http://communications.science.ankara.edu.tr

Commun.Fac.Sci.Univ.Ank.Ser. A1 Math. Stat.
Volume 71, Number 1, Pages 133–152 (2022)
DOI:10.31801/cfsuasmas.895598
ISSN 1303-5991 E-ISSN 2618-6470

Research Article; Received: March 12, 2021; Accepted: August 5, 2021

ON THE MATRIX REPRESENTATION OF 5^{th} ORDER BÉZIER CURVE AND ITS DERIVATIVES IN E^3

Şeyda KILIÇOĞLU[1] and Süleyman ŞENYURT[2]

[1]Department of Mathematics, Başkent University, Ankara, TURKEY
[2]Department of Mathematics, Ordu University, Ordu, TURKEY

Abstract. Using the matrix representation form, the first, second, third, fourth, and fifth derivatives of 5th order Bézier curves are examined based on the control points in E^3. In addition to this, each derivative of 5th order Bézier curves is given by their control points. Further, a simple way has been given to find the control points of a Bézier curves and its derivatives by using matrix notations. An example has also been provided and the corresponding figures which are drawn by Geogebra v5 have been presented in the end.

1. Introduction

French engineer Pierre Bézier, who used Bézier curves to design automobile bodies studied with them in 1962. But the study of these curves was first developed in 1959 by mathematician Paul de Casteljau using deCasteljau's algorithm, a numerically stable method to evaluate Bézier curves. A Bézier curve is frequently used in computer graphics and related fields, in vector graphics, and in animations as a tool to control motion. To guarantee smoothness, the control points at which two curves meet must be on the line between two control points on either side. In animation applications, such as Adobe Flash and Synfig, Bézier curves are used to outline, for example, movement. Users outline the wanted path in Bézier curves, and the application creates the needed frames for the object to move along the path. For 3D animation Bézier curves are often used to define 3D paths as well as 2D curves for key frame interpolation. We have been motivated by the following studies. First Bézier-curves with curvature and torsion continuity has been examined in [6]. Also in [4], [7] and [10], Bézier curves and surfaces has been given. In [1] and [5], Bézier curves are designed for Computer-Aided Geometric Designs.

2020 *Mathematics Subject Classification.* 53A04-53A05.
Keywords. Bézier curve, cubic Bézier curve, derivatives of Bézier curves, matrix representation.
seyda@baskent.edu.tr-Corresponding author; senyurtsuleyman52@gmail.com
0000-0001-8535-944X; 0000-0003-1097-5541.

Recently equivalence conditions of control points and application to planar Bézier curves have been examined in [8] and [9].The Serret-Frenet frame and curvatures of Bézier curves are examined those in E^4 in [3]. Frenet apparatus of the cubic Bézier curves and involute of the cubic Bezier curve by using matrix representation have been examined in E^3, in [11] and [12], respectively.

2. Preliminaries

A Bézier curve is defined by a set of control points P_0 through P_n, where n is called its order. If $n = 1$ for linear, if $n = 2$ for quadratic, if $n = 3$ for cubic Bézier curve, etc. The first and last control points are always the end points of the curve; however, the intermediate control points (if any) generally do not lie on the curve. Generally, Béziers curve can be defined by $n + 1$ control points $P_0, P_1, ..., P_n$ and has the following form:

$$\mathbf{B}(t) = \sum_{I=0}^{n} \binom{n}{I} t^I (1-t)^{n-I} (t) \quad [P_I] \ , \quad t \in [0,1] \, ,$$

where $\binom{n}{I} = \frac{n!}{I!(n-I)!}$ are the binomial coefficients [2]. The points P_I are called control points for the Bézier curve. The polygon formed by connecting the Bézier points with lines, starting with P_0 and finishing with P_n, is called the Bézier polygon (or control polygon). The convex hull of the Bézier polygon contains the Bézier curve.

The derivatives of the any Bézier curve $\mathbf{B}(t)$ is

$$\mathbf{B}'(t) = \sum_{i=0}^{n-1} \binom{n-1}{i} t^i (1-t)^{n-i-1} Q_i$$

where $Q_0 = n(P_1 - P_0), Q_1 = n(P_2 - P_1), Q_2 = n(P_3 - P_2), ...$ [2].

Given points P_0 and P_1, a linear Bézier curve is simply a straight line between those two points. Linear Bézier curve is given by

$$\boldsymbol{\alpha}(t) = (1-t) P_0 + t P_1$$

and also it has the matrix form with control points P_0 and P_1

$$\boldsymbol{\alpha}(t) = \begin{bmatrix} t & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \end{bmatrix}.$$

A quadratic Bézier curve is the path traced by the function $\boldsymbol{\alpha}(t)$, given points P_0, P_1 and P_2, which can be interpreted as the linear interpolant of corresponding points on the linear Bézier curves from P_0 to P_1 and from P_1 to P_2, respectively. A quadratic Bézier curve has also the matrix form with control points P_0 , P_1 and P_2

$$\boldsymbol{\alpha}(t) = \begin{bmatrix} t^2 & t & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \end{bmatrix}.$$

Four points P_0, P_1, P_2, P_3, and P_4 in the plane or in higher-dimensional space define a cubic Bézier curve with the following equation

$$\alpha(t) = (1-t)^3 P_0 + 3t(1-t)^2 P_1 + 3t^2(1-t) P_2 + t^3 P_3.$$

We have already examined the cubic Bézier curves and involutes in [11] and [12], respectively. The matrix form of the cubic Bézier curve with control points P_0, P_1, P_2, and P_3 is

$$\alpha(t) = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{bmatrix}.$$

The matrix form of the first derivative of a cubic Bézier curve based on the control points P_0, P_1, P_2, and P_3 is

$$\alpha'(t) = \begin{bmatrix} t^2 & t & 1 \end{bmatrix} \begin{bmatrix} -3 & 9 & -9 & 3 \\ 6 & -12 & 6 & 0 \\ -3 & 3 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{bmatrix}.$$

The first derivative of a cubic Bézier curve is a quadratic Bézier curve with control points $Q_0 = 3(P_1 - P_0)$, $Q_1 = 3(P_2 - P_1)$, and $Q_2 = 3(P_3 - P_2)$,

$$\alpha'(t) = \begin{bmatrix} t^2 & t & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 3(P_1 - P_0) \\ 3(P_2 - P_1) \\ 3(P_3 - P_2) \end{bmatrix}.$$

