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The Theory of Integration on Manifold
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The Volume For Parallel Hypersurfaces

by

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30

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The Theory of Integration on Manifold and The Volume For Parallel Hypersurfaces

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ABSTRACT

We generalized the theorem about the hyper area element dA for the manifold of orient μ in E^n and by using this theorem we proved the generalized Divergence Theorem.

First we defined the outward unit normal vector filed of manifold M in a hypersurface as:

$$\vec{N}\,|_x = \frac{\vec{X}u_1 \wedge \vec{X}u_2 \wedge \ldots \wedge \vec{X}u_{n-1}}{\| \vec{X}u_1 \wedge \vec{X}u_2 \wedge \ldots \wedge \vec{X}u_{n-1} \|}$$

and secondly we defiend the hyper area element dA as:

$$dA = \| \vec{X}u_1 \wedge \vec{X}u_2 \wedge \ldots \wedge \vec{X}u_{n-1} \| \, du_1 \wedge du_2 \wedge \ldots \wedge du_{n-1}$$

and finally by using these definitions, we obtained the area A (t) of the hypersurface M_t which is parallel to the hypersurface M.

In addition we obtained the Wolume of an orbit-house, generated by H/H' motion in the active H space.

I.1. INTRODUCTION

Let M be a k-manifold with orientation μ in E^n and μ_x be an orient at the point $x \in M$. $<,>$ shows the inner product and $dA\,(x) \in \Lambda^k\,(T_M(x))$. The non – zero k– manifold dA is called the area element defined by the orientation μ, where μ is defined by the inner product $<,>$ on M.

If the orthonormal base system $\{u_1,..., u_k\}$ in $T_M\,(x)$ and the outward unit normal vector field of M at x is $N\,|_x$, (for the simplicity we will show $N\,|_x$ by N), then

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$$(1.1)\quad dA\,(u_1,\ldots,u_k) = \det \begin{bmatrix} u_1 \\ u_2 \\ \cdot \\ \cdot \\ \cdot \\ u_{k-1} \\ N \end{bmatrix} = < u_1 \Lambda u_2 \Lambda \ldots \Lambda u_{k-1}, N >$$

is defined like this [1]. According to this definition it is clear that the vector $u_1 \Lambda u_2 \Lambda \ldots u_{k-1}$ is the direction of N.

THEOREM I. 1.

If M is an (n–1)-manifold with orientation μ in E^n and the outward unit normal vector field of M is $N = (n_1, n_2, \ldots n_n)$ and $\{x_1, \ldots, x_n\}$ is an Euclidean coordinate system of E^n, then

$$dA = \sum_{j=1}^{n} (-1)^{j+1}\, dx_1 \Lambda dx_2 \Lambda \ldots \Lambda \widehat{dx_j} \Lambda \ldots \Lambda dx_n$$

and

$$n_1\, dA = \widehat{dx_1} \Lambda dx_2 \Lambda \ldots \Lambda dx_n,$$

$$n_2\, dA = -dx_1 \Lambda \widehat{dx_2} \Lambda dx_3 \Lambda \ldots \Lambda dx_n,$$

$$\cdot$$
$$\cdot$$
$$\cdot$$

$$n_j\, dA = (-1)^{j+1}\, dx_1 \Lambda \ldots \Lambda \widehat{dx_j} \Lambda \ldots \Lambda dx_n,$$

$$\cdot$$
$$\cdot$$
$$\cdot$$

$$n_n\, dA = (-1)^{n+1}\, dx_1 \Lambda dx_2 \Lambda \ldots \Lambda \widehat{dx_n}\,.$$

Here dA shows the hyper area element and the (<) terms will be ommited.

PROOF: We know that the dim $\Lambda^{n-1}\,(T^{*}_{E^n}{}^{(x)}) = \binom{n}{n-1} = n$, $(u_1 \Lambda u_2 \Lambda \ldots\ldots \Lambda u_{n-1})$ is an element of $T_{E^n}{}^{(x)}$. If $\{u_1, \ldots, u_{n-1}\}$ is an orthonormal base system in $T_M(x)$, an orient μ_x in M is

$\mu_x = [u_1,\ldots,u_{n-1}]$.

