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by

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Proper Pincherle bases in the space of entire fnnctions having fast growth

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1. A classical problem of fundamental interest is to study the representability of analytic functions as infinite series in a given sequence of functions. In other words, the expansion problem in the space of entire functions Γ is just the problem of determining conditions under which sequence $\{\alpha_n\}_{n=0}^{\infty}$ of entire functions in Γ constitutes a basis for the space. Considerable interest attaches to the bases functions known as Pincherle bases, of the form

$$\alpha_n(z) = z^n \{1 + \lambda_n(z)\} \tag{1.1}$$

where each λ_n is an entire function vanishing at origin. Sufficient conditions for $\{\alpha_n\}$ defined by (1.1) to be a proper Pincherle basis in Γ, have been established by Arsove [1].

He also gave a method for constructing proper Pincherle bases from entire functions of exponential type. Later on, Krishnamurthy [5] obtained a sufficient condition for a sequence $\{\alpha_n\}$ given by (1.1) to form a proper Pincherle basis in the spaces $\Gamma(\rho)$, $\Gamma(\rho,T)$ and $\Gamma(0)$, where $\Gamma(\rho)$, $\Gamma(\rho,T)$ and $\Gamma(0)$ are the spaces of entire functions of order less than ρ, of growth (ρ,T) and of order zero respectively.

The present work is in continuation of the earlier works done by Arsove [1], Krishnamurthy [5] and others. In this paper, we obtain a sufficient condition for a sequence $\{\alpha_n\}$ of the type (1.1) to be a proper Pincherle basis in the space of entire functions having fast growth and and then establish a method to construct such bases.

The result of this paper generalises the corresponding results of Arsove and Krishnamurthy.

2. In this section, we recall a few of relevant concepts.

Let $\Gamma_{(p,q)}(\rho,T)$ denote the class of entire functions which are either constants or whose index pairs are less than (p,q) or which are of (p,q)-growth (ρ,T). It is easily seen that $\Gamma_{(q,q)}(\rho,T)$ is a linear space over the complex field D with usual addition and scalar multiplication.

Further, any element $f(z) = \sum_{0}^{\infty} a_n z^n \in \Gamma_{(p,q)}(\rho,T)$ is characterized by the relation

$$(2.1)\quad \limsup_{n\to\infty}\,(\log^{[p-2]} n)\ .\ (\log^{[q-1]} |a_n|^{-1/n})^{-(\rho-A)} \leq T/M \text{ or by}$$

the condition,

$$(2.2)\quad |a_n|^{1/n} \exp^{[q-1]} \left(\frac{M}{T+\delta} \log^{[p-2]} n\right)^{1/\rho-A} \to 0 \text{ as } n \to \infty$$

for every $\delta > 0$,

where

$$M = M(p,q) = \begin{cases} (\rho-1)^{\rho-1}/\rho^{\rho} & \text{if } (p,q) = (2,2) \\ 1/e\rho & \text{if } (p,q) = (2,1) \\ 1 & \text{if } p \geq 3 \end{cases}$$

and

$$A = 1 \quad \text{for } (p,q) = (2,2)$$
$$= 0 \quad \text{for all other pairs.}$$

[For details regarding index pair, (p,q) -order and (p,q)-type etc., see [2], [3]].

Define

$$(2.3)\quad \|f,\rho,T+\delta\| = \sum_{n=0}^{\infty} |a_n| \exp\left(n \exp^{[q-2]} \left(\frac{M}{T+\delta} \log^{[p-2]} \chi_n\right)^{1/\rho-A}\right.$$

where

$$(2.4)\quad \chi_n = N_0 \quad \text{for } 0 \leq n \leq N_0$$
$$= n \quad \text{for } n > N_0$$
$$\text{and } N_0 = [\exp^{[p-3]} 1] + 1.$$