The matrix form of the second derivative of a cubic Bézier curve based on the control points P_0, P_1, P_2, and P_3 is

$$\alpha''(t) = \begin{bmatrix} t & 1 \end{bmatrix} \begin{bmatrix} -6 & 18 & -18 & 6 \\ 6 & -12 & 6 & 0 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{bmatrix}.$$

The second derivative of a cubic Bézier curve is a linear Bézier curve with control points $6(P_2 - 2P_1 + P_0)$, and $6(P_3 - 2P_2 + P_1)$,

$$\alpha''(t) = \begin{bmatrix} t & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 6(P_2 - 2P_1 + P_0) \\ 6(P_3 - 2P_2 + P_1) \end{bmatrix}.$$

Five points P_0, P_1, P_2, P_3, and P_4 in the plane or in higher-dimensional space define a $4th$ *order* Bézier curve with the following equation

$$\boldsymbol{\alpha}(t) = \sum_{I=0}^{4} \binom{4}{I} t^I (1-t)^{4-I} (t) [P_I] \ , \quad t \in [0,1].$$

The matrix form of the $4th$ *order* Bézier curve based on the control points is

$$\alpha(t)=\left[\begin{array}{ccccc} t^4 & t^3 & t^2 & t & 1 \end{array}\right]\left[\begin{array}{ccccc} 1 & -4 & 6 & -4 & 1 \\ -4 & 12 & -12 & 4 & 0 \\ 6 & -12 & 6 & 0 & 0 \\ -4 & 4 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{array}\right]\left[\begin{array}{c} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \end{array}\right].$$

3. 5^{th} Order Bézier Curve and Its Derivatives

Definition 1. *In the plane or in higher-dimensional space define a 5^{th} order Bézier curve with six points P_0, P_1, P_2, P_3, P_4, and P_5 and it has the following equation*

$$\boldsymbol{\alpha}(t)=\sum_{I=0}^{5}\binom{5}{I}t^I\left(1-t\right)^{5-I}(t)\left[P_I\right]\ ,\quad t\in[0,1]\,.$$

Theorem 1. *The matrix representation of 5^{th} order Bézier curve with control points P_0, P_1, P_2, P_3, P_4, and P_5 is*

$$\alpha(t)=\left[\begin{array}{cccccc} t^5 & t^4 & t^3 & t^2 & t & 1 \end{array}\right]\left[\begin{array}{cccccc} -1 & 5 & -10 & 10 & -5 & 1 \\ 5 & -20 & 30 & -20 & 5 & 0 \\ -10 & 30 & -30 & 10 & 0 & 0 \\ 10 & -20 & 10 & 0 & 0 & 0 \\ -5 & 5 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{array}\right]\left[\begin{array}{c} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \end{array}\right].$$

Proof. We have already found that

$$\alpha(t)=\left[\begin{array}{cccccc} t^5 & t^4 & t^3 & t^2 & t & 1 \end{array}\right][5Bc]\left[\begin{array}{c} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \end{array}\right],$$

where $[5Bc]_{6\times 6}$ is the coefficient matrix of 5^{th} *order* of Bézier curve. "$[5Bc]_{6\times 6}$ " is obtained by the initial letters of "5^{th} *order* Bézier curve" , and the coefficient matrix of 5^{th}*degree* Bézier curve is

$$\left[\begin{array}{cccccc} -\binom{5}{0}\binom{5}{5} & \binom{5}{1}\binom{4}{4} & -\binom{5}{2}\binom{3}{3} & \binom{5}{3}\binom{2}{2} & -\binom{5}{4}\binom{1}{1} & \binom{5}{5}\binom{0}{0} \\ \binom{5}{0}\binom{5}{4} & -\binom{5}{1}\binom{4}{3} & \binom{5}{2}\binom{3}{2} & -\binom{5}{3}\binom{2}{1} & \binom{5}{4}\binom{1}{0} & 0 \\ -\binom{5}{0}\binom{5}{3} & \binom{5}{1}\binom{4}{2} & -\binom{5}{2}\binom{3}{1} & \binom{5}{3}\binom{2}{0} & 0 & 0 \\ \binom{5}{0}\binom{5}{2} & -\binom{5}{1}\binom{4}{1} & \binom{5}{2}\binom{3}{0} & 0 & 0 & 0 \\ -\binom{5}{0}\binom{5}{1} & \binom{5}{1}\binom{4}{0} & 0 & 0 & 0 & 0 \\ \binom{5}{0}\binom{5}{0} & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Inverse matrix $[5Bc]$, *of* 5^{th} *order* of Bézier curve is

$$[5Bc]^{-1} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & \frac{1}{5} & 1 \\ 0 & 0 & 0 & \frac{1}{10} & \frac{2}{5} & 1 \\ 0 & 0 & \frac{1}{10} & \frac{3}{10} & \frac{3}{5} & 1 \\ 0 & \frac{1}{5} & \frac{2}{5} & \frac{3}{5} & \frac{4}{5} & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}_{6\times 6}.$$

□

Theorem 2. *The matrix representation of the first derivative of* 5^{th} *order of a Bézier curve with control points* P_0, P_1, P_2, ..., *and* P_5 *is*

$$\alpha'(t) = \begin{bmatrix} t^4 \\ t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} -5 & 25 & -50 & 50 & -25 & 5 \\ 20 & -80 & 120 & -80 & 20 & 0 \\ -30 & 90 & -90 & 30 & 0 & 0 \\ 20 & -40 & 20 & 0 & 0 & 0 \\ -5 & 5 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \end{bmatrix},$$

$$= \begin{bmatrix} t^4 \\ t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} 1 & -4 & 6 & -4 & 1 \\ -4 & 12 & -12 & 4 & 0 \\ 6 & -12 & 6 & 0 & 0 \\ -4 & 4 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -5 & 5 & 0 & 0 & 0 & 0 \\ 0 & -5 & 5 & 0 & 0 & 0 \\ 0 & 0 & -5 & 5 & 0 & 0 \\ 0 & 0 & 0 & -5 & 5 & 0 \\ 0 & 0 & 0 & 0 & -5 & 5 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \end{bmatrix}.$$

