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TOPOLOGICAL FOLDINGS

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ABSTRACT

When a sheet of paper is crumpled in the hands and then crushed flat against a desk-top, the pattern of creases so formed is governed by certain simple rules. These rules are generalized to theorems on folding manifolds isometrically into one another which has been examined independently by Robertson [1977] and Sewell [1973]. In this paper we have constructed a more general theory of purely topological character. To achieve this, we abandon the definition of isometric folding, which has no obvious analogue in the topological case, and instead we adopt an inductive procedure.

MANIFOLDS WITHOUT BOUNDADRY

We define the following standard subsets of Euclidean n-space E^n for any $n > 0$:

$$D^n = \{ x \in E^n : | x | \leq 1 \};$$
$$S^{n-1} = \{ y \in E^n : | y | = 1 \}.$$

We call D^n and S^{n-1} the unit disc and the unit sphere in Euclidean n-space, respectively. Thus $S^{n-1} = D^n$. It follows from the definition that for each $x \in D^n$ with $x \neq 0$, there is a unique real number t and a unique point $y \in S^{n-1}$, such, that $x = t\,y$, $0 \leq t \leq 1$. Of course, for all $y \in S^{n-1}$, $0 = 0\,y$. See figure (1).

Now suppose that $f : S^{n-1} \to S^{n-1}$ is any map. Then f induces a map $f_* : D^n \to D^n$, given by $f_* (tx) = tf(x)$ where $0 \leq t \leq 1$, $x \varepsilon S^{n-1}$ and $f_* (0) = 0$.

By using this construction we can define a topological folding by the following induction. Let M and N be toplogical manifolds. Where $\dim M = \dim N = n > 0$ and $\partial M = \partial N = \varnothing$. For all $x \varepsilon M$, a disc chart

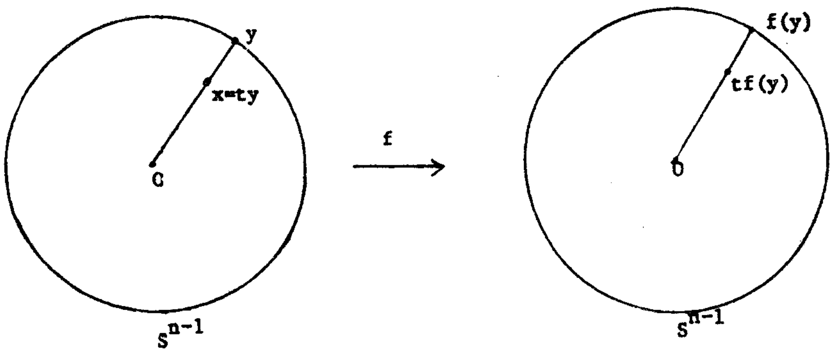

Figure (1)

at x is a homeomorphism $\xi : D^n \rightarrow V_x$, where V_x is a neighbourhood of x in M and $\xi(0) = x$. Hence, every $x \varepsilon M$ has a disc chart.

Now let $\Phi : M \rightarrow N$ be a continuous map. We say that Φ is a topological folding of M into N, iff for each $x \varepsilon M$, there are disc charts $\xi : D^n \rightarrow V_x$ for M at x and $\eta : D^n \rightarrow W_y$ for N at $y = \Phi(x)$ together with a topological folding $f : S^{n-1} \rightarrow S^{n-1}$ such that $\eta \circ f_* = \Phi \circ \xi$.

$$\begin{array}{ccc} D^n & \xrightarrow{\quad f_* \quad} & D^n \\ \xi \Big\downarrow & & \Big\downarrow \eta \\ V_x & \xrightarrow[\quad \Phi/V_x \quad]{} & W_y \end{array}$$

To complete the definition we say that any map $f : S^o \rightarrow S^o$ is a topological folding. Since S^o consists of the two real numbers 1, –1, there are exactly four topological foldings of S^o to itself. We denote by τ (M,N) the set of all topological foldings of M into N, ant put $\tau(M) = \tau(M,M)$.

If $\Phi \varepsilon \tau(M,N)$, then $x \varepsilon M$ is said to be a singularity of Φ iff Φ is not alocal homeomorphism at x. The set of all singularities of Φ is denoted by $\Sigma(\Phi)$.

FOLDINGS OF 1–MANIFOLDS

PROPOSITION: Let $\Phi \varepsilon \tau(M,N)$, where M and N are 1–manifolds without boundary. Then $\Sigma(\Phi)$ is a discrete subset of M.

