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RICCI CURVATURE TENSOR OF (K+1)-RULED SURFACE IN E^n.

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ABSTRACT

If we choose a natural companian basis (naturliche begleitbasis) for (k+1)-ruled surface in the Euclidean space E^n, then the metric coefficients are $g_{\nu\mu} = \delta_{\nu\mu}$, $1 \leq \nu, \mu \leq k$.

The Ricci curvature tensor S for a manifold is defined by

$$S(X,Y) = \Sigma\, R\, (e_i, X, Y, e_i).$$

In this paper we show that the Ricci curvature tensor of a (k+1) - ruled surface in E^n is

$$S = \sum_{\nu,\mu=0}^{k} [g_{00} R^{o}{}_{\mu o \nu} + \sum_{i=1}^{k} (R^{i}{}_{\mu i \nu} + g_{io} R^{o}{}_{\mu i \nu} + g_{io} R^{i}{}_{\mu o \nu}]\, \theta\nu \otimes \theta\mu.$$

Here, $\{\theta\nu\}$ is the dual basis of the local coordinate basis $\{e\nu\}$.

I. INTRODUCTION

(k+1)-dimensional ruled surfaces in E^n are studied by H. Frank and O. Giering, [1], [2]. Several properties of two-dimensional ruled surfaces are also given by C. Thas, [3]. The purpose of this paper is to calculate the Ricci curvature tensor of the (k+1)-ruled surfaces in terms of metric coefficients $g_{\nu\mu}$'s and $\theta_o, \theta_1, \ldots, \theta_k$ 1-forms where $\{\theta_i\}$ is the dual of the coordinate frame field $\{e_o, e_1, \ldots, e_n\}$.

2. FUNDAMENTAL CONCEPTS

Let the orthonormal field system $\{e_1(t), \ldots, e_k(t))\}$ defined at a point of the curve

$$\eta : I \rightarrow E^n$$

$$\eta : t \rightarrow \eta(t)$$

in E^n, be given. Let us now define

$$M = \bigcup_{t \in I} E_k(t)$$

where $Sp\{e_1(t),...,e_k(t))\} = E_k(t)$. It is known that M is a submanifold of (k+1)-dimension in E^n.

$$\varphi(t,u_1,...,u_k) = \eta(t) + \sum_{\nu=1}^{k} u_\nu e_\nu(t) \tag{2.1}$$

is a parameterization for M. $E_k(t)$ is called the generating space of M at the point $\eta(t)$ and M is called a ruled surface [1]. The vector subspace

$$Sp\ \{e_1(t),...,e_k(t),\ \dot{e}_1(t),...,\dot{e}_k(t))\}$$

is called the asymtotic bundle of M in $E_k(t)$, and it is denoted by A(t). We have

$$\dim A(t) = k+m, \qquad 0 \leq m \leq k.$$

There exists an orthonormal basis of A(t) which we denote as follows

$$\{e_1, e_1, ..., e_k, a_{k+1}, ..., a_{k+m}\}\ . \tag{2.2}$$

We may also write

$$\dot{e}_\nu = \sum_{\mu=1}^{k} \alpha_{\nu\mu} e_\mu + \sum_{\iota=1}^{m} \sigma_{\nu\iota}\, a_{k+\iota} \tag{2.3}$$

Since $<e_\nu, e_\nu> = \delta_{\nu\mu}$, $\alpha_{\nu\mu} = -\alpha_{\nu\mu}$. We can find a basis $\{e_1(t), ..., e_k(t)\}$ of the space $E_k(t)$ such that

$$\dot{e}_\nu = \sum_{\nu=1}^{k} \alpha_{\nu\mu} e_\mu + \varkappa_\nu a_{k+\nu} \qquad (1 \leq \nu \leq m),\ (\varkappa_1 \geq ... \geq \varkappa_m > 0)$$

and

$$\dot{e}_\nu = \sum_{\nu=1}^{k} \alpha_{\nu\mu}\, e_\mu, \qquad (m < \nu \leq k),$$

[1]. The basis $\{e_1(t),...,e_k(t)\}$ is said to be the natural companion basis (natürliche Begleitbasis) of $E_k(t)$. Let $P = \varphi(t,u_1,...,u_k) \in M$. The set

$$\{\dot{\eta}(t) + \sum_{\nu=1}^{k} u_\nu \dot{e}_\nu,\ e_1(t),...,e_k(t)\}$$

is a basis of the tangent space at the point P. We can define any point P of $E_k(t)$ by changing $u_1, u_2, .., u_k$ for a fixed value of t. The space

$$Sp\ \{\dot{\eta},\ \dot{e}_1,\ \dot{e}_2,\ ...,\ \dot{e}_k,\ e_1,\ ...,\ e_k\}$$

includes the union of all the tangent spaces of $E_k(t)$ at a point P. This space is denoted by T(t) and called the tangential bundle of M in $E_k(t)$. It can be easily seen that

$k+m \leqq \dim T(t) \leqq k+m+1.$

If $\dim T(t) = k+m,$

$$\dot{\eta}(t) = \sum_{\nu=1}^{k} \xi_\nu e_\nu + \sum_{\iota=1}^{m} \eta_\iota a_{k+\iota} . \tag{2.4}$$