Also as a 4^{th} *order Bézier curve, the matrix representation of the first derivative of* 5^{th} *order of a Bézier curve with control points* Q_0, Q_1, ..., Q_4 *is*

$$\alpha'(t) = \begin{bmatrix} t^4 & t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} 1 & -4 & 6 & -4 & 1 \\ -4 & 12 & -12 & 4 & 0 \\ 6 & -12 & 6 & 0 & 0 \\ -4 & 4 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} Q_0 \\ Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{bmatrix},$$

where the control points, $(5P_1 - 5P_0)$, $(5P_2 - 5P_1)$, $(5P_3 - 5P_2)$, $(5P_4 - 5P_3)$, *and* $(5P_5 - 5P_4)$, *respectively.*

Proof. We have already found that

$$\alpha'(t) = \begin{bmatrix} t^4 & t^3 & t^2 & t & 1 \end{bmatrix} [5Bc]' \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \end{bmatrix},$$

where $[5Bc]'$ is the coefficient matrix of the first derivative of 5^{th} *order* of a Bézier curve defined by following

$$[5Bc]' = \begin{bmatrix} -5\binom{5}{0}\binom{5}{5} & 5\binom{5}{1}\binom{4}{4} & -5\binom{5}{2}\binom{3}{3} & 5\binom{5}{3}\binom{2}{2} & -5\binom{5}{4}\binom{1}{1} & 5\binom{5}{5}\binom{0}{0} \\ 4\binom{5}{0}\binom{5}{4} & -4\binom{5}{1}\binom{4}{3} & 4\binom{5}{2}\binom{3}{2} & -4\binom{5}{3}\binom{2}{1} & 4\binom{5}{4}\binom{1}{0} & 0 \\ -3\binom{5}{0}\binom{5}{3} & 3\binom{5}{1}\binom{4}{2} & -3\binom{5}{2}\binom{3}{1} & 3\binom{5}{3}\binom{2}{0} & 0 & 0 \\ 2\binom{5}{0}\binom{5}{2} & -2\binom{5}{1}\binom{4}{1} & 2\binom{5}{2}\binom{3}{0} & 0 & 0 & 0 \\ -\binom{5}{0}\binom{5}{1} & \binom{5}{1}\binom{4}{0} & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$= \begin{bmatrix} -5 & 25 & -50 & 50 & -25 & 5 \\ 20 & -80 & 120 & -80 & 20 & 0 \\ -30 & 90 & -90 & 30 & 0 & 0 \\ 20 & -40 & 20 & 0 & 0 & 0 \\ -5 & 5 & 0 & 0 & 0 & 0 \end{bmatrix},$$

and thus,

$$\alpha'(t) = \begin{bmatrix} t^4 & t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} -5 & 25 & -50 & 50 & -25 & 5 \\ 20 & -80 & 120 & -80 & 20 & 0 \\ -30 & 90 & -90 & 30 & 0 & 0 \\ 20 & -40 & 20 & 0 & 0 & 0 \\ -5 & 5 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \end{bmatrix}. \quad (1)$$

Also the first derivative of 5^{th} *order* of a Bézier curve is a 4^{th} *order* Bézier curve. Hence, the matrix representation of 4^{th} *order* Bézier curve with control points Q_0, Q_1, ..., Q_4 is

$$\alpha'(t) = \begin{bmatrix} t^4 & t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} 1 & -4 & 6 & -4 & 1 \\ -4 & 12 & -12 & 4 & 0 \\ 6 & -12 & 6 & 0 & 0 \\ -4 & 4 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} Q_0 \\ Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{bmatrix}, \quad (2)$$

where $Q_0 = 5P_1 - 5P_0, Q_1 = 5P_2 - 5P_1, Q_2 = 5P_3 - 5P_2, Q_3 = 5P_4 - 5P_3$ and $Q_4 = 5P_5 - 5P_4$ are the control points. From (1) and (2), we write

$$\begin{bmatrix} 1 & -4 & 6 & -4 & 1 \\ -4 & 12 & -12 & 4 & 0 \\ 6 & -12 & 6 & 0 & 0 \\ -4 & 4 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} Q_0 \\ Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{bmatrix} = \begin{bmatrix} -5 & 25 & -50 & 50 & -25 & 5 \\ 20 & -80 & 120 & -80 & 20 & 0 \\ -30 & 90 & -90 & 30 & 0 & 0 \\ 20 & -40 & 20 & 0 & 0 & 0 \\ -5 & 5 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \end{bmatrix}.$$

Since,

$$\begin{bmatrix} 1 & -4 & 6 & -4 & 1 \\ -4 & 12 & -12 & 4 & 0 \\ 6 & -12 & 6 & 0 & 0 \\ -4 & 4 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & \frac{1}{4} & 1 \\ 0 & 0 & \frac{1}{6} & \frac{1}{2} & 1 \\ 0 & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix},$$

we have

$$\begin{bmatrix} Q_0 \\ Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & \frac{1}{4} & 1 \\ 0 & 0 & \frac{1}{6} & \frac{1}{2} & 1 \\ 0 & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -5 & 25 & -50 & 50 & -25 & 5 \\ 20 & -80 & 120 & -80 & 20 & 0 \\ -30 & 90 & -90 & 30 & 0 & 0 \\ 20 & -40 & 20 & 0 & 0 & 0 \\ -5 & 5 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \end{bmatrix},$$

$$= \begin{bmatrix} -5 & 5 & 0 & 0 & 0 & 0 \\ 0 & -5 & 5 & 0 & 0 & 0 \\ 0 & 0 & -5 & 5 & 0 & 0 \\ 0 & 0 & 0 & -5 & 5 & 0 \\ 0 & 0 & 0 & 0 & -5 & 5 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \end{bmatrix},$$

$$= \begin{bmatrix} 5P_1 - 5P_0 \\ 5P_2 - 5P_1 \\ 5P_3 - 5P_2 \\ 5P_4 - 5P_3 \\ 5P_5 - 5P_4 \end{bmatrix},$$

or equivalently we may write

$$\alpha'(t) = \begin{bmatrix} t^4 \\ t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} 1 & -4 & 6 & -4 & 1 \\ -4 & 12 & -12 & 4 & 0 \\ 6 & -12 & 6 & 0 & 0 \\ -4 & 4 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -5 & 5 & 0 & 0 & 0 & 0 \\ 0 & -5 & 5 & 0 & 0 & 0 \\ 0 & 0 & -5 & 5 & 0 & 0 \\ 0 & 0 & 0 & -5 & 5 & 0 \\ 0 & 0 & 0 & 0 & -5 & 5 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \end{bmatrix}.$$

