi_{ij}(t, g)] \in \mathbb{H}(n)$ then we can write

$$\sum_{j=1}^{n} \Psi_{ij}(t, g) = 1, \ 1 \leq i \leq n. \tag{8}$$

From (8)

$$\sum_{j=1}^{n} \frac{d\Psi_{ij}}{dt} \Bigg|_{(t,g)} = 0 \Rightarrow \sum_{j=1}^{n} \frac{d\Psi_{ij}}{dt} \Bigg|_{(0,g)} = 0. \tag{9}$$

On the other hand

$$\frac{d\Psi_{ij}}{dt} \Bigg|_{(0,g)} = \xi_{ij}(\Psi_{11}, \Psi_{21}, \ldots, \Psi_{nn}) \Bigg|_{(0,g)}.$$

From (7)

$$\begin{aligned} \frac{d\Psi_{ij}}{dt} \Big|_{(0,g)} &= \xi_{ij}(x_{11}o\Psi, x_{21}o\Psi, \ldots, x_{nn}o\Psi) \Big|_{(0,g)} \\ &= \xi_{ij}(x_{11}(\Psi_0(g)), x_{21}(\Psi_0(g)), \ldots, x_{nn}(\Psi_0(g)) \\ &= \xi_{ij}(x_{11}(g), x_{21}(g), \ldots, x_{nn}(g)) \\ &= \xi_{ij} \big|_g. \end{aligned} \tag{10}$$

From (9) and (10) we get

$$\sum_{j=1}^{n} \xi_{ij} \big|_g = 0, 1 \leq i \leq n. \tag{11}$$

($\Rightarrow$): For all $g \in \mathbb{H}(n)$

$$\sum_{j=1}^{n} \xi_{ij}\Big|_{g} = 0 \Rightarrow \sum_{j=1}^{n} \frac{d\Psi_{ij}}{dt}\Big|_{(0,g)} = 0$$

$$\Rightarrow \sum_{j=1}^{n} \Psi_{ij}(t,g) = c_1,\ c_1 \in \mathbb{R},$$

$$\Rightarrow \sum_{j=1}^{n} \Psi_{ij}(0,g) = c_1,\ t = 0 \in \mathbb{R},$$

$$\Rightarrow \sum_{j=1}^{n} g_{ij} = c_1. \tag{12}$$

From (12) and since $\sum_{j=1}^{n} g_{ij} = 1$ for all $g \in \mathbb{H}(n)$, we get

$$\sum_{j=1}^{n} \Psi_{ij}(t,g) = 1. \tag{13}$$

Consequently, from (13) $\Psi(t,g) \in \mathbb{H}(n)$. Therefore we can find a vector field such that $x \in \mathfrak{X}^{\Psi}(\mathbb{H}(n))$.

IV. THE LIE SUBGROUPS OF $\mathbb{H}(n)$

In this section, we will state definitions and theorems from [13] and [9].

DEFINITION IV. 1: The set of Umbrella Matrices which are chosen from 0(n) is denoted by A(n) and they are defined in [13] by

$$A(n) = \{A \in O(n) \mid AS = S,\ S = [11 \ldots 1]^t \in IR_1{}^n\}$$

where 0(n) is the set of all nxn real orthogonal matrices.

THEOREM IV. 1: A(n) is a Lie subgroup of O(n), [13].

As a consequence of Theorem IV. 1, we have the following theorems:

THEOREM IV. 2: A(n) is a subgroup of $\mathbb{H}(n)$.

Proof: Let $A \in A(n)$. Then we can write $AS = S$ and $A \in O(n)$. If $A \in O(n)$ then $A \in GL(n, \mathbb{R})$. Hence $AS = S$ and $A \in GL(n, \mathbb{R})$. Therefore $A \in \mathbb{H}(n)$ for all $A \in A(n)$. So that $A(n) \subset \mathbb{H}(n)$. Consequently A(n) is a subgroup of $\mathbb{H}(n)$.

As a consequence of Theorem IV. 1 and Theorem IV. 2, we can give the following theorem:

THEOREM IV. 3: A(n) is a Lie subgroup of $\mathbb{H}(n)$.