PROOF: Let $x \varepsilon M$ and $y = \Phi(x)$. Then there are disc charts $\xi : I \to V_x$, $\eta : I \to W_y$ on M and N, respectively, and a topological folding $f : S^o \to S^o$, such that $\eta \circ f_* = \Phi \circ \xi$, where $I = \{-1, 1\} = D^1$. Now suppose that $x \varepsilon \Sigma(\Phi)$. Then $f(1) = f(-1) = \pm 1$, say $f(1) = f(-1) = 1$. Then $f_*(t) = |t|$. Hence Φ is a local homeomorphism on $V_x \setminus \{x\}$. Hence x is an isolated point of $\Sigma(\Phi)$, and so $\Sigma(\Phi)$ is discrete.

COROLLARY: Let $\Phi \varepsilon \tau(M,N)$. If $M \approx R$, then $\Sigma(\Phi)$ is countable. If $M \approx S^1$, then is finite, and $\# \Sigma(\Phi)$ is even.

PROOF: The first statement follows immediately from the proposition. Suppose then that $M \approx S_1$, and let $x \varepsilon \Sigma(\Phi)$, then there are disc charts $\xi : I \to S^1$, $\eta : I \to S^1$ such that $\xi(0) = x$, $\eta(0) = \Phi(x) = y$ and $\Phi \circ \xi = \eta \circ f_*$, where $f_* : I \to I$ is given by $f_*(t) = |t|$. Hence f_* induces orientations on rays $I_-(0 < t < 1)$ and $I_+ (-1 < t < 0)$ and hence local opposite orientations on $\xi(I_-)$ and $\xi(I_+)$. These local orientations can be chosen so that each region has a unique orientation induced by disc charts. This shows that the singularities of Φ partition S^1 into arcs in such a way that successive arcs have opposite orientations. Thus the number of arcs is even, and so the number of singularities is also even.

It should be noted that, topological foldings of S^1 to itself can be of any degree. For example, the power map $\Phi_k : S^1 \to S^1$ given by $\Phi_k(e^{i\theta}) = e^{ik\theta}$, is a topological folding (without singularities) for any $k \neq 0$.

FOLDINGS OF SURFACES

Consider now any topological folding $\Phi \varepsilon \tau(M,N)$, where M and N are connected surfaces without boundary. The disc charts provide local models for the set of singularities $\Sigma(\Phi)$, as follows. Let $f : S^1 \to S^1$ be a topological folding. Then $\Sigma(f)$ consists of 2k points $p_1, .. p_{2k}$. Hence $\Sigma(f_*)$ consists o the rays joining each p_i to 0. That is, $\Sigma(f_*) = \{ t\, p_i : 0 \leq t \leq 1, i=1, \ldots, 2k\}$.

It follows that the set $\Sigma(\Phi)$ has the structure of a locally finite graph K_Φ embedded in M, for which every vertex has even valency.

A connected subset of $M \setminus K_\Phi$ is called a Φ-region. We note that the Φ-regions, together with the edges and vertices of K_Φ constitute a topological stratification of M, (where a topological stratification of a topological manifold M is a partition of M as a disjoint union of disjoint connected manifolds called strata, such that the frontier in M of each stratum is a union of finitely many strata of lower dimension).

Note also that if M is compact, then K_Φ is finite and the number of Φ - regions is finite. Moreover, every Φ-region is bounded by a closed polygon in K_Φ.

FOLDINGS OF MANIFOLDS

From the previous two sections, we can begin to form a picture of how the structure of $\Sigma(\Phi)$ may be described, for any $\Phi \varepsilon \tau(M,N)$, where M and N are topological n-manifolds without boundary. We proceed inductively as in the case of isometric foldings {1} and conclude that $\Sigma(\Phi)$ partitions M into disjoint strata that fit together to form a topological stratification S of M. We refer to the r-dimensional strata as r-strata, and to the n-strata as Φ-regions. This stratification is locally finite and, if M is compact, is finite.

We remark that, if $\Phi \varepsilon \tau$ (M,N) and $\Psi \varepsilon \tau$ (P,Q). then $\Phi \times \Psi \varepsilon \tau$ $(M\times P, N\times Q)$. Also, it is easy to check that

$$\Sigma(\Phi \times \Psi) = (\Sigma(\Phi)\times P) \cup (M \times \Sigma(\Psi)).$$

For example, let $\Phi \varepsilon \tau$ (I) and $\Psi \varepsilon \tau$ (S^1) be the topological foldings given by for all $x \varepsilon I$, $\Phi(x) = |x|$ and for all $(y, z) \varepsilon S^1$, $\Psi(y,z) = (y,—z)$. Then $\Phi \times \Psi \varepsilon \tau$ $(I \times S^1)$. The set $\Sigma(\Phi \times \Psi)$, and its relation to $\Sigma(\Phi)$ and $\Sigma(\Psi)$, is indicated in Figure (2).