Let $U_i = (Ui_1 ,\ldots, Ui_n)$, then

$$(1.2) \qquad dA(u_1,\ldots,u_{n-1}) = \det \begin{bmatrix} u_1 \\ \vdots \\ u_{n-1} \\ N \end{bmatrix}, 1 \leqq i \leqq n-1,$$

$$= \begin{bmatrix} u_{11} \, u_{12} \cdots u_{1n} \\ u_{21} \, u_{22} \cdots u_{2n} \\ \vdots \\ u_{n-11} \cdots u_{n-1n} \\ n_1 \, n_2 \cdots n_n \end{bmatrix}$$

$$= n_1 \begin{bmatrix} u_{12} \, u_{13} \cdots u_{1n} \\ u_{22} \, u_{23} \cdots u_{2n} \\ \vdots \\ u_{n-12} \cdots n_{n-1n} \end{bmatrix}$$

$$+ -n_2 \begin{bmatrix} u_{11} \, u_{13} \cdots u_{1n} \\ u_{21} \, u_{23} \cdots u_{2n} \\ \vdots \\ u_{n-11} \cdots u_{n-1n} \end{bmatrix} + \cdots +$$

$$+ (-1)^{n+1} n_n \begin{bmatrix} u_{11} \, u_{12} \cdots u_{1n-1} \\ u_{21} \, u_{22} \cdots u_{2n-1} \\ \vdots \\ u_{n-11} \cdots u_{n-1\, n-1} \end{bmatrix}.$$

On the other hand

$$dx_{i_1} \wedge dx_{i_2} \wedge \ldots \wedge \widehat{dx}_{i_j} \wedge \cdots \wedge dx_{i_n}$$

$$= \frac{\overbrace{(1+1+\ldots+1)}^{n}}{1!1!\ldots!1!} \text{Alt} (dx_{i_1} \otimes \ldots \otimes \widehat{dx}_{i_j} \otimes \ldots \otimes dx_{i_n})$$

$$= \sum_{\sigma \varepsilon \mathcal{S}_{n-1}} s(\sigma) dx_{\sigma(i_1)} \otimes \ldots \otimes dx_{\sigma(i_j)} \otimes \ldots \otimes dx_{\sigma(i_n)}$$

[2], Where $\sigma = \begin{pmatrix} 1 & 2 & \dots & n \\ i_1 & i_2 & \dots & i_n \end{pmatrix}$). Thus it can be written as

$$(1.3)\quad (dx_{i_1} \Lambda \dots \Lambda \hat{dx}_{i_j} \Lambda \dots \Lambda dx_{i_n})(u_1, \dots, u_{n-1}) = \begin{bmatrix} dx_{i_1}(u_1) dx_{i_1}(u_2) \dots dx_{i_1}(u_{n-1}) \\ dx_{i_2}(u_1) dx_{i_2}(u_2) \dots dx_{i_2}(u_{n-1}) \\ \cdot \qquad\qquad \cdot \\ \cdot \qquad\qquad \cdot \\ \cdot \qquad\qquad \cdot \\ dx_{i_j}(u_1) dx_{i_j}(u_2) \dots dx_{i_j}(u_{n-1}) \\ \cdot \qquad\qquad \cdot \\ \cdot \qquad\qquad \cdot \\ \cdot \qquad\qquad \cdot \\ dx_{i_n}(u_1) dx_{i_n}(u_2) \dots dx_{i_n}(u_{n-1}) \end{bmatrix}.$$

Let $dx_{i_p}(u_q) = u_{qp}$ and $dx_{i_p} = dx_p$

$$(\hat{dx}_1 \Lambda dx_2 \Lambda \dots \Lambda dx_n)(u_1, \dots, u_{n-1}) = \begin{bmatrix} dx_2(u_1) \; dx_2(u_2) \dots dx_2(u_{n-1}) \\ dx_3(u_1) \; dx_3(u_2) \dots dx_3(u_{n-1}) \\ \cdot \\ \cdot \\ \cdot \\ dx_n(u_1) \; dx_n(u_2) \dots dx_n(u_{n-1}) \end{bmatrix}$$

$$= \begin{bmatrix} u_{12} & u_{22} \dots & u_{(n-1)2} \\ u_{12} & u_{23} \dots & u_{(n-1)3} \\ \cdot & & \\ \cdot & & \\ \cdot & & \\ u_{1(n-1)} & u_{2(n-1)} \dots & u_{(n-1)(n-1)} \end{bmatrix}.$$

Its transpose is

$$\begin{bmatrix} u_{12} & u_{13} \dots & u_{1n-1} \\ u_{22} & u_{23} \dots & u_{2(n-1)} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ u_{(n-1)2} & u_{(n-1)3} \dots & u_{(n-1)(n-1)} \end{bmatrix}.$$

