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A GENERALIZATION FOR LAGUERRE FUNCTION OF A HYPERSURFACE IN A RIEMANNIAN MANIFOLD

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ABSTRACT

A generalization relative to a congruence of curves for the Laguerre function of a hypersurface in a Riemannian space has obtained by Nirmala (1965) . In this article we generalize the Laguerre function relative to any vector field without using a congruence of curves.

CURVATURE FUNCTIONS

Let $\{y^1, \ldots, y^{n+1}\}$ be a coordinate system of a C^∞ Riemannian (n+1)-manifold $\bar{M}$ whose Riemannian metric is

$$a = \sum_{\alpha,\beta}^{n+1} a_{\alpha\beta}\, dy^\alpha \otimes dy^\beta ,$$

and $\{x^1, \ldots, x^n\}$ be a coordinate system of a hypersurface M of $\bar{M}$ with

$$g = \sum_{i,j}^{n} g_{ij}\, dx^i \otimes dx^j$$

as the fundamental metric. Let V be a mapping which attaches to each point p of M, a tangent vector V_p in $T_p(\bar{M})$. Then V is called an $\bar{M}$-vector field defined on M. V is said to be C^∞ on M if about each point p of M there is a coordinate neighbourhood $\bar{U}$ of p in $\bar{M}$ with coordinate functions $y^1, \ldots, y^{n+1}$ such that

$$V = \sum_{\alpha}^{n+1} V^\alpha E_\alpha , \qquad \left(E_\alpha = \frac{\partial}{\partial y^\alpha} \right)$$

on U where V^{α}' s are C^{∞} functions on the neighbourhood U of M, and $\{E_1, \ldots, E_{n+1}\}$ is a (local) basis of T ($\bar{M}$). The set $\bar{T}$ (M) of all such (smooth) vector fields is a module over C^{∞} (M,IR). For any $Y \in T(\bar{M})$ the restriction Y /M is in $\bar{T}$ (M). T (M) is a submodule of $\bar{T}$ (M).

Let $\bar{D}$ denote the Riemannian connection on $\bar{M}$. Then the connection $\bar{D}$ gives rise in a natural way to a function

$$T(M) \times \bar{T}(M) \longrightarrow \bar{T}(M)$$

called the induced connection on M. Since the induced connection is so closely related to the Riemannian connection of $\bar{M}$ we will use the same notation for both. If X, Y $\in$ T(M), then

$$D_X Y = \tan \bar{D}_X Y$$

where D is the Riemannian connection of M, and

$$\tan\colon \bar{T}(M) \longrightarrow T(M)$$

is C^{∞} (M,IR)-linear.

Now consider $V \in \bar{T}(M)$. We can decompose V uniquely into its tangential and normal components given by

$$V = \tan V + \operatorname{nor} V, \tag{1}$$

where

$$\operatorname{nor}\colon \bar{T}(M) \longrightarrow T(M)^{\perp}$$

is C^{∞} (M,IR)-linear. Let C: $x^i = x^i(s)$ be a C^{∞} curve passing through a point p on M and T be the unit tangent vector field of C on M. Covariant derivative of V in the direction T gives

$$\bar{D}_T V = \bar{D}_T(\tan V) + \bar{D}_T(\operatorname{nor} V). \tag{2}$$

If h is a real valued C^{∞} function on U, then we have

$$\operatorname{nor} V = hN, \tag{3}$$

where N is the unit normal vector field to M. Hence using the Gauss' equation in (2), we can write

$$\bar{D}_T V = (D_T(\tan V) + hL(T)) + (T(h) + \langle L(T), \tan V \rangle)N$$

or putting

$$D_T(\tan V) + hL(T) = \tan \bar{D}_T V,$$
$$(T(h) + \langle L(T), \tan V \rangle)N = \operatorname{nor} \bar{D}_T V$$