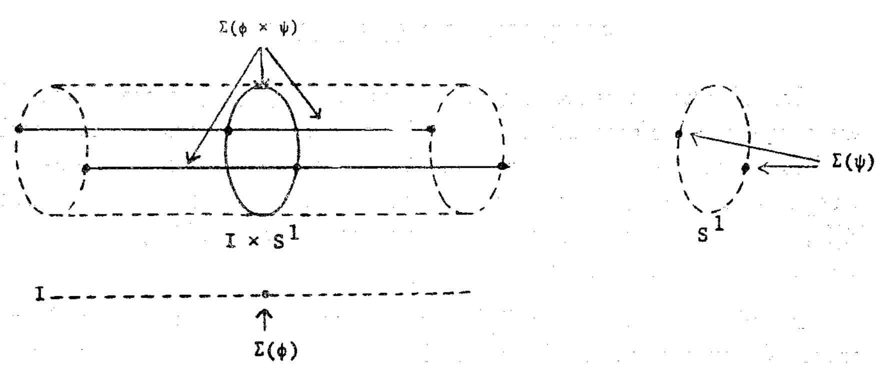

Figure (2)

However, the composite of any two topological foldings is not in general a topological folding. We give an example to illustrate the phenomenon.

Let $\Phi : S^2 \to S^2$ be given by $\Phi(x, y, z) = (x, y, |z|)$. Then $\Phi \varepsilon \tau(S^2)$, the image of this topological folding being the 'Northern' hemisphere H. Let η be an embedding of the equator $z=0$ of S^2 into S^2, given by $\eta(x,y,0) = \left((x,y, \varepsilon x \sin \frac{1}{x}\right)$, where $0 < \varepsilon < 1, x \neq 0$ and $\eta(0,y,0) = (0,y,0)$. By the Schoenflies theorem, since $\eta : \partial H \to S^2$ is a topological embedding, η extends to a homeomorphism $\bar{\eta} : S^2 \to S^2$. Let $\Psi = \eta \circ \Phi$. Then $\Psi \varepsilon \tau(S^2)$. But $\Phi \circ \Psi \notin \tau(S^2)$, since $\Sigma(\Phi \circ \Psi)$ has infinitely many strata.

We observe that for any $\Phi \varepsilon \tau(M,N)$ and for each stratum $\sigma \varepsilon S$, $\Phi \mid \sigma$ is a topological immersion of σ in N. Suppose now that $\Psi \varepsilon \tau(N,P)$ is a topological folding. Then $\Psi \circ \Phi$ will be a topological folding if for each stratum $\sigma \varepsilon S$, where S in the topological stratification induced by Φ on M, and $\Phi(\sigma)$ is topologically transverse to each stratum of Ψ. This condition is not, however, necessary.

MANIFOLDS WITH BOUNDARY

Let M and N be topological manifolds, where dim M = dim N = $n > 0$, and $\partial M = \partial N \neq \Phi$. For all $x \varepsilon$ Int M (Int M means interior of M), a disc chart at x can be defined as before. If $x \varepsilon \partial M$ a disc chart at x is a homeomorphism $\bar{\xi} = \bar{D}^n \to \bar{V}_x$, where $\bar{V}_x$ is a half disc neighbourhood of x in M, $\bar{D}^n = \{x \varepsilon E^n : |x| \leq 1, x_n \geq o\}$ and $\xi(0) = x$. Hence every $x \varepsilon M$ has a disc chart

Now, let Φ: M $\to$ N be a continuous map. We say that Φ is a topological folding of M into N, iff for each $x \varepsilon M$, there are disc charts $\xi : D^n \to V_x$ or $\bar{\xi} : \bar{D}^n \to \bar{V}_x$ for M at $x \varepsilon$ Int M or $x \varepsilon \partial M$, respectively, and $\eta : D^n \to W_y$ or $\bar{\eta} : \bar{D}^n \to \bar{W}_y$ for N at $y = \Phi(x) \varepsilon$Int N or $y = \Phi(x) \varepsilon \partial N$, together with one of the following topological foldings:

(i) $f : S^{n-1} \to S^{n-1}$ such that $\eta \circ f_* = \Phi \circ \xi$, ($x \varepsilon$ Int M any $y \varepsilon$ Int N);

(ii) $\bar{f} : \bar{S}^{n1} \to \bar{S}^{n-1}$, where $\bar{S}^{n-1} = \{x \varepsilon E^n : |x| = 1, x_n \geq 0\}$, such that $\bar{\eta} \circ \bar{f}_* = \Phi \circ \bar{\xi}$, ($x \varepsilon \partial M$ and $y \varepsilon \partial N$);

(iii) $f_1 : S^{n-1} \to \bar{S}^{n-1}$, such that $\bar{\eta} \circ f_{*1} = \Phi \circ \xi$, ($x\varepsilon$ Int M and $y \varepsilon \partial N$);

(iv) $f_2 : \bar{S}^{n-1} \to S^{n-1}$ such that $\eta \circ f_{*2} = \Phi_o, \xi$ ($x \varepsilon \partial M$ and $y \varepsilon$ IntN).