If we continue in this way we can find the following result

$$(dx_1 \Lambda \ldots \Lambda\, dx_{n-1} \Lambda \widehat{dx_n})(u_1,\ldots,u_{n-1}) = \begin{bmatrix} dx_1(u_1)\, dx_1(u_2) \ldots dx_1(u_{n-1}) \\ dx_2(u_1)\, dx_2(u_2) \ldots dx_2(u_{n-1}) \\ \vdots \\ dx_{n-1}(u_1) dx_{n-1}(u_2) \ldots dx_{n-1}(u_{n-1}) \end{bmatrix}$$

$$= \begin{bmatrix} u_{11} & u_{21} & \cdots & u_{(n-1)1} \\ u_{12} & u_{22} & \cdots & u_{(n-1)2} \\ \vdots & \vdots & & \vdots \\ u_{1(n-1)} & u_{2(n-1)} & \cdots & u_{(n-1)(n-1)} \end{bmatrix}$$

and its transpose is

$$\begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1(n-1)} \\ u_{21} & u_{22} & \cdots & u_{2(n-1)} \\ \vdots & \vdots & & \vdots \\ u_{(n-1)1} & u_{(n-1)2} & \cdots & u_{(n-1)(n-1)} \end{bmatrix}.$$

If we compare the above result with terms on the right of the equation (1.2) one by one, we can find the following

$$(1.4) \quad dA(u_1,\ldots,u_{n-1}) = n_1 \widehat{dx_1}\, \Lambda dx_2 \Lambda \ldots \Lambda dx_n\, (u_1,\ldots,u_{n-1})$$
$$-n_2 dx_1\, \Lambda\, \widehat{dx_2} \Lambda \ldots \Lambda dx_n\, (u_1,\ldots,u_{n-1}) + \ldots +$$
$$(-1)^{n+1}\, n_n\, dx_1 \Lambda dx_2 \Lambda \ldots \Lambda \widehat{dx_n}\, (u_1,\ldots u_{n-1})$$
$$= (n_1 dx_1 \Lambda dx_2 \Lambda \ldots \Lambda dx_n - n_2 dx_1 \Lambda \widehat{dx_2} \Lambda \ldots \Lambda dx_n + \ldots +$$
$$+ (-1)^{n+1}_{n_n}\, dx_1\, \Lambda \ldots \Lambda \widehat{dx_n})(u_1,\ldots,u_{n-1}).$$

From the right side of the equation (1.4), for each $u_i \varepsilon T_M(x)$, $1 \leqq i \leqq n\text{-}1$, we get

$$dA = \sum_{j=1}^{n} (-1)^{j+1}\, n_j dx_1\, \Lambda\, dx_2\, \Lambda \ldots \Lambda\, \widehat{dx_j}\, \Lambda \ldots \Lambda\, dx_n.$$

For each $u_i \varepsilon T_M(x)$ and $z \varepsilon T_{E^n}(x)$

$$\begin{aligned} <z, N><u_1 \Lambda u_2 \Lambda ... \Lambda u_{n-1}, N> &= <z,N><\lambda N,N> \\ &= <z,N> \lambda \\ &= <z,\lambda N> \\ &= <z,u_1 \Lambda u_2 \Lambda ... \Lambda u_{n-1}> \end{aligned}$$

and

$$<z,N> dA(u_1,...,u_{n-1}) = < z,u_1 \Lambda u_2 \Lambda ... \Lambda u_{n-1} >.$$

If we take

$$z = \frac{\partial}{\partial x_1} = (1,0,...,0), \text{ we get}$$

$$n_1 dA (u_1 ,..., u_{n-1}) = \begin{bmatrix} 1 & 0 & ... & 0 \\ u_{11} & u_{12} & ... & u_{1n} \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ u_{(n-1)1} & u_{(n-1)2} & ... & u_{(n-1)n} \end{bmatrix}$$

$$= (dx_2 \Lambda ... \Lambda dx_n) (u_1 ,..., u_{n-1}).$$

Then

$$n_1 dA = dx_2 \Lambda ... \Lambda dx_n$$

is obtained. Now, if we take

$$z = \frac{\partial}{\partial x_2} = (0,1,0,...,0),..., z = \frac{\partial}{\partial x_n} = (0,0,...,0,1)$$

respectively, we can find

$$n_2 dA = -dx_1 \Lambda dx_3 \Lambda ... \Lambda dx_n$$

$\cdot$

$\cdot$

$\cdot$

$$n_n dA = (-1)^{n+1} dx_1 \Lambda ... \Lambda dx_{n-1} .$$

THEOREM I.1 (Generalized Green's Theorem):