in this equation we have

$$\bar{D}_T V = \tan \bar{D}_T V + \text{nor } \bar{D}_T V, \tag{4}$$

where L is the Weingarten map and T(h) is the derivative of h in the direction T. Now let V be unit vector field. Then

$$\bar{D}_T V, \ \tan \bar{D}_T V \text{ and nor } \bar{D}_T V$$

are called the absolute curvature vector field, geodesic curvature vector field and normal curvature vector field of the vector field V with respect to C, respectively. In addition, if we put $\|\tan V\| = t$, then real valued C^∞ functions

$$\|\bar{D}_T V\| = \bar{K}_{V/1}, \ (1/t)\|\tan \bar{D}_T V\| = K_{V/g} \text{ and } (1/t)\|\text{nor } \bar{D}_T V\| = K_{V/n} \tag{5}$$

are called the absolute curvature function, geodesic curvature function and normal curvature function of the vector field V with respect to C, respectively. Hence the equation (I.4) can be written as

$$\bar{K}_{V/1} \bar{N}_1 = t\,(K_{V/g} X + K_{V/n} N), \tag{6}$$

where $\bar{N}_1$, X and N are the unit vector fields along the absolute curvature vector field, geodesic curvature vector field and normal curvature vector field of V with respect to C, respectively.

In the particular case when V = T, the expressions $\bar{K}_{V/1}$, $K_{V/n}$, and $K_{V/g}$ reduce to geodesic curvature function of C in $\bar{M}$, normal curvature function and geodesic curvature function of C in M respectively, and therefore the equation (1.6) takes the form,

$$\bar{D}_T T = K_g\, b + K_n\, N,$$

where b is a unit vector field along the geodesic curvature vector field of C in M [Weatherburn (1957)].

THE OPERATOR S

Let V be an $\bar{M}$-vector field defined on M and

$$V = \sum_{\beta}^{n+1} V^\beta E_\beta$$

be its local expression. We define

$$d_j V^\beta = \sum_{j}^{n} V^\beta{}_{;j}\, dx^j, \tag{7}$$

where

$$V^{\beta}{}_{;j} = \frac{\partial V^{\beta}}{\partial x^{j}} + \sum_{\gamma,\mu}^{n+1} y^{\gamma}{}_{;j} V^{\mu} \Gamma^{\beta}{}_{\gamma\mu} . \tag{8}$$

Then we can write that

$$d_j = \sum_{j}^{n} \frac{\delta}{\delta x^{j}} dx^{j} , \tag{9}$$

where the operator $\frac{\delta}{\delta x^{j}}$ is the symbol of covariant differentiation defined by

$$\frac{\delta}{\delta x^{j}} (V^{\beta}) = \frac{\delta V^{\beta}}{\delta x^{j}} = V^{\beta}{}_{;j} .$$

Since

$$\frac{\delta y^{\gamma}}{\delta x^{j}} = \frac{\partial y^{\gamma}}{\partial x^{j}}$$

we get

$$d_j y^{\gamma} = dy^{\gamma} . \tag{10}$$

Now let us define

$$S = \sum_{\alpha}^{n+1} < d_j y^{\alpha} , d_j > E_{\alpha} . \tag{11}$$

Hence

$$S \otimes V = \sum_{\alpha,\beta}^{n+1} < d_j y^{\alpha} , d_j V^{\beta} > E_{\alpha} \otimes E_{\beta} . \tag{12}$$

If X any vector field and

$$X = \sum_{\nu}^{n+1} a^{\nu} E_{\nu}$$

is its local expression, then the direct product (or dot product) of X with $S \otimes V$ gives

$$X . S \otimes V = \sum_{\beta}^{n+1} \left(\sum_{\nu,\alpha}^{n+1} a_{\nu\alpha} a^{\nu} < d_j y^{\alpha} , d_j V^{\beta} > \right) E_{\beta} . \tag{13}$$

The following theorem gives us a relation between the operator S and the covariant derivative.