Clearly, for each $\delta > 0$ and $f \in \Gamma_{(p,q)}(\rho,T)$, (2.3) defines a norm. Denote the corresponding normed space by $\Gamma_{(p,q)}(\rho,T,\delta)$ and let $\Gamma_{(p,q)}(\rho,T)$ be the weakest topology which is stronger than each $\Gamma_{(p,q)}(\rho,T,\delta)$. Obviously, $\Gamma_{(p,q)}(\rho,T)$ is generated by the family $\{\Gamma_{(p,q)}(\rho,T,\delta);\ \delta > 0)\}$. Further, it can be easily verified that $\Gamma_{(p,q)}(\rho,T)$ is an F-space under the induced metric

$$(2.5)\quad d(f,g) = |||f-g||| = \sum_{p=1}^{\infty} \frac{1}{2^p} \frac{\|f-g;\rho;T+1/p\|}{1+\|f-g;\rho;T+1/p\|}$$

It is well known that a basis in $\Gamma_0 \subset \Gamma_{(p,q)}(\rho,T)$ is a linearly independent set spanning the closed subspace Γ_0 whereas a proper basis is a basis which has in addition the property;

For all sequences $\{c_n\}$ of complex numbers, $\sum_0^{\infty} c_n\alpha_n$ converges in $\Gamma_{(p,q)}(\rho,T)$ if and only if $\sum_0^{\infty} c_ne_n$ converges in $\Gamma_{(p,q)}(\rho,T)$, where $e_n(z) = z^n$ for $n = 1,2,\ldots,$ $e_0(z) = 1$.

Now it can be easily seen that

(2.6) $\sum_0^{\infty} c_ne_n$ converges in $\Gamma_{(p,q)}(\rho,T)$ if and only if

$$\limsup_{n\to\infty} \frac{\log^{[p-2]}\chi_n}{(\log^{[q-1]}|c_n|^{-1/n})^{\rho-A}} \leq \frac{T}{M}$$

$$\text{OR}\quad |c_n|^{1/n} \exp^{[q-1]}\left(\frac{M}{T+\delta}\log^{[p-2]}\chi_n\right)^{1/\rho-A} \to 0 \text{ as } n\to\infty$$

for every $\delta > 0$.

Thus, $\sum_0^{\infty} c_n\alpha_n$ converges in $\Gamma_{(p,q)}(\rho,T)$ if and only if

$$\limsup_{n\to\infty} \frac{\log^{[p-2]}\chi_n}{(\log^{[q-1]}|c_n|^{-1/n})^{\rho-A}} \leq \frac{T}{M}.$$

A characterisation of proper bases in $\Gamma_{(p,q)}(\rho,T)$ has been given by Juneja et al. [4]. In fact. they proved the following theorem.

THEOREM 2.1. A basis $\{\alpha_n\}$ in a closed subspace Γ_0 of $\Gamma_{(p,q)}(\rho,T)$ is proper if and only if the following conditions hold:

(a) $$\limsup_{n\to\infty} \frac{\log^{[q-1]} \|\alpha_n; \rho; T+\delta\|^{1/n}}{(\log^{[p-2]}\chi_n)^{1/\rho-A}} < \left(\frac{M}{T}\right)^{1/\rho-A} \text{ for every } \delta>0$$

and

(b) $$\lim_{\delta\to 0} \left\{ \liminf_{n\to\infty} \frac{\log^{[q-1} \|\alpha_n; \rho; T+\delta\|^{1/n}}{(\log^{[p-2}\chi_n)^{1/\rho-A}} \right\} \geq \left(\frac{M}{T}\right)^{1/\rho-A}$$

3. A Pincherle basis in $\Gamma_{(p,1)}(\rho,T)$ is a basis $\{\alpha_n\}$ in $\Gamma_{(p,1)}(\rho,T)$ as given in (1.1). Obviously $\lambda_n(z)$ is also in $\Gamma_{(p,1)}(\rho,T)$.