DEFINITION IV. 2: A matrix $A \in GL(n, \mathbb{R})$ is a Double Umbrella matrix if $AS = S$ and $A^tS = S$, where $S = [11 \ldots 1]^t \in \mathbb{R}^n{}_1$, $[\ ]^t$ denotes the transpose of a matrix and $GL(n, \mathbb{R})$ is the set of all nxn real non-singular matrices. The set of Double Umbrella matrices is denoted by DU(n), [9].

THEOREM IV. 4: DU(n) is a Lie subgroup of $GL(n, \mathbb{R})$, [9].

THEOREM IV. 5: DU(n) is a subgroup of $\mathbb{H}(n)$.

Proof: Let $A \in DU(n)$. Then $AS = S$ and $A \in GL(n, \mathbb{R})$. From definition of $\mathbb{H}(n)$, $A \in \mathbb{H}(n)$ for all $A \in DU(n)$. Hence $DU(n) \subset \mathbb{H}(n)$. Therefore DU(n) is a subgroup of $\mathbb{H}(n)$.

As a consequence of Theorem IV. 4 and Theorem IV. 5, we can give the following theorem:

THEOREM IV. 6: DU(n) is a Lie subgroup of $\mathbb{H}(n)$.

V. THE CONNECTEDNESS AND COMPACTNESS OF THE LIE GROUP OF UMBRELLA MATRICES

In this section, we will show that the Lie group of Umbrella Matrices is not connected and non-compact.

THEOREM V. 1: The Lie group of Umbrella Matrices is not connected.

Proof: For the proof of theorem, we should find at least two nonempty subset of $\mathbb{H}(n)$ as $\mathbb{H}_1(n)$ and $\mathbb{H}_2(n)$ such that the set of Umbrella Matrices can be written as follows:

$$\mathbb{H}(n) = \mathbb{H}_1(n) \cup \mathbb{H}_2(n) \text{ and } \mathbb{H}_1(n) \cap \mathbb{H}_2(n) = \varnothing.$$

Let us define the function of determinant as $f: \mathbb{H}(n) \to \mathbb{R}$, $f(A) = \det A$, $\forall A = [a_{ij}] \in \mathbb{H}(n)$. Then we can define the following sets.

$$\mathbb{H}_1(n) = \{A \in \mathbb{H}(n) \mid \det A > 0\}$$

and

$$\mathbb{H}_2(n) = \{A \in \mathbb{H}(n) \mid \det A < 0\}.$$

Therefore $\mathbb{H}_1(n) \neq \emptyset$, $\mathbb{H}_2(n) \neq \emptyset$ and $\mathbb{H}_1(n) \cap \mathbb{H}_2(n) = \emptyset$ and also

$$\mathbb{H}_1(n) \cup \mathbb{H}_2(n) = \mathbb{H}(n).$$

Hence the set of Umbrella Matrices is not connected. Consequently, we can say the Lie group of Umbrella Matrices is not connected.

THEOREM V. 2: The Lie group of Umbrella Matrices is a closed Lie subgroup of $GL(n, \mathbb{R})$.

Proof. Now, let us define the following continuous functions from

$GL(n, \mathbb{R})$ to $\mathbb{R}$ as $f_i: GL(n, \mathbb{R}) \rightarrow \mathbb{R}$; $f_i([a_{ij}]) = \sum_{j=1}^{n} a_{ij}$, $1 \leq i \leq n$.

Then $f_i^{-1}(1)$ are closed subsets of $GL(n, \mathbb{R})$, for all $A \in \mathbb{H}(n)$. Therefore

$\bigcap_{i=1}^{n} f_i^{-1}(1)$ is a closed subset of $GL(n, \mathbb{R})$ and also $\bigcap_{i=1}^{n} f_i^{-1}(1) = \mathbb{H}(n)$.

Hence $\mathbb{H}(n)$ is a closed subset of $GL(n, \mathbb{R})$. Consequently, we can say that the Lie group of Umbrella Matrices is a closed Lie subgroup of $GL(n, \mathbb{R})$. Next, we can give the following theorem about compacthness.