Any base curve p(t) can be written in terms of $\eta(t)$ as

$$p(t) = \eta(t) + \sum_{\nu=1}^{k} u_\nu(t)\, e_\nu(t) \tag{2.5}$$

Then we obtain

$$\dot{p}(t) = \dot{\eta}(t) + \sum_{\nu=1}^{k} [\dot{u}_\nu(t) e_\nu(t) + u_\nu(t)\, \dot{e}_\nu(t)].$$

Using (2.4) we find

$$\dot{p}(t) = \sum_{\nu=1}^{k} (\xi_\nu + \dot{u}_\nu + \sum_{\mu} u_\mu \alpha_{\mu\nu}) e_\nu + \sum_{\iota=1}^{m} (\varkappa_\iota u_\iota + \eta_\iota)\, \alpha_{k+\iota}. \tag{2.6}$$

If the point p(t) satisfy

$$\varkappa_\iota u_\iota + \eta_\iota = 0, \ (\iota = 1,\dots,m), \tag{2.7}$$

then the vector $\dot{p}(t)$ is in the space $E_k(t)$. As it is known the coefficients $\varkappa_1,\dots,\varkappa_m$ are different from zero. Therefore, the scalars u_1, u_1, ..., u_m can be uniquely from the linear system (2.7). The k-m variables can be choosen arbitrarily. So the set of the points p(t) satisfying the system (2.7) for a fixed t, construct a (k-m)-dimensional vector subspace $K_{k-m}(t)$ of $E_k(t)$.

If $\dim T(t) = k+m+1$, then

$$\dot{\eta}(t) = \Sigma_\nu \zeta_\nu e_\nu + \sum_{\iota}^{m} \eta_\iota a_{k+\iota} + \eta_{m+1} a_{k+m+1}. \tag{2.8}$$

In this case

$$\dot{p}(t) = \sum_{\nu}^{k} (\dot{u}_\nu + \Sigma \alpha_{\mu\nu} u_\nu + \zeta_\nu) e_\nu + \sum_{\iota}^{m} (\eta_\iota + \varkappa_\iota u_\iota) a_{k+\iota} + \eta_{m+1} a_{k+m+1}. \tag{2.9}$$

The (k-m)-dimensional subspace $Z_{k-m}(t)$ defined by the linear system

$$\varkappa_l u_l + \eta_l = 0, \qquad (l=1,...,m) \tag{2.10}$$

is said to be the centralspace of M in $E_k(t)$.

Theorem 2.1: Let the metric coefficients of the (k+1)-dimensional ruled surface in E^n be $g_{\nu\mu}$. Then

$$g_{00} = \sum_{\nu=1}^{k} (\zeta_\nu + \sum_{\nu=1}^{k} \alpha_{\nu\mu} u_\nu)^2 + \sum_{\iota=1}^{m} (\eta_\iota + \varkappa_\iota u_\iota)^2 + (\eta_{k+1})^2$$

$$g_{\nu o} = \zeta_\nu + \sum_{\nu=1}^{k} \alpha_{\nu\mu} u_\nu \tag{2.11}$$

$$g_{\nu\mu} = \delta_{\nu\mu}, \qquad (\nu,\nu = 1,...,k)$$

[1].

Theorem 2.2: Let the dual of frame field $\{e_o, e_1, ..., e_k\}$ be $\{\theta_o, \theta_1, ..., \theta_k\}$ where $\{e_1, e_2, ..., e_k\}$ is the natural companion basis of $E_k(t)$ and

$e_o = \varphi_* \left(\frac{\partial}{\partial t}\right)$. Then, the first fundamental form of M is

$$I = g_{00}\, \theta_o \otimes \theta_o + \sum_{\nu=1}^{k} g_{oo} (\theta_\nu \otimes \theta_o + \theta_o \otimes \theta_\nu) + \sum_{\nu=1}^{k} \theta_\nu \otimes \theta_\nu. \tag{2.12}$$

3. RICCI CURVATURE TENSOR OF (K+1)-RULED SURFACE

The Ricci curvature of a manifold M is the tensor field S which is defined by

$$S(X_p, Y_p) = \sum_i R(e_{ip}, X_p, Y_p, e_{ip}) \tag{3.1}$$

[4]. Since,

$$R(e_{ip}, X_p, Y_p, e_{ip}) = < R(e_{ip}, X_p) Y_p, e_{ip} > \tag{3.2}$$

then

$$S(X_p, Y_p) = \Sigma < R(e_{ip}, X_p) Y_p, e_p >. \tag{3.3}$$

We have

$$R(e_k, e_\iota) e_i = \sum_j R^j{}_{ik\iota} e_j. \tag{3.4}$$

Theorem 3.1: The Ricci curvature tensor of (k+1)-Ruled surface is

$$S = \sum_{\nu,\nu=0}^{k} [R^{o}{}_{\mu o\nu} g_{oo} + \sum_{i=1}^{k} (R^{i}{}_{\mu i\nu} + g_{io}(R^{o}{}_{\mu i\nu} + R^{i}{}_{\mu o\nu})] \, \theta_{\nu} \otimes \theta_{\mu}. \quad (3.5)$$