Again we say that any map $f : S^o \to S^o$, $\bar{f} : \bar{S^o} \to \bar{S^o}$, $f_1 : S^o \to \bar{S}^o$ or $f_2 : \bar{S}^o \to S^o$ is a topological folding.

These definitions imply immediately that : If $\Phi \varepsilon \tau$ (M,N) is a topological folding of M onto N where $\partial M = \partial N \neq \varnothing$, and $\Phi(\partial M) \subset \partial N$, then $\Phi | \partial M \varepsilon \tau (\partial M, \partial N)$.

As before, any such topological folding determines a stratification S on M in which each stratum is a manifold without boundary, and S restricts to a stratification ∂S on ∂M. In constructing this stratification we have considered points in ∂M separately. Thus the set $\Sigma(\Phi)$ of singularities of Φ is a proper subset of the union of the strata of dimension $\leq$ m—1. This is because the $\Phi | \partial M$-regions of ∂M are (m—1)-strata in S, but Φ is not singular on these strata.

THE GRAPH TOPOLOGICAL FOLDING

Let $\Phi \varepsilon J$ (M,N). Then, as we saw in the fourht section above, there is a topological stratification S on M by singularities of Φ. In this section we show that there is a graph Γ_Φ associated with this stratification in a natural way. In fact the vertices of Γ_Φ are just the n-strata of S, and its edges are the (n—1)-strata. If $E \varepsilon S_{n-1}$, then E lies in the frontiers of exactly two n-strata σ, $\sigma' \varepsilon S_n$. We then say that E is an edge in Γ_Φ with end points σ, σ'.

The graph Γ_Φ can be realised as a graph $\tilde{\Gamma}_\Phi$ embedded in M, as follows. For each n-stratum $\sigma \varepsilon S_n$, choose any point $\tilde{\sigma} \varepsilon \sigma$. If σ, σ' εS_n are end-points of $E \varepsilon S_{n-1}$, then we can join $\tilde{\sigma}$ to $\tilde{\sigma}'$ by an arc $\tilde{E}$ in M that runs from $\tilde{\sigma}$ through σ and σ' to $\tilde{\sigma}$, crossing E transversely at a single point. Trivially, the correspondence $\sigma \rightarrow \sigma$, $\tilde{E} \rightarrow \tilde{E}$ is a graph isomorphism from Γ_Φ to $\tilde{\Gamma}_\Phi$. Figure (3) below illustrate this relationship in case n=2.

In this case, the cell complex subdivision of the surface M induced by $\tilde{\Gamma}_\Phi$ is the dual of that induced by K_Φ. These constructions have a greater significance in the case of a special sort of foldings which we call neat.

It should be noted that the graph Γ_Φ may have more than one edge joining a given pair of verticet. For instance, consider the topological folding Φ of the torus T into itself shown in Figure (4) below, induced by the map $\Phi : R_3$ given by Φ (x,y,z) = (x,y, |z|). The graph Γ_Φ has just two vertices but has two edges. See figure (4).

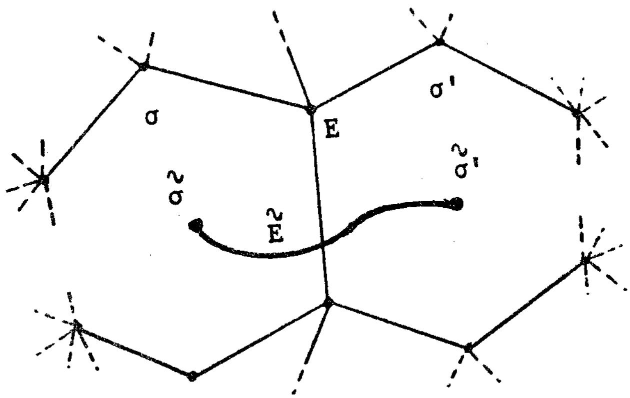

Figure (3)

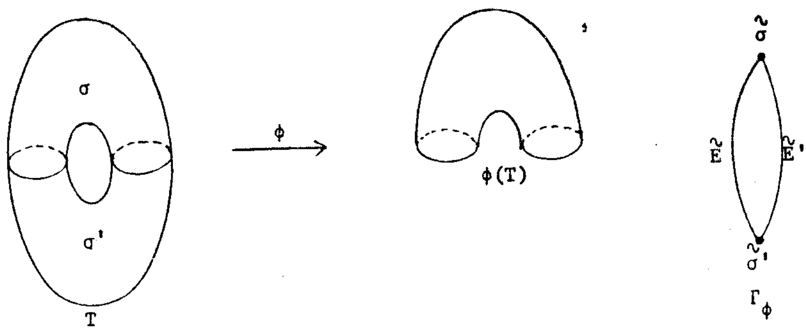

Figure (4)

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