□

Theorem 3. *The matrix representation of the second derivative of* 5^{th} *order of a Bézier curve with control points* P_0, P_1, P_2, ..., P_5 *is*

$$\alpha''(t) = \begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} -20 & 100 & -200 & 200 & -100 & 20 \\ 60 & -240 & 360 & -240 & 60 & 0 \\ -60 & 180 & -180 & 60 & 0 & 0 \\ 20 & -40 & 20 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \end{bmatrix},$$

$$= \begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 20 & -40 & 20 & 0 & 0 & 0 \\ 0 & 20 & -40 & 20 & 0 & 0 \\ 0 & 0 & 20 & -40 & 20 & 0 \\ 0 & 0 & 0 & 20 & -40 & 20 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \end{bmatrix}.$$

Also as a cubic Bézier curve, it has the following form

$$\alpha^{''}(t) = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} R_0 \\ R_1 \\ R_2 \\ R_3 \end{bmatrix},$$

where the control points R_0*,* R_1*,, ...,* R_3 *are given by*

$$\begin{bmatrix} R_0 \\ R_1 \\ R_2 \\ R_3 \end{bmatrix} = \begin{bmatrix} 20P_0 - 40P_1 + 20P_2 \\ 20P_1 - 40P_2 + 20P_3 \\ 20P_2 - 40P_3 + 20P_4 \\ 20P_3 - 40P_4 + 20P_5 \end{bmatrix}.$$

Proof. We have already found $\alpha^{''}(t)$, therefore the coefficient matrix of the second derivative of $5^{th}order$ of a Bézier curve is

$$[5Bc]^{''} = \begin{bmatrix} -5.4\binom{5}{0}\binom{5}{5} & 5.4\binom{5}{1}\binom{4}{4} & -5.4\binom{5}{2}\binom{3}{3} & 5.4\binom{5}{3}\binom{2}{2} & -5.4\binom{5}{4}\binom{1}{1} & 5.4\binom{5}{5}\binom{0}{0} \\ 4.3\binom{5}{0}\binom{5}{4} & -4.3\binom{5}{1}\binom{4}{3} & 4.3\binom{5}{2}\binom{3}{2} & -4.3\binom{5}{3}\binom{2}{1} & 4.3\binom{5}{4}\binom{1}{0} & 0 \\ -3.2\binom{5}{0}\binom{5}{3} & 3.2\binom{5}{1}\binom{4}{2} & -3.2\binom{5}{2}\binom{3}{1} & 3.2\binom{5}{3}\binom{2}{0} & 0 & 0 \\ 2\binom{5}{0}\binom{5}{2} & -2\binom{5}{1}\binom{4}{1} & 2\binom{5}{2}\binom{3}{0} & 0 & 0 & 0 \end{bmatrix},$$

$$= \begin{bmatrix} -20 & 100 & -200 & 200 & -100 & 20 \\ 60 & -240 & 360 & -240 & 60 & 0 \\ -60 & 180 & -180 & 60 & 0 & 0 \\ 20 & -40 & 20 & 0 & 0 & 0 \end{bmatrix}.$$

By the definition of a cubic Bézier curve that

$$\alpha^{''}(t) = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} R_0 \\ R_1 \\ R_2 \\ R_3 \end{bmatrix},$$

and by using the equality of these, we get

$$\begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} R_0 \\ R_1 \\ R_2 \\ R_3 \end{bmatrix} = \begin{bmatrix} -20 & 100 & -200 & 200 & -100 & 20 \\ 60 & -240 & 360 & -240 & 60 & 0 \\ -60 & 180 & -180 & 60 & 0 & 0 \\ 20 & -40 & 20 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \end{bmatrix}.$$

Since inverse is

$$\begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{1}{3} & 1 \\ 0 & \frac{1}{3} & \frac{2}{3} & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix},$$

we have

$$\begin{bmatrix} R_0 \\ R_1 \\ R_2 \\ R_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{1}{3} & 1 \\ 0 & \frac{1}{3} & \frac{2}{3} & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -20 & 100 & -200 & 200 & -100 & 20 \\ 60 & -240 & 360 & -240 & 60 & 0 \\ -60 & 180 & -180 & 60 & 0 & 0 \\ 20 & -40 & 20 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \end{bmatrix},$$

$$= \begin{bmatrix} 20 & -40 & 20 & 0 & 0 & 0 \\ 0 & 20 & -40 & 20 & 0 & 0 \\ 0 & 0 & 20 & -40 & 20 & 0 \\ 0 & 0 & 0 & 20 & -40 & 20 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \end{bmatrix}.$$

Here,

$$R_0 = 20P_0 - 40P_1 + 20P_2, \quad R_1 = 20P_1 - 40P_2 + 20P_3,$$
$$R_2 = 20P_2 - 40P_3 + 20P_4, \quad R_3 = 20P_3 - 40P_4 + 20P_5$$

are the control points. By combining the calculations above, we finally write

$$\alpha^{''}(t) = \begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 20 & -40 & 20 & 0 & 0 & 0 \\ 0 & 20 & -40 & 20 & 0 & 0 \\ 0 & 0 & 20 & -40 & 20 & 0 \\ 0 & 0 & 0 & 20 & -40 & 20 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \end{bmatrix},$$

which completes the proof. □

Theorem 4. *The matrix representation of the third derivative of a* 5^{th} *order Bézier curve with control points* P_0, P_1, P_2, ..., P_5 *is*

$$\alpha'''(t) = \begin{bmatrix} t^2 & t & 1 \end{bmatrix} \begin{bmatrix} -60 & 300 & -600 & 600 & -300 & 60 \\ 120 & -480 & 720 & -480 & 120 & 0 \\ -120 & 360 & -360 & 120 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \end{bmatrix},$$

$$= \begin{bmatrix} t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} 1 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} -120 & 360 & -360 & 120 & 0 & 0 \\ -60 & 120 & 0 & -120 & 60 & 0 \\ -60 & 180 & -240 & 240 & -180 & 60 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \end{bmatrix}.$$