Proof. Let $X = \sum_{\nu=0}^{k} x_{\nu} e_{\nu}$, $Y = \sum_{\nu=0}^{k} y_{\nu} e_{\nu}$. Since R is a tensor field, we have

$$R(e_i, X)\, Y = R(e_i, \sum_{\nu=0}^{k} x_{\nu} e_{\nu}) (\sum_{\mu=0}^{k} y_{\mu} e_{\mu}) = \sum_{\nu,\mu=0}^{k} x_{\nu} y_{\nu} R(e_i, e_{\nu}) e_{\mu}.$$

By the equation (3.4), we find

$$R\,(e_i, X)\, Y = \sum_{\nu,\mu,h}^{k} x_{\nu} y_{\mu} R^{h}{}_{\mu i\nu}\, e_h. \quad (3.6)$$

From (3.2), we obtain

$$S\,(X, Y) = \sum_{i=o}^{k} < \sum_{\nu,\mu,h} x_{\nu} y_{\mu} R^{h}{}_{\mu i\nu} e_h, e_i >$$

$$= \sum_{\nu,\mu} x_{\nu} y_{\mu} R^{h}{}_{\mu i\nu} < e_h, e_i >$$

$$= \sum_{\nu,\mu} x_{\nu} y_{\mu} (\sum_{i,h} R^{h}{}_{\mu i\nu} < e_h, e_i >).$$

Since $< e_h, e_i > = g_{ih}$, using (2.11), we find

$$S\,(X, Y) = \sum_{\nu,\mu} x_{\nu} y_{\mu} (\sum_{i,h} R^{h}{}_{\mu i\nu}\, g_{ih})$$

$$= \sum_{\nu,\mu} x_{\mu} y_{\nu} \, [R^{o}{}_{\mu o\nu} g_{oo} + \sum_{i=1}^{k} R^{i}{}_{\mu i\nu} + \sum_{i=1}^{k} g_{io}\, (R^{o}{}_{\mu i\nu} + R^{i}{}_{\mu o\nu})]$$

$$= \sum_{\nu,\mu=o}^{k} x_{\mu} y_{\nu} \, [R^{o}{}_{\mu o\nu} g_{oo} + \sum_{i=1}^{k} (R^{i}{}_{\mu i\nu} + g_{io}\, (R^{o}{}_{\mu i\nu} + R^{i}{}_{\mu o\nu})] \, .$$

Let $\{\theta_o, \theta_1, ..., \theta_k\}$ be the dual of the basis $\{e_o, e_1, ..., e_k\}$. Since $(\theta_{\nu} \otimes \theta_{\nu})\,(X, Y) = \theta_{\nu}\,(X).\, \theta_{\mu}\,(Y) = x_{\nu} y_{\mu}$, we have (3.5).

If we calculate the Christoffel symbols $\Gamma^i{}_{jk}$ for a (k+1)-ruled surface, we find

$$\Gamma^o{}_{oo} = \frac{1}{2g}\left[\frac{\partial g}{\partial s} + \sum_{\nu=1}^{k}\left(\zeta_\nu + \sum_{\mu=1}^{k}\alpha_{\nu\mu}u_\mu\right)\frac{\partial g}{\partial u_\nu}\right]$$

$$\Gamma^\lambda{}_{oo} = \frac{1}{2g}\Bigg[-\left(\zeta_\lambda + \sum_\mu \alpha_{\lambda\nu}u_\mu\right)\left(\frac{\partial g}{\partial t} + \sum_\nu\left(\zeta_\nu + \sum_\mu \alpha_{\nu\mu}u_\mu\right)\frac{\partial g}{\partial u_\nu}\right.$$

$$\left.\left. + 2g\left(\dot\zeta +_\lambda \sum_\mu \dot\alpha_{\lambda\nu}u_\mu + \sum_\nu\left(\zeta_\nu + \sum_\mu \alpha_{\nu\mu}u_\mu\right)\alpha_{\lambda\nu} - \frac{1}{2}\frac{\partial g}{\partial u_\lambda}\right)\right)\right]$$

$$\Gamma^o{}_{\nu\mu} = \Gamma^\lambda{}_{\nu\mu} = 0,\ (1 \leq \lambda, \nu, \mu \leq k)$$

$$\Gamma^o{}_{\lambda o} = \Gamma^o{}_{o\lambda} = \frac{1}{2g}\frac{\partial g}{\partial u_\lambda}$$

$$\Gamma^\nu{}_{\lambda o} = \Gamma^\nu{}_{o\lambda} = \frac{1}{2g}\left[-\left(\zeta_\nu + \sum_\mu \alpha_{\nu\mu}u_\mu\right)\frac{\partial g}{\partial u_\lambda} + 2g\,\alpha_{\nu\lambda}\right].$$

So, the Ricci curvature of the Ruled surface can be given in terms of the functions α_{ij} and metric coefficients of the surface.

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