Let M be a compact n-dimensional manifold-with-boundary [1] in E_n and suppose that

$$f_1, f_2 ,..., f_n: M \longrightarrow IR$$

are n-variable differentiable functions. Then

$$\underbrace{\int\int\cdots}_{(n-1)\,-\,\text{fold}}\int_{\partial M}\{f_1 dx_2 \wedge\ldots\wedge dx_n + f_2 dx_1 \wedge dx_3 \wedge\ldots\wedge dx_n$$

$$+\cdots+ f_n\, dx_1 \wedge\ldots\wedge dx_{n-1}\} =$$

$$\underbrace{\int\int\cdots\int_M}_{n\text{-fold}}\left\{ \frac{\partial f_1}{\partial x_1} - \frac{\partial f_2}{\partial x_n} +\ldots+ (-1)^{n+1} \frac{\partial f_n}{\partial x_n} \right\} dx_1 \wedge\ldots\wedge dx_n$$

PROOF: If an Euclidean coordinate system in E^n is

$\{x_1,\ldots,x_n\}$

and a base of

$\Lambda^{n-1} (T^*_M (x))$

is

$\{dx_1 \wedge\ldots\wedge \widehat{dx_j} \wedge\ldots\wedge dx_n,\ 1 \leqq j \leqq n \}$

We can write

$$W = \sum_{j=1}^{n} f_j dx_1 \wedge\ldots\wedge \widehat{dx_j} \wedge\ldots\wedge dx_n$$

$$dW = \frac{\partial f_1}{\partial x_1} dx_1 \wedge dx_2 \wedge\ldots\wedge dx_n$$

$$+ \frac{\partial f_2}{\partial x_2} dx_2 \wedge dx_1 \wedge dx_3 \wedge\ldots\wedge dx_n$$

$$+\cdots+ \frac{\partial f_n}{\partial x_n} dx_n \wedge dx_1 \wedge dx_2 \wedge\ldots\wedge dx_{n-1}$$

$$= \left\{ \frac{\partial f_1}{\partial x_1} - \frac{\partial f_2}{\partial x_2} +\ldots+ (-1)^{n+1} \frac{\partial f_n}{\partial x_n} \right\} dx_1 \wedge dx_2 \wedge\ldots\wedge dx_n.$$

According to Stokes theorem, we have

$$\underbrace{\int\int\cdots}_{(n-1)\,-\,\text{fold}}\int_{\partial M} w = \underbrace{\int\int\cdots}_{n\text{-fold}}\int_M dw.$$

Thus, the theorem is proved.

THEOREM: I.3 (Generalized Divergence Theorem):

Let $M \subset E^n$ be a compact n-dimensional manifol-with-boundary and N be the outward unit normal vector field on ∂M. If a differentiable vector field on M is F then

$$\underbrace{\int\int\cdots\int_M}_{n\text{-fold}} \operatorname{div} F\, dV = \underbrace{\int\int\cdots\int_{\partial M}}_{(n-1)\text{-fold}} \langle F,N\rangle\, dA.$$

Here the hyper volume element on M is dV the hyper area element on M is dA.

PROOF: Firstly, for the vector field

$$F:M \xrightarrow[\text{onto}]{1:1} \bigcup_{P\varepsilon M} T_M(p)$$

$$p = (x_1,\ldots,x_n) \longrightarrow F(x_1,\ldots,x_n) = f_1\,(x_1,\ldots,x_n)\frac{\partial}{\partial x_1} + \ldots +$$

$$+ f_n\, x(_1,\ldots,x_n)\frac{\partial}{\partial x_n},$$

the divergence of F is

$$\operatorname{div} F = \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i},$$

the Euclidean coordinate system in E^n is

$\{x_1,\ldots,x_n\}$

and if we choose

$\{(-1)^{j+1}\, dx_1 \Lambda\, dx_2\, \Lambda\ldots\Lambda\, \widehat{dx_j}\, \Lambda\ldots\Lambda\, dx_n,\ 1 \leqq j \leqq n\}$

as a base of

$\Lambda^{n-1}\,(T^*_M(x))$

then we can write (n–1)–from on M, which is shown as W

$$W = \sum_{j=1}^{n} (-1)^{j+1} f_j dx_1\, \Lambda\ldots\Lambda\, \widehat{dx_j}\, \Lambda\ldots\Lambda\, dx_n\ .$$