Also, since the third derivative of 5^{th} *order of a Bézier curve is a quadratic Bézier curve, with control points* S_0, S_1, *and* S_2 , $\alpha'''(t)$ *has the following representation*

$$\alpha^{'''}(t) = \begin{bmatrix} t^2 & t & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} S_0 \\ S_1 \\ S_2 \end{bmatrix},$$

where

$$\begin{aligned} S_0 &= -60P_0 + 180P_1 - 180P_2 + 60P_3, \\ S_1 &= -60P_1 + 180P_2 - 180P_3 + 60P_4, \\ S_2 &= -60P_2 + 180P_3 - 180P_4 + 60P_5. \end{aligned}$$

Proof. We have already found that

$$\alpha'''(t) = \begin{bmatrix} t^2 & t & 1 \end{bmatrix} [5Bc]''' \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \end{bmatrix},$$

where the coefficient matrix of the third derivative of $5^{th}order$ of a Bézier curve is

$$[5Bc]''' = \begin{bmatrix} -5.4.3\binom{5}{0}\binom{5}{5} & 5.4.3\binom{5}{1}\binom{4}{4} & -5.4.3\binom{5}{2}\binom{3}{3} & 5.4.3\binom{5}{3}\binom{2}{2} & -5.4.3\binom{5}{4}\binom{1}{1} & 5.4.3\binom{5}{5}\binom{0}{0} \\ 4.3.2\binom{5}{0}\binom{5}{4} & -4.3.2\binom{5}{1}\binom{4}{3} & 4.3.2\binom{5}{2}\binom{3}{2} & -4.3.2\binom{5}{3}\binom{2}{1} & 4.3.2\binom{5}{4}\binom{1}{0} & 0 \\ -3.2\binom{5}{0}\binom{5}{3} & 3.2\binom{5}{1}\binom{4}{2} & -3.2\binom{5}{2}\binom{3}{1} & 3.2\binom{5}{3}\binom{2}{0} & 0 & 0 \end{bmatrix},$$

$$= \begin{bmatrix} -60 & 300 & -600 & 600 & -300 & 60 \\ 120 & -480 & 720 & -480 & 120 & 0 \\ -60 & 180 & -180 & 60 & 0 & 0 \end{bmatrix}.$$

Hence

$$\alpha'''(t)=\begin{bmatrix} t^2 & t & 1 \end{bmatrix}\begin{bmatrix} -60 & 300 & -600 & 600 & -300 & 60 \\ 120 & -480 & 720 & -480 & 120 & 0 \\ -60 & 180 & -180 & 60 & 0 & 0 \end{bmatrix}\begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \end{bmatrix}.$$

Also Bézier curve is a quadratic curve with control points S_0, S_1 and S_2, it has the following form

$$\alpha^{'''}(t)=\begin{bmatrix} t^2 & t & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} S_0 \\ S_1 \\ S_2 \end{bmatrix}.$$

By using the equality of these, we get

$$\begin{bmatrix} 1 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} S_0 \\ S_1 \\ S_2 \end{bmatrix}=\begin{bmatrix} -60 & 300 & -600 & 600 & -300 & 60 \\ 120 & -480 & 720 & -480 & 120 & 0 \\ -60 & 180 & -180 & 60 & 0 & 0 \end{bmatrix}\begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \end{bmatrix}.$$

Since again the inverse is

$$\begin{bmatrix} 1 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix}^{-1}=\begin{bmatrix} 0 & 0 & 1 \\ 0 & \frac{1}{2} & 1 \\ 1 & 1 & 1 \end{bmatrix},$$

we have

$$\begin{bmatrix} S_0 \\ S_1 \\ S_2 \end{bmatrix}=\begin{bmatrix} 0 & 0 & 1 \\ 0 & \frac{1}{2} & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} -60 & 300 & -600 & 600 & -300 & 60 \\ 120 & -480 & 720 & -480 & 120 & 0 \\ -60 & 180 & -180 & 60 & 0 & 0 \end{bmatrix}\begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \end{bmatrix},$$

$$=\begin{bmatrix} -60 & 180 & -180 & 60 & 0 & 0 \\ 0 & -60 & 180 & -180 & 60 & 0 \\ 0 & 0 & -60 & 180 & -180 & 60 \end{bmatrix}\begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \end{bmatrix}.$$

or correspondingly,

$$\alpha'''(t)=\begin{bmatrix} t^2\\ t\\ 1\end{bmatrix}^T\begin{bmatrix} 1 & -2 & 1\\ -2 & 2 & 0\\ 1 & 0 & 0\end{bmatrix}\begin{bmatrix} -60 & 180 & -180 & 60 & 0 & 0\\ 0 & -60 & 180 & -180 & 60 & 0\\ 0 & 0 & -60 & 180 & -180 & 60\end{bmatrix}\begin{bmatrix} P_0\\ P_1\\ P_2\\ P_3\\ P_4\\ P_5\end{bmatrix}.$$

□

Theorem 5. *The matrix representation of the fourth derivative of a* 5^{th} *order Bézier curve with control points* P_0, P_1, P_2, ..., *and* P_5 *is*

$$\alpha^{(4)}(t)=\begin{bmatrix} t & 1\end{bmatrix}\begin{bmatrix} -120 & 600 & -1200 & 1200 & -600 & 120\\ 120 & -480 & 720 & -480 & 120 & 0\end{bmatrix}\begin{bmatrix} P_0\\ P_1\\ P_2\\ P_3\\ P_4\\ P_5\end{bmatrix},$$

$$=\begin{bmatrix} t & 1\end{bmatrix}\begin{bmatrix} -1 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix} 120P_0-480P_1+720P_2-480P_3+120P_4\\ 120P_1-480P_2+720P_3-480P_4+120P_5\end{bmatrix}.$$

Also the fourth derivative of a 5^{th} *order Bézier curve is a linear Bézier curve, with control points* T_0, *and* T_1, *and it has the following equation*

$$\alpha^{(4)}(t)=\begin{bmatrix} t & 1\end{bmatrix}\begin{bmatrix} -1 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix} T_0\\ T_1\end{bmatrix},$$

where

$$\begin{aligned} T_0 &= 120P_0-480P_1+720P_2-480P_3+120P_4,\\ T_1 &= 120P_1-480P_2+720P_3-480P_4+120P_5\end{aligned}$$

are the control points of the fourth derivative of a 5^{th}*order Bézier curve based on the points* P_0, P_1, P_2, ..., *and* P_5.