THEOREM V. 3: The Lie group of Umbrella Matrices is non-compact.

Proof. For the proof of theorem, we must show that this Lie group is not closed or not bounded. We know that the Lie group of Umbrella Matrices is closed by the Theorem V.2. So that we will show that the Lie group of Umbrella Matrices is not bounded.
Let A be a matrix such that

$$A = \begin{bmatrix} B & O \\ O & I_{n-2} \end{bmatrix}, \text{ where } B = \begin{bmatrix} a & 1\text{-}a \\ b & 1\text{-}b \end{bmatrix}, \; a,b \in \mathbb{R},\; a \neq b$$

and I_{n-2} is a identity matrix in $GL(n-2, \mathbb{R})$.

Then $B \in \mathbb{H}(2)$ for all $a,b \in \mathbb{R}$ with $a \neq b$. Therefore $A \in \mathbb{H}(n)$ for all $a,b \in \mathbb{R}$, $a \neq b$. Hence $\mathbb{H}(n)$ is not bounded. Consequently, we can say that the Lie group of Umbrella Matrices is non-compact.

VI. THE MAURER-CARTAN FORMS ON $\mathbb{H}(n)$ AND THE DIMENSION OF $\mathbb{H}(n)$

We will define Maurer-Cartan forms on $\mathbb{H}(n)$ and will calculate the dimension of $\mathbb{H}(n)$ using the principal 1-forms on the Lie group of Umbrella Matrices. If the set of Maurer-Cartan forms on $GL(n, \mathbb{R})$ is $\Omega_1(GL(n, \mathbb{R}))$ and if $W \in \Omega_1(GL(n, \mathbb{R}))$ then the Maurer-Cartan forms satisfy the following equation for all $g_o \in GL(n, \mathbb{R})$ and any $g \in GL(n, \mathbb{R})$.

$$(L_{g^{-1}})^* (W|_{go}) = W|_{ggo}$$

Now, let $\{x_{11}, x_{21}, \ldots, x_{n1}, \ldots, x_{nn}\}$ be a local coordinate system on $GL(n, \mathbb{R})$ and let $w_{k1} \in \Omega_1(GL(n, \mathbb{R}))$ be principal 1-forms on $GL(n, \mathbb{R})$. Then we can write the following equations for all $g \in GL(n, \mathbb{R})$ by [8].

$$[(L_{g^{-1}})(dx_{k1}|_e)] = [w_{k1}|_g] = [g_{kj}]^{-1} [dx_{j1}|_g]$$

where $e \in GL(n, \mathbb{R})$ is a unit matrix and L_g is a left-translation at $g \in GL(n, \mathbb{R})$. Since the group of Umbrella matrices is a subgroup of $GL(n, \mathbb{R})$ we can define the inclusion map as follows:

$$i : \mathbb{H}(n) \to GL(n, \mathbb{R})\ ;\ i(g) = g, \forall\, g \in \mathbb{H}(n).$$

If $g \in GL(n, \mathbb{R})$ with $i(g) = g$ for all $g \in \mathbb{H}(n)$, then we can define the Maurer-Cartan forms on $\mathbb{H}(n)$ using the map i^* as $i^*(W) = \xi$. for all $W \in \Omega_1(GL(n, \mathbb{R}))$. Then $\xi \in \Omega_1(\mathbb{H}(n))$ by [8] where i^* is a transformation between the space of cotangent vectors as follows;

$$ig: \quad \underset{GL(n, \mathbb{R})}{T^*(g)} \longrightarrow \underset{\mathbb{H}(n)}{T^*(g)}$$

DEFINITION VI. 1.: The above Maurer-Cartan forms which are defined with $i(W) = \xi$ for all $W \in \Omega_1(GL(n, \mathbb{R}))$, will be called Maurer-Cartan form on $\mathbb{H}(n)$.