Then,

$$dW = \frac{\partial f_1}{\partial x_1} dx_1\, \Lambda\ldots\Lambda\, dx_n - \frac{\partial f_2}{\partial x_2} dx_2\, \Lambda\, dx_1\, \Lambda\, dx_3\, \Lambda\ldots\Lambda\, dx_n +$$

$$+\dots+(-1)^{n+1}\frac{\partial f_n}{\partial x_n}\, dx_n \wedge dx_1 \wedge \dots \wedge dx_{n-1}$$

$$=\left\{\frac{\partial f_1}{\partial x_1}+\frac{\partial f_2}{\partial x_n}+\dots+\frac{\partial f_n}{\partial x_n}\right\} dx_1 \wedge dx_2 \wedge \dots \wedge dx_n$$

$$=\text{div } F\, dV.$$

In addition, we have

$$<F,N> = f_1\, n_1\, dA + f_2\, n_2\, dA + \dots + f_n\, n_n\, dA$$

on ∂M, [1]. Now, from theorem (1.1), we can write

$$n_j\, dA = (-1)^{j+1}\, dx_1 \wedge \dots \wedge \widehat{dx_j} \wedge \dots \wedge dx_n$$

and

$$<F,N>\, dA = f_1\, dx_2 \wedge \dots \wedge dx_n - f_2\, dx_1 \wedge dx_3 \wedge \dots \wedge dx_n + \dots$$
$$+ (-1)^{n+1} f_n\, dx_1 \wedge dx_2 \wedge \dots \wedge dx_{n-1}$$
$$= W$$

is obtained. Again from Stokes theorem

$$\underbrace{\int\int\dots}_{n\text{-fold}}\int_M \text{div } F\, dA = \underbrace{\int\int\dots}_{(n-1)\text{-fold}}\int_{\partial M} <F,N>\, dA$$

is obtained.

II.1. HYPERSURFACES

Each (n–1)–submanifold of a n-manifold is called an hypersurface [3]. A (n–1) real parameter hypersurface M in E^n is given by the vector

$$\vec{X} = \vec{X}(u_1,\dots,u_{n-1}) = x_1(u_1,\dots,u_{n-1})\frac{\partial}{\partial x} + \dots + x_n(u_1,\dots,u_{n-1})\frac{\partial}{\partial x_n}$$

and the outward unit vector field of M is defined as

$$\text{(II.1)} \quad \vec{N}|_x = \frac{\vec{X}u_1 \wedge \dots \wedge \vec{X}u_{n-1}}{\| \vec{X}_{u_1} \wedge \dots \wedge \vec{X}u_{n-1} \|}$$

Let us choose the coordinate neighborhood U in M in such a way that

$$x: (u_1,...,u_{n-1}) \varepsilon U \subset E^{n-1} \longrightarrow X (u_1,...,u_{n-1}) = x\varepsilon M$$

is statisfied. Let the system

$$\{\vec{X}_{u_1},..., \vec{X}u_{n-1}\}$$

be an orthonormal base for $T_M(x)$. In this case, the orientation in M is given by

$$[\vec{N}(x), \vec{X}_{u_1}, \vec{X}_{u_2},...,\vec{X}u_{n-1}] = \mu_x$$

and

$$\vec{N}(x) = \vec{X}_{u_1} \Lambda \vec{X}_{u_2} \Lambda ... \Lambda \vec{X}u_{n-1} .$$

dA hyper area element at p ε M point neigborhood of

$$\vec{X} = \vec{X} (u_1 ,..., u_{n-1})$$

is

$$\text{(II.2)} \qquad dA = \| \vec{X}_{u_1} \Lambda ... \Lambda \vec{X}_{u_{n-1}} \| du_i \Lambda du_2 \Lambda ... \Lambda du_{n-1}$$

and the vector differential form of the same vector is

$$\text{(II.3)} \qquad d\vec{X} = \sum_{i=1}^{n-1} \vec{X}_{ui} du_i$$

and, again by multiplying

$$\text{(II.4)} \qquad \underbrace{d\vec{X} \Lambda \dots \Lambda d\vec{X}}_{(n-1)}$$

is defined as the vector differential form of the (n–1)–th order.