Proof. We have already found that

$$\alpha^{(4)}(t)=\begin{bmatrix} t & 1\end{bmatrix}[5Bc]^{(4)}\begin{bmatrix} P_0\\ P_1\\ P_2\\ P_3\\ P_4\\ P_5\end{bmatrix},$$

where the coefficient matrix of the fourth derivative of $5^{th}order$ of a Bézier curve is

$$[5Bc]^{(4)} = \begin{bmatrix} -5.4.3.2\binom{5}{0}\binom{5}{5} & 5.4.3.2\binom{5}{1}\binom{4}{4} & -5.4.3.2\binom{5}{2}\binom{3}{3} & 5.4.3.2\binom{5}{3}\binom{2}{2} & -5.4.3.2\binom{5}{4}\binom{1}{1} & 5.4.3.2\binom{5}{5}\binom{0}{0} \\ 4.3.2\binom{5}{0}\binom{5}{4} & -4.3.2\binom{5}{1}\binom{4}{3} & 4.3.2\binom{3}{2} & -4.3.2\binom{5}{3}\binom{2}{1} & 4.3.2\binom{5}{4}\binom{1}{0} & 0 \end{bmatrix},$$

$$= \begin{bmatrix} -120 & 600 & -1200 & 1200 & -600 & 120 \\ 120 & -480 & 720 & -480 & 120 & 0 \end{bmatrix}.$$

Hence

$$\alpha^{(4)}(t) = \begin{bmatrix} t & 1 \end{bmatrix} \begin{bmatrix} -120 & 600 & -1200 & 1200 & -600 & 120 \\ 120 & -480 & 720 & -480 & 120 & 0 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \end{bmatrix}.$$

And also as a linear Bézier curve it has the matrix form with control points T_0 and T_1

$$\alpha^{(4)}(t) = \begin{bmatrix} t & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} T_0 \\ T_1 \end{bmatrix}.$$

By using the equality of these, we get

$$\begin{bmatrix} -120 & 600 & -1200 & 1200 & -600 & 120 \\ 120 & -480 & 720 & -480 & 120 & 0 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} T_0 \\ T_1 \end{bmatrix},$$

Since the inverse matrix is

$$\begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix},$$

we get

$$\begin{bmatrix} T_0 \\ T_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -120 & 600 & -1200 & 1200 & -600 & 120 \\ 120 & -480 & 720 & -480 & 120 & 0 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \end{bmatrix}.$$

Therefore, the control points of the fourth derivative of a $5^{th}order$ Bézier curve based on the points P_0, P_1, P_2, ..., and P_5 are given by

$$T_0 = 120P_0 - 480P_1 + 720P_2 - 480P_3 + 120P_4,$$
$$T_1 = 120P_1 - 480P_2 + 720P_3 - 480P_4 + 120P_5,$$

and accordingly the matrix represented form of the curve is

$$\alpha^{(4)}(t)=\begin{bmatrix} t & 1 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 120P_0-480P_1+720P_2-480P_3+120P_4 \\ 120P_1-480P_2+720P_3-480P_4+120P_5 \end{bmatrix}.$$

□

Theorem 6. *The matrix representation of the fifth derivative of a 5^{th} order Bézier curve with control points P_0, P_1, P_2, ..., and P_5 is*

$$\alpha^{(5)}(t)=600P_1-120P_0-1200P_2+1200P_3-600P_4+120P_5.$$

Proof. It is clear that

$$\alpha^{(5)}(t)=[5Bc]^{(5)}\begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \end{bmatrix},$$

where $[5Bc]^{(5)}=\begin{bmatrix} -120 & 600 & -1200 & 1200 & -600 & 120 \end{bmatrix}$. □

Now, we may consider an example of a curve given by its parametric form. Our first attempt is to find its control points with the help of matrix representation. Second we examine its derivatives and their control points. Finally, we represent each control point of every derivatives by the control points of initial curve, and draw their correspondence figures by using a free-ware program Geogebra v5.

Example 1. *Let us consider the 5th order Bézier curve parameterized as*

$$\begin{aligned}\alpha(t)=\big(&74t^5-210t^4+180t^3-50t^2+5t+1,\\ &-79t^5+185t^4-130t^3+10t^2+10t+1,\\ &-63t^5+95t^4-30t^3-5t+2\big).\end{aligned}$$

To find the control points, we first write it as in the matrix product form by following:

$$\alpha(t)=\begin{bmatrix} t^5 & t^4 & t^3 & t^2 & t & 1 \end{bmatrix}\begin{bmatrix} 74 & -79 & -63 \\ -210 & 185 & 95 \\ 180 & -130 & -30 \\ -50 & 10 & 0 \\ 5 & 10 & -5 \\ 1 & 1 & 2 \end{bmatrix}.$$

Hence

$$\begin{bmatrix} 74 & -79 & -63 \\ -210 & 185 & 95 \\ 180 & -130 & -30 \\ -50 & 10 & 0 \\ 5 & 10 & -5 \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} -1 & 5 & -10 & 10 & -5 & 1 \\ 5 & -20 & 30 & -20 & 5 & 0 \\ -10 & 30 & -30 & 10 & 0 & 0 \\ 10 & -20 & 10 & 0 & 0 & 0 \\ -5 & 5 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & \frac{1}{5} & 1 \\ 0 & 0 & 0 & \frac{1}{10} & \frac{2}{5} & 1 \\ 0 & 0 & \frac{1}{10} & \frac{3}{10} & \frac{3}{5} & 1 \\ 0 & \frac{1}{5} & \frac{2}{5} & \frac{3}{5} & \frac{4}{5} & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 74 & -79 & -63 \\ -210 & 185 & 95 \\ 180 & -130 & -30 \\ -50 & 10 & 0 \\ 5 & 10 & -5 \\ 1 & 1 & 2 \end{bmatrix} = \mathbf{I} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \end{bmatrix},$$

$$\Rightarrow \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 1 \\ -2 & 6 & 0 \\ 7 & -3 & -4 \\ 5 & 0 & 5 \\ 0 & -3 & -1 \end{bmatrix},$$