THEOREM 1: Let Z be any vector field in T(M). Then

$$Z \,.\, S \otimes V = \bar{D}_Z V \quad . \tag{14}$$

PROOF: If

$$Z = \sum_{k}^{n} b^k e_k \;, \quad \left(e_k = \frac{\partial}{\partial x^k} \right) ,$$

then we have

$$Z = \sum_{\nu}^{n+1} \left(\sum_{k}^{n} b^k y^{\nu}{}_{;k} \right) E_{\nu} \,.$$

Hence

$$Z \,.\, S \otimes V = \sum_{\beta}^{n+1} \left(\sum_{\nu,\alpha}^{n+1} a_{\nu\alpha} \left(\sum_{k}^{n} b^k y^{\nu}{}_{;k} < d_j y^{\alpha}, d_j V^{\beta} > \right) \right) E_{\beta}.$$

Since

$$< d_j y^{\alpha} , d_j V^{\beta} > = \sum_{i'j}^{n} y^{\alpha}{}_{;i} g^{ij} V^{\beta}{}_{;j} \,,$$

it follows that

$$Z \,.\, S \otimes V = \sum_{\beta}^{n+1} \left[\sum_{j,k}^{n} \left(\sum_{i}^{n} \left(\sum_{\nu,\alpha}^{n+1} a_{\nu\alpha} y^{\nu}{}_{;k} y^{\alpha}{}_{;i} \right) g^{ij} \right) b^k V^{\beta}{}_{;j} \right] E_{\beta}$$

$$= \sum_{\beta}^{n+1} \left[\sum_{j,k}^{n} \left(\sum_{i}^{n} g_{ik} g^{ij} \right) b^k V^{\beta}{}_{;j} \right] E_{\beta}$$

$$= \sum_{\beta}^{n+1} \left(\sum_{j}^{n} b^j V^{\beta}{}_{;j} \right) E_{\beta}$$

$$= \bar{D}_Z V \,.$$

QED.

THEOREM 2: Let Y be an $\bar{M}$-vector field defined on M. Then

$$Y \,.\, S \otimes V = \bar{D}_{\tan Y} V \quad . \tag{15}$$

PROOF: We can decompose V uniquely into its tangential and normal components given by

$$Y = \tan Y + \text{nor } Y \,.$$

Then

$$\begin{aligned} Y \,.\, S \otimes V &= (\tan Y + \text{nor } Y) \,.\, S \otimes V \\ &= \tan Y \,.\, S \otimes V + \text{nor } Y \,.\, S \otimes V \\ &= \tan Y \,.\, S \otimes V + (hN) \,.\, S \otimes V \\ &= \tan Y \,.\, S \otimes V + h\,(N \,.\, S \otimes V)\,, \end{aligned}$$

and since

$$N \,.\, S \otimes V = 0$$

we find

$$Y \,.\, S \otimes V = \tan Y \,.\, S \otimes V \,.$$

Since tan $Y \in T(M)$, using (14) we obtain

$$Y \,.\, S \otimes V = \bar{D}_{\tan Y} V \,.$$

QED.

GENERALISED LAGUERRE FUNCTION

Let V be a unit $\bar{M}$-vector field defined on M and C be a C^∞ curve passing rhrough a point p on M. Let T denote the unit tangent vector fieldof C oń M. The function $K_{V/}$ defined by

$$-K_{V/} = < \bar{D}_T V \,,\, T > \tag{16}$$

is called the generalised normal curvature of the curve C relative to the vector field V [Singal and Behari (1955)]. Covariant derivative of (16) in the direction T gives

$$-T(K_{V/}) = < \bar{D}_T(\bar{D}_T V) \,,\, T > + < \bar{D}_T V \,,\, \bar{D}_T T > . \tag{17}$$