THEOREM II.1

With the (n–1)–real parameter

$$\vec{X} = \vec{X} (u_1,...,u_{n-1}) = x_1 (u_1,...,u_{n-1}) \frac{\partial}{\partial x_1} + ... + x_2 (u_1,...,u_{n-1}) \frac{\partial}{\partial x_{n-1}}$$

for the M regular hypersurface in E^n

$$1.\ \underbrace{d\vec{X} \wedge \ldots \wedge d\vec{X}}_{(n-1)\text{-fold}} = (n-1)!\ \vec{N} dA$$

$$2.\ \underbrace{(d\vec{X} \wedge \ldots \wedge d\vec{X})}_{(n-2)\text{-fold}} \wedge dN = -(n-1)!\ H\vec{N} dA$$

$$3.\ \underbrace{d\vec{N} \wedge \ldots \wedge d\vec{N}}_{(n-1)\text{-fold}} = (-1)^{n-1} K\vec{N} dA ,$$

where, $\vec{N}$ is the unit normal vector field of the regular hypersurface M given by the vector $\vec{X}$, and for the mean curvature function H on hypersurface M, we have

$$(n-1)\ H = \sum_{i=1}^{n-1} k_i ,$$

where k_i are the principal curvature functions. And, for the higher order Gaussian curvature function K on the hypersurface, we have

$$K = \prod_{i=1}^{n-1} k_i .$$

PROOF: 1. As

$$du_i \wedge du_j = \begin{cases} 0 & i = j \\ -du_j \wedge du_i & i \neq j \end{cases}$$

then, the number of the orders of the multipliers

$$\underbrace{d\vec{X} \wedge \ldots \wedge d\vec{X}}_{(n-1)}$$

is (n–1) and all of them equal to each other. That is,

$$\underbrace{d\vec{X} \wedge \ldots \wedge d\vec{X}}_{(n-1)} = \left(\sum_{i=1}^{n-1} \vec{X}u_i\ du_i\right) \wedge \ldots \wedge \left(\sum_{i=1}^{n-1} \vec{X}u_i\ du_i\right)$$

$$= (n-1)\,!\,\vec{X}_{u_1} \Lambda \vec{X}_{u_2} \Lambda \ldots \Lambda \vec{X}_{u_{n-1}} du_1 \Lambda du_2 \Lambda \ldots \Lambda du_{n-1}$$

$$= (n-1)\,!\,\vec{N} dA\,.$$

2. To prove this we take the parameter curves as the curvature line. Thus, according to Olinde Rodrigues formulae

$$\vec{N}u_i = -k_i \vec{X}u_i.$$

$$\underbrace{(d\vec{X}\Lambda \ldots \Lambda d\vec{X})}_{(n-2)\text{-fold}} \Lambda d\vec{N} = \left(\sum_{i=1}^{n-1} \vec{X}u_i du_i \Lambda \ldots \Lambda \sum_{i=1}^{n-1} \vec{X}u_i du_i\right) \Lambda \sum_{i=1}^{n-1} (-k_i \vec{X}u_i du_i)$$

$$= (n-2)!(-k_1 - \ldots - k_{n-1}) \vec{X}u_i \Lambda \ldots \Lambda \vec{X}_{u_{n-1}} du_1 \Lambda \ldots \Lambda du_{n-1}$$

$$= -(n-1)\,!\,H\vec{N}dA\,.$$

3. $$\underbrace{d\vec{N}\Lambda \ldots \Lambda d\vec{N}}_{(n-1)\text{-fold}} = \left(\sum_{i=1}^{n-1} -k_i \vec{X}u_i du_i\right) \Lambda \ldots \Lambda \left(\sum_{i=1}^{n-1} -k_i \vec{X}u_i du_i\right)$$

$$= (-1)^{n-1} k_i \vec{X}_{u_1} \Lambda k_2 \vec{X}_{u_2} \Lambda \ldots \Lambda k_{n-1} \vec{X}_{u_{n-1}} du_1 \Lambda \ldots \Lambda du_{n-1}$$

$$= (-1)^{n-1}\, K\vec{N}dA$$

are obtained.

II.2. PARALLEL HYPERSURFACES IN E^n

Suppose M an hypersurface in E^n given by the equation

$$\vec{X} = \vec{X}(u_1, \ldots, u_{n-1}) = x_1 (u_1, \ldots, u_{n-1}) \frac{\partial}{\partial x_1} + \ldots + x_n (u_1, \ldots, u_{n-1}) \frac{\partial}{\partial x_{n-1}}$$