where $\mathbf{I}$ *is a six by six identity matrix.*

Inversely, we find the parametric form of a $5th$ *order* Bézier curve, $\alpha(t)$ with control points $P_0 = (1,1,2)$, $P_1 = (2,3,1)$, $P_2 = (-2,6,0)$, $P_3 = (7,-3,-4)$, $P_4 = (5,0,5)$, $P_5 = (0,-3,-1)$ as follows:

$$\begin{aligned} \alpha(t) &= \begin{bmatrix} t^5 \\ t^4 \\ t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} -1 & 5 & -10 & 10 & -5 & 1 \\ 5 & -20 & 30 & -20 & 5 & 0 \\ -10 & 30 & -30 & 10 & 0 & 0 \\ 10 & -20 & 10 & 0 & 0 & 0 \\ -5 & 5 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 1 \\ -2 & 6 & 0 \\ 7 & -3 & -4 \\ 5 & 0 & 5 \\ 0 & -3 & -1 \end{bmatrix}, \\ &= \big(74t^5 - 210t^4 + 180t^3 - 50t^2 + 5t + 1, -79t^5 + 185t^4 - 130t^3 + 10t^2 + 10t + 1, \\ &\quad - 63t^5 + 95t^4 - 30t^3 - 5t + 2\big). \end{aligned}$$

Let us find the control points of the first derivative $\alpha'(t)$

$$\begin{aligned} \alpha'(t) = \big(&370t^4 - 840t^3 + 540t^2 - 100t + 5, -395t^4 + 740t^3 - 390t^2 + 20t + 10, \\ &- 315t^4 + 380t^3 - 90t^2 - 5\big). \end{aligned}$$

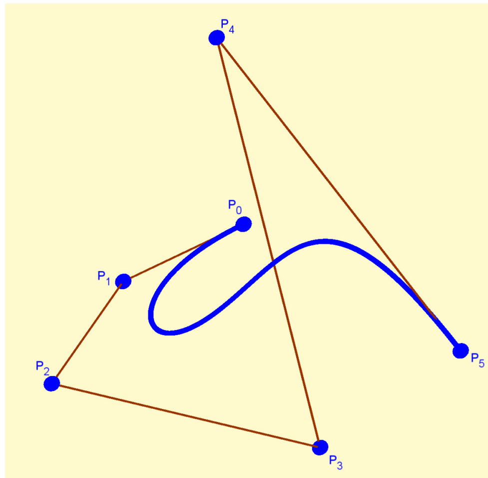

FIGURE 1. 5^{th} order Bézier curve with control points P_j $(j = 0, ..., 5)$

First we need to write its matrix product form as:

$$\alpha'(t) = \begin{bmatrix} t^4 & t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} 370 & -395 & -315 \\ -840 & 740 & 380 \\ 540 & -390 & -90 \\ -100 & 20 & 0 \\ 5 & 10 & -5 \end{bmatrix}.$$

Next, by equating the terms we have

$$\begin{bmatrix} t^4 \\ t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} 370 & -395 & -315 \\ -840 & 740 & 380 \\ 540 & -390 & -90 \\ -100 & 20 & 0 \\ 5 & 10 & -5 \end{bmatrix} = \begin{bmatrix} t^4 \\ t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} 1 & -4 & 6 & -4 & 1 \\ -4 & 12 & -12 & 4 & 0 \\ 6 & -12 & 6 & 0 & 0 \\ -4 & 4 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} Q_0 \\ Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{bmatrix},$$

$$\begin{bmatrix} 370 & -395 & -315 \\ -840 & 740 & 380 \\ 540 & -390 & -90 \\ -100 & 20 & 0 \\ 5 & 10 & -5 \end{bmatrix} = \begin{bmatrix} 1 & -4 & 6 & -4 & 1 \\ -4 & 12 & -12 & 4 & 0 \\ 6 & -12 & 6 & 0 & 0 \\ -4 & 4 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} Q_0 \\ Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{bmatrix},$$

$$\Longrightarrow \begin{bmatrix} Q_0 \\ Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{bmatrix} = \begin{bmatrix} 5 & 10 & -5 \\ -20 & 15 & -5 \\ 45 & -45 & -20 \\ -10 & 15 & 45 \\ -25 & -15 & -30 \end{bmatrix},$$

$$\begin{bmatrix} Q_0 \\ Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{bmatrix} = \begin{bmatrix} -5 & 5 & 0 & 0 & 0 & 0 \\ 0 & -5 & 5 & 0 & 0 & 0 \\ 0 & 0 & -5 & 5 & 0 & 0 \\ 0 & 0 & 0 & -5 & 5 & 0 \\ 0 & 0 & 0 & 0 & -5 & 5 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \end{bmatrix},$$

$$\begin{bmatrix} 5 & 10 & -5 \\ -20 & 15 & -5 \\ 45 & -45 & -20 \\ -10 & 15 & 45 \\ -25 & -15 & -30 \end{bmatrix} = \begin{bmatrix} -5 & 5 & 0 & 0 & 0 & 0 \\ 0 & -5 & 5 & 0 & 0 & 0 \\ 0 & 0 & -5 & 5 & 0 & 0 \\ 0 & 0 & 0 & -5 & 5 & 0 \\ 0 & 0 & 0 & 0 & -5 & 5 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \end{bmatrix},$$

$$\begin{bmatrix} 5 & 10 & -5 \\ -20 & 15 & -5 \\ 45 & -45 & -20 \\ -10 & 15 & 45 \\ -25 & -15 & -30 \end{bmatrix} = \begin{bmatrix} 5P_1 - 5P_0 \\ 5P_2 - 5P_1 \\ 5P_3 - 5P_2 \\ 5P_4 - 5P_3 \\ 5P_5 - 5P_4 \end{bmatrix}.$$

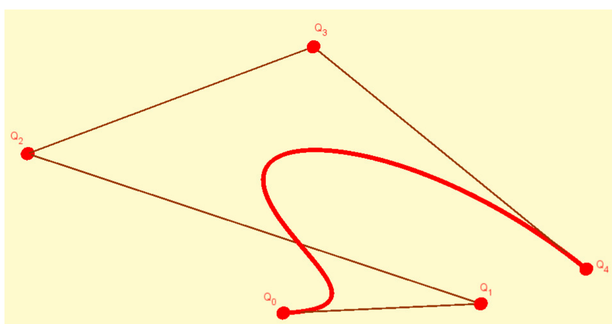

FIGURE 2. 1^{st} derivative of a 5^{th} order Bézier curve with control points Q_j $(j = 0, ..., 4)$