Moreover since

$$\begin{aligned} < \bar{D}_T(\bar{D}_T V) \,,\, T > &= < \bar{D}_T\,(T.S \otimes V) \,,\, T > \\ &= < \bar{D}_T T.S \otimes V \,,\, T > + < T.\bar{D}_T\,(S \otimes V),\, T > \\ &= < \bar{D}_{\tan \bar{D}_T T} V \,,\, T > + < T.\bar{D}_T\,(S \otimes V),\, T > \end{aligned}$$

the equation (17) reduces to

$$-T(K_{V/}) = < \bar{D}_{\tan \bar{D}_T T}\, V,\, T > + < T.\bar{D}_T\,(S \otimes V),\, T > + < \bar{D}_T V,\, \bar{D}_T T > .$$

Hence we obtain

$$-< T.\bar{D}_T\,(S \otimes V),\, T > = T(K_{V/}) + < \bar{D}_{\tan \bar{D}_T T}\, V,\, T > + < \bar{D}_T V,\, \bar{D}_T T > . \tag{18}$$

We shall call $- < T \,.\, \bar{D}_T (S \otimes V) \,,\, T > = L_V$ as the generalised Laguerre function for the direction T and a curve in M such that the generalised Laguerre function in the direction of the curve vanishes at each point of the curve as a generalised Laguerre line.

Now let us deseribe L_V in terms of curvatures. We have

$$
\begin{aligned}
< \bar{D}_{\tan \bar{D}_T T} V \,,\, T > &= < \bar{D}_{K_g b} V \,,\, T > \\
&= K_g < \bar{D}_b V \,,\, T > \\
&= K_g < t\bar{K}_{V/g}\, \bar{a} + t\bar{K}_{V/n}\, \bar{N} \,,\, T > \\
&= tK_g \bar{K}_{V/g} < \bar{a} \,,\, T > ,
\end{aligned}
$$

where $\bar{K}_{V/g}$ is the geodesic cnrvature function of V with respect to a curve $C' : x^i = x^i (s')$ whose the unit tangent is b, and $\bar{a}$ is the unit vector field along the geodesic curvature vector field of V with respect to C'. Let us say

$$< \bar{a} \,,\, T > = \cos \theta \,.$$

Then

$$< \bar{D}_{\tan \bar{D}_T T} V \,,\, T > = tK_g \bar{K}_{V/g} \cos \theta \,. \tag{19}$$

Moreover

$$
\begin{aligned}
< \bar{D}_T V \,,\, \bar{D}_T T > &= < tK_{V/n}\, N + tK_{V/g}\, X, \;\; K_n N + K_g b > \\
&= tK_{V/n}\, N + tK_{V/g}\, K_g < X \,,\, b > ,
\end{aligned}
$$

and if we say $< X \,,\, b > = \cos \varphi$ then we have

$$< \bar{D}_T V \,,\, \bar{D}_T T > = tK_{V/n}\, K_n + tK_{V/g}\, K_g \cos \varphi \,. \tag{20}$$

Hence (18) reduces to

$$L_V = T(K_{V/}) + tK_{V/n} K_n + tK_g(\bar{K}_{V/g} \cos \theta + K_{V/g} \cos \varphi)$$

which is in terms of curvatures.

SPECIAL CASES

1. When V is normal to M and $n > 2$, (18) can be written as

$$L_V = T(K_n) + 2 < \bar{D}_T N \,,\, \bar{D}_T T > \tag{21}$$

since

$$K_{V/} = K_n \text{ and } < \bar{D}_{\tan} \bar{D}_T T\, N \,,\, T > = < \bar{D}_T N, \bar{D}_T T > .$$

In terms of curvatures, (21) takes the form

$$L_V = T(K_n) + 2\, T_g\, K_g \cos \psi, \tag{22}$$

where T_g is the geodesic torsion of the curve C, $\cos\psi = \langle N_2, b \rangle$ and N_2 is the unit 2–th normal vector field of the curve C.

2. When V is normal to M and n=2, we have $N_2 = b$ and $\psi = 0$. Therefore

$$L_V = T(K_n) + 2\, T_g\, K_g \tag{23}$$

which is an expression obtained for the Laguerre function of a surface in a 3–manifold, by Weatherburn [1957].

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