If another M_t hypersurface in E^n is given as

$$\vec{X}_t = \vec{X} + t\vec{N}.$$

Then, the hypersurfaces M and M_t are called parallel hypersurfaces. Here t is a real parameter and N is also the unit normal vector field of M. According to this definition, it is clear that $\vec{X}_t$ and $\vec{X}$ have the same unit normal vector filed. Thus,

$$\underbrace{d\vec{X}_t \wedge \ldots \wedge d\vec{X}_t}_{(n-1)\text{-fold}} = \sum_{i=1}^{n-1} (\vec{X}_t)u_i\, du_i \wedge \ldots \wedge \sum_{i=1}^{n-1} (\vec{X}_t)u_i\, du_i$$

$$= (n-1)\,!\,(\vec{X}_t)_{u_1} \wedge \ldots \wedge (\vec{X}_t)_{u_{n-1}}\, du_1 \wedge \ldots \wedge du_{n-1}$$

$$= (n-1)\,!\,\vec{X}_t dA_t$$

$$= (n-1)\,!\,\vec{X} dA_t\ .$$

On the other hand,

$$\underbrace{d\vec{X}_t \wedge \ldots \wedge d\vec{X}_t}_{(n-1)\text{-fold}} = (d\vec{X} + td\vec{N}) \wedge (d\vec{X} + td\vec{N}) \wedge \ldots \wedge (d\vec{X} + td\vec{N})$$

$$= \underbrace{d\vec{X} \wedge \ldots \wedge d\vec{X}}_{(n-1)\text{-fold}} + \binom{n-1}{1} t\, \underbrace{(d\vec{X} \wedge \ldots \wedge d\hat{a}\vec{X})}_{(n-2)\text{-fold}} \wedge d\vec{N} +$$

$$+ \binom{n-1}{2} t^2 \underbrace{(d\vec{X} \wedge \ldots \wedge d\vec{X})}_{(n-3)\text{-fold}} \wedge d\vec{X} \wedge d\vec{N} + \ldots +$$

$$+ \binom{n-1}{n-2} t^{n-2}\, d\vec{X} \wedge \underbrace{(d\vec{N} \wedge \ldots \wedge d\vec{N})}_{(n-2)\text{-fold}} + \binom{n-1}{n-1} t^{n-1} \underbrace{d\vec{N} \wedge \ldots \wedge d\vec{N}}_{(n-1)\text{-fold}}\ .$$

Thus

$$(n-1)!\vec{N}dA_t = (n-1)!\vec{N}dA - \binom{n-1}{1} t(n-1)!\vec{N}HdA - \binom{n-1}{2} t^2 2!(n-3)! \sum_{i_1<i_2=1}^{n-1} (-1)^2$$

$$k_{i_1} k_{i_2} \vec{N}dA + \ldots + \binom{n-1}{p} t^p\, p\,!\,(n-1-p)\,! \sum_{i_1<\ldots<i_p=1}^{n-1} (-1)^p k_{i_1} \ldots k_{i_p} \vec{N}dA$$

$$+ \ldots + \binom{n-1}{n-2} t^{n-2} (n-2)\,! \sum_{i_1<\ldots<i_{n-2}=1}^{n-1} (-1)^{n-2} k_{i_1} \ldots k_{i_{n-2}} \vec{N}dA$$

$$+ \ldots + \binom{n-1}{n-1} t^{n-1} (n-1)\,!\,(-1)^{n-1} K\vec{N}dA$$

is obtained. If we multiply both sides of the equation by $\vec{N}$ as scalar product and taking the higher order Gaussian curvatures [4] and if we intergrate the above expression on M, the area A (t) of M_t hypersurface parallel to M can be written as

$$A(t) = A(0) - \binom{n-1}{1} t \int_M HdA + t^2 \int_M K_2 (k_1,...,k_{n-1}) dA +$$

(II.5)

$$+ ... + (-1)^{n-2} t^{n-2} \int_M K_{n-2} (k_1,...,k_{n-1}) dA + (-1)^{n-1} t^{n-1} \int_M KdA.$$

If M is closed the colume, formed by all the points of M_t hypersurfaces are formed by changing t, is V, then

$$\text{(II.6)} \quad V = tA(0) - \frac{n-1}{2} t^2 \int_M HdA + \frac{t^3}{3} \int_M K_2 (k_1,...,k_{n-1}) dA +$$

$$+ ... + (-1)^{n-2} \frac{t^{n-1}}{n-1} \int_M K_{n-2} (k_1,...,k_{n-1}) dA + (-1)^{n-1} \frac{t^n}{n} \int_M KdA$$

is obtained.

III.1. THE VOLUME OF ORBIT-HOSE

Let us represent the moving space H with the positive oriented orthonormal system $\{0,\vec{E}_i\}$ and the fixed space H′ with the positive oriented orthonormal system $\{Q',\vec{E'}_1\}$. Besides, lets take a third positive oriented orthonormal system $\{Q,\vec{R}_i\}$, relative system, representing space H_1.