By following same steps given above, we may find the control points of the second and third derivative of the curve $\alpha(t)$ and draw them as in Fig. 3 and Fig. 4.

$$\alpha''(t) = \big(1080t - 2520t^2 + 1480t^3 - 100, -780t + 2220t^2 - 1580t^3 + 20$$

$$-180t + 1140t^2 - 1260t^3\big) ,$$

$$\alpha'''(t) = \big(-5040t + 4440t^2 + 1080, 4440t - 4740t^2 - 780, 2280t - 3780t^2 - 180\big) .$$

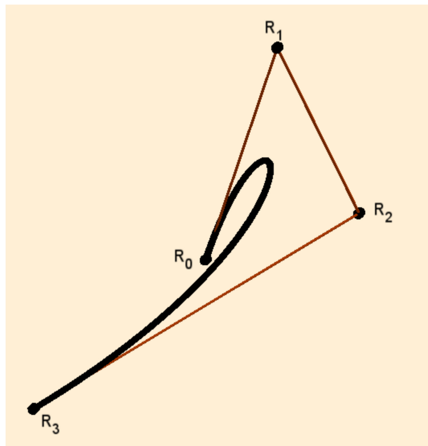

FIGURE 3. 2^{nd} derivative of a 5^{th} order Bézier curve with control points R_j $(j = 0, ..., 3)$

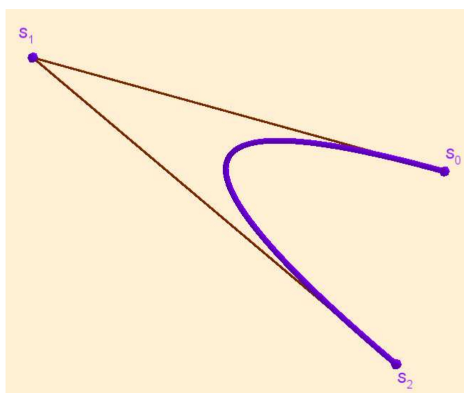

FIGURE 4. 3^{th} derivative of a 5^{th} order Bézier curve with control points S_j $(j = 0, ..., 2)$

The fourth derivative of the curve, $\alpha(t)$ is simply draws a line while the fifth derivative is a single point:

$$\alpha^{(4)}(t) = (8880t - 5040, -9480t + 4440, -7560t + 2280) ,$$

$$\alpha^{(5)}(t) = (8880, -9480, -7560) .$$

4. Conclusion

We can write the parametric form of 5^{th} order Bézier curve using a simple matrix product. Further, we can find the control points using a simple matrix product, inversely. Also the second derivative of a 5^{th} order Bézier curve with the control points P_i, $(i = 0, ..., 4)$ can be considered another 4^{th} order Bézier curve having $(5+1)-2 = 4$ control points as $R_j = n(n-1)\left(P_j - 2P_{j+1} + P_{j+2}\right)$, $j = 0, ..., 3$.The third derivative of a 5^{th} order Bézier curve with the control points P_i, $(i = 0, ..., 5)$ can be considered another cubic Bézier curve having $(5+1)-3 = 3$ control points as $S_j = n(n-1)(n-2)(-P_j + 3P_{j+1} - 3P_{j+2} + P_{j+3})$, $j = 0, ..., 2$.The third derivative of an 5^{th} order Bézier curve with the control points P_i, $(i = 0, ..., 5)$, can be considered a quadratic Bézier curve having $(5+1)-3 = 5-2 = 3$ control points as $N_j = n(n-1)(n-2)\left(-P_j + 3P_{j+1} - 3P_{j+2} + P_{j+3}\right)$, $j = 0, ..., 2$.

Author Contribution Statements All authors jointly worked on the results and findings. They both read and approved the final manuscript.

Declaration of Competing Interests There is no competing interest between the authors to declare.

References

[1] Marsh, D., Applied Geometry for Computer Graphics and CAD, Springer Science and Business Media, 2006.

[2] Derivatives of a Bézier Curve, https://pages.mtu.edu/ 126 shene/COURSES/cs3621 /NOTES/spline /Bezier/bezier-der.html

[3] Erkan, E., Yüce, S., Some notes on geometry of Bézier curves in Euclidean 4-space, *Journal of Engineering Technology and Applied Sciences*, 5(3) (2020), 93-101. https://doi.org/10.30931/jetas.837921

[4] Taş, F., İlarslan, K., A new approach to design the ruled surface, *International Journal of Geometric Methods in Modern Physics*, 16(6) (2019), 1950093. https://doi.org/10.1142/S0219887819500932

[5] Farin, G., Curves and Surfaces for Computer-Aided Geometric Design, Academic Press, 1996.

[6] Hagen, H., Bézier-curves with curvature and torsion continuity, *Rocky Mountain J. Math.*, 16(3) (1986), 629-638. https://doi.org/10.1216/RMJ-1986-16-3-629

[7] Zhang, J. W. C., Jieqing, F., Bézier curves and surfaces, *Graphical Models and Image Processing*, 61(1) (1999), 2-15.

[8] İncesu, M., LS (3)-equivalence conditions of control points and application to spatial Bézier curves and surfaces, *AIMS Mathematics*, 5(2) (2020) 1216-1246. https://doi.org/10.3934/math.2020084

[9] İncesu, M., Gursoy, O., LS (2)-Equivalence conditions of control points and application to planar Bézier curves, *New Trends in Mathematical Sciences*, 5(3) (2017), 70-84. https://doi.org/10.20852/ntmsci.2017.186

[10] Michael, S., Bézier Curves and Srfaces, Lecture 8, Floater Oslo, 2003.

[11] Kılıçoğlu, Ş., Şenyurt, S., On the cubic Bézier curves in E^3, *Ordu University Journal of Science and Technology*, 9(2) (2019) 83-97.

[12] Kılıçoğlu, Ş. Şenyurt, S., On the involute of the cubic Bézier curve by using matrix representation in E^3, *European Journal of Pure and Applied Mathematics*, 13 (2020), 216-226. https://doi.org/10.29020/nybg.ejpam.v13i2.3648