THEOREM VI. 1: Let $\{x_{11}, x_{21}, \ldots, x_{n1}, \ldots, x_{nn}\}$ be a local coordinate system on $GL(n, \mathbb{R})$ and let $w_{k1}|_g$ be principal 1-forms on $GL(n, \mathbb{R})$ with

$$(L_{g^{-1}})^* (dx_{k1}|_e) = w_{k1}|_g,\ 1 \leq k, 1 \leq n$$

for all $g \in GL(n, \mathbb{R})$. If $\xi_{k1}|_g$ are Maurer-Cartan forms on $\mathbb{H}(n)$ with $i^*(w_{k1}|_g) = \xi_{k1}|_g$ gor all $g \in \mathbb{H}(n)$ then

$$\sum_{l=1}^{n} \xi_{kl}|_g = 0, \; 1 \leq k \leq n$$

for all $g \in \mathbb{H}(n)$.

Proof: We know that if $w_{kl}|_g$ are principal 1-forms on $GL(n, \mathbb{R})$ for all $g \in GL(n, \mathbb{R})$ then

$$[w_{kl}|_g] = [g_{kj}]^{-1} [dx_{jl}|_g] = g^{-1}dg, \; 1 \leq k, l, j \leq n \tag{1}$$

for all $g \in GL(n, \mathbb{R})$. If $g \in \mathbb{H}(n)$ then $gS = S$. Therefore we can write

$$dgS = 0 \text{ or } g^{-1} dgS = 0. \tag{2}$$

From (1) and (2) we get

$$[w_{kl}|_g] \, S = 0, \; 1 \leq k,l \leq n.$$

Hence

$$\sum_{l=1}^{n} w_{kl}|_g = 0, \; 1 \leq k \leq n. \tag{3}$$

Using the equation (3) and the map i^*, we can get

$$i^*\left(\sum_{l=1}^{n} w_{kl}|_g\right) = i^*(0)$$

and since the map i^* is a linear transformation, we can write

$$\sum_{l=1}^{n} i^*(w_{kl}|_g) = 0. \tag{4}$$

From (3) and the definition of the Maurer-Cartan forms on $\mathbb{H}(n)$ we get

$$\sum_{l=1}^{n} \xi_{kl}|_g = 0, \; 1 \leq k \leq n. \tag{5}$$

DEFINITION VI. 2: The Maurer-Cartan forms on $\mathbb{H}(n)$ which are given by equation (5) will be called principal 1-forms on $\mathbb{H}(n)$.

As a consequence of Theorem VI. 1 we have the following theorem about the dimension of the group of Umbrella Matrices.

THEOREM VI. 2: dim $\mathbb{H}(n) = n(n-1)$.

Proof: We have n linear equations among the principal 1-forms on IH(n) which are induced from GL(n, $\mathbb{R}$) by Theorem VI. 1 as follows:

$$\sum_{i=1}^{n} \xi_{ki}|_g = 0, \forall g \in \mathbb{H}(n), 1 \leq k \leq n.$$

Since the dimension of GL(n, $\mathbb{R}$) is n^2, we can write

$$\dim \mathbb{H}(n) = n^2 - n = n(n-1).$$

ÖZET

Bu çalışmadaGL (n, $\mathbb{R}$) den seçilen şemsiye matrisleri tanımlandı ve bu matrislerin matris çarpımı işlemine göre GL (n, $\mathbb{R}$) nin bir alt grubu olduğu gösterildi.

$\mathbb{H}$(n) ile gösterilen bu matris grubu ile ilgili olarak bir karakteristik özellik verildi ve bu matris grubunun [13] ve [9] tanımlanan A(n) ve DU(n) matris gruplarından daha genel olduğu gösterildi.

Daha sonra $\mathbb{H}$ (n) matris grubunun GL (n, $\mathbb{R}$) nin bir Lie alt grubu olduğu gösterildi ve bu Lie grubu ile ilgili bir karakterizasyon verildi.

Son olarak $\mathbb{H}$ (n) Lie grubunun $n \geq 2$ için kompakt ve irtibatlı olmadığı ispatlandı ve asli 1- formlar kullanılarak boyut hesabı yapıldı.

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