Let

$$\vec{OQ} = \vec{q} \text{ and } \vec{O'Q} = \vec{q}'.$$

Variation, of the initial point Q of the relative system according to the moving space H is

$$d\vec{q} = \sum_{i=1}^{n} w^*_i \vec{R}_i = \vec{w}^*$$

and again, variation of the initial point Q of the relative system according to the fixed space H' is

$$d\vec{q}' = \sum_{i=1}^{n} w_i^{*\prime} \vec{R}_i = \vec{w}'^*$$

If the coordinates of the point $x \varepsilon H_1$ according to $\{Q,\vec{R_i}\}$ is $(x_1,...,x_n)$, the change of point according to H_1 will be

$$\vec{dx} = d\left(\sum_{i=1}^{n} x_i\vec{R_i}\right).$$

As $\vec{X}$ and $\vec{X'}$ are

$$\vec{X} = \vec{OX} = \vec{OQ} + \vec{QX} = \vec{q} + X^T R$$

$$\vec{X'} = \vec{O'X} = \vec{O'Q} + \vec{QX} = \vec{q'} + X^T R$$

the change of the point X according to H is

$$\begin{aligned}\vec{dX} = \vec{dq} + d(X^T R) &= w^{*T} R + X^T dR + (dX^T) R .\\ &= w^{*T} R + X^T \Omega R + (dX^T)R\end{aligned}$$

and the change of the point X according to H′ is

$$\vec{dX'} = \vec{dq'} + dX^TR = w'^{*T} R + X^T \Omega' R + (dX^T)R.$$

If X is a fixed point at H, then the sliding velocity of the point X will be

$$d_f\vec{X} = \vec{dX'} - \vec{dX}$$

$$d_fX^T R = (w'^* - w^*)^T + X^T (\Omega' - \Omega)R.$$

Since $\Omega' - \Omega$ is a skewsymetric matrix. Now, if we represent

$$\Omega' - \Omega = \Psi$$

the equation of the matrix form of the sliding velocity is

(III.1) $d_fX^T = (w'^* - w^*)^T + X^T\Psi$.

Here the column matrix $w'^* - w^*$ correspond to the moment of the Darboux tensor Ψ, with respect to the point Q and so we denote it by

$$\Psi^* = w'^* - w^*$$

and

(III.2) $d_fX = \Psi^* + \Psi^T X$

is obtained [5].

The hypersurface M, whose boundary is ∂M, in the moving space H generates a space shaped like a hypertorus in the space H' in the motion H/H' = B. This space is called the orbit-hose of M. The dV hyper volume element of this orbit-hose is

$$dV = <\vec{d_f X}, \vec{dA}>$$

$$= <\Psi^* - \vec{\Psi^T} \Lambda \vec{X}, \frac{1}{(n-1)!} \underbrace{\vec{dX} \Lambda ... \Lambda \vec{dX}}_{(n-1)-\text{fold}} >$$

$$= \frac{1}{(n-1)!} <\vec{\Psi^*}, \underbrace{\vec{dX} \Lambda ... \Lambda \vec{dX}}_{(n-1)-\text{fold}} > + \frac{1}{(n-1)!} < \vec{\Psi^T} \Lambda \vec{X}), (\underbrace{\vec{dX} \Lambda ... \Lambda \vec{dX}}_{(n-1)-\text{fold}}) >$$

and

$$\text{(III.3)} \quad V = < \int_B \vec{\Psi^*}, \int_M N\vec{dA} > + < \int_B \vec{\Psi^T} \Lambda \vec{X}, \int_M N\vec{dA} >$$

is obtained.

ÖZET

E^n de μ yönlü (n-1) – manifold için dA hiper alan elemanı ile ilgili teorem genelleştirilerek, genelleştirilmiş Divergens Teoreminin ispatında kullanıldı ve hiperyüzeylerde M manifoldunun dış birim normal vektör alanı

$$\vec{N}\Big|_x = \frac{\vec{Xu_1} \Lambda \dots \Lambda \vec{Xu_{n-1}}}{\| \vec{Xu_1} \Lambda \dots \Lambda \vec{Xu_{n-1}} \|}$$

biçiminde ve dA hiper alan elemanı da

$$dA = \| \vec{Xu_1} \Lambda \dots \Lambda \vec{Xu_{n-1}} \| du_1 \Lambda \dots \Lambda du_{n-1}$$

biçiminde tanımlanarak M hiperyüzeyine paralel M_t hiperyüzeyinin A (t) alanı hesaplandı.

Ayrıca hareketli H uzayının H/H' hareketinde meydana gelen yörünge hortumunun hacmi hesaplandı.

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