Let $\lambda_n(z) = \sum_{k=0}^{\infty} h_{n,k}z^k$, $n = 0,1,2,\ldots$ with each $h_{n,0} = 0$ where for each n,

$$\limsup_{k\to\infty} (\log^{[p-2]}k)\,(|h_{n,k}|^{-1/k})^{-\rho} \leq T/M$$

So $$\|\alpha_n, \rho, T+\delta\| = \|z^n + z^n\lambda_n(z), \rho, T+\delta\|$$

$$= \|z^n, \rho, T+\delta\| + \sum_{k=1}^{\infty} \|h_{n,k}z^{n+k}, \rho, T+\delta\|$$

$$\geq \|z^n, \rho, T+\delta\|$$

$$= \left[\exp\left(\exp^{[-1]}\left(\frac{M}{T+\delta}\right)\log^{[p-2]}\chi_n)^{1/\rho}\right)\right]^n$$

for each $\delta > 0$.

$$\therefore \|\alpha_n, \rho, T+\delta\|^{1/n} \geq \exp\left(\exp^{[-1]}\left(\frac{M}{T+\delta}\log^{[p-2]}\chi_n\right)^{1/\rho}\right)$$

or $$\frac{\|\alpha_n, \rho, T+\delta\|^{1/n}}{(\log^{[p-2]}\chi_n)^{1/\rho}} \geq \left(\frac{M}{T+\delta}\right)^{1/\rho}.$$

So $\underset{\delta\to 0}{\text{lt}} \left\{ \underset{n\to\infty}{\text{ltinf}} \dfrac{\|\alpha_n, \rho, T+\delta\|^{1/n}}{(\log^{[p-2]}\chi_n)^{1/\rho}} \right\} \geq \left(\dfrac{M}{T}\right)^{1/\rho}$

Now for a Pincherle basis to be proper, it is necessary and sufficient that only condition

$$(3.1) \quad \underset{n\to\infty}{\text{ltsup}} \frac{\|\alpha_n, \rho, T+\delta\|^{1/n}}{(\log^{[p-2]}\chi_n)^{1/\rho}} > \left(\frac{M}{T}\right)^{1/\rho}$$

holds good for each $\delta > 0$.

THEOREM 3.1. If $\{\alpha_n\}$ as defined by (1.1) satisfied

$$(3.2) \quad \underset{(n+k)\to\infty}{\text{ltsup}} (\log^{[p-2]}(n+k)) \; |h_{n,k}|^{\rho/n+k} \leq \frac{T}{M}$$

then it constitutes a proper basis in $\Gamma_{(p,1)}(\rho,T)$.

PROOF. First we shall show that $\{\alpha_n\}$ satisfies (3.1) and therefore, if it is a basis in $\Gamma_{(p,1)}(\rho,T)$, it is as a proper basis. To see this, we have, for each $\delta' > 0$, we can find $N(\delta') \geq N_0$ such that from (3.2).

$$(3.3) \quad |h_{n,k}| \leq \exp \left\{-(n+k) \exp^{[-1]} \left(\frac{M}{T+\delta'} \log^{[p-2]}(n+k)\right)^{1/\rho} \right\}$$

for all $(n+k) \geq N$, where $N = N(\delta')$ is independent of n and k. So for each $\delta > 0$ and for a fixed n,

$$\|\alpha_n(z), \rho, T+\delta\| = \|z^n + \sum_{k=o}^{\infty} h_{n,k} z^{n+k}, \rho, T+\delta\|$$

$$= \|(1+h_{n,o}) z^n + \sum_{k=1}^{\infty} h_{n,k} z^{n+k}, \rho, T+\delta\|$$

$$= \|z^n, \rho, T+\delta\| + \sum_{k=1}^{\infty} \|z^{n+k}, \rho, T+\delta\| \; |h_{n,k}|$$

$$= \exp \left(n \exp^{[-1]} \left(\frac{M}{T+\delta} \log^{[p-2]}\chi_n \atop \rho\right)^{1/\rho}\right)$$

$$+ \sum_{k=1}^{\infty} |h_{n,k}| \left[\exp \left\{(n+k) \exp^{[-1]} \left(\frac{M}{T+\delta} \log^{[p-2]}\chi_{n+k}\right)^{1/\rho} \right\}\right]$$

$$\therefore \quad \|\alpha_n, \rho, T+\delta\| \leq \left(\frac{M}{T+\delta} \log^{[p-2]}\chi_n\right)^{n/\rho}$$

$$+ \sum_{\substack{k \\ (n+k)<N}} |h_{n,k}| \left(\frac{M}{T+\delta} \log^{[p-2]}\chi_{n+k}\right)^{(n+k)/\rho}$$

$$+ \sum_{\substack{k \\ (n+k)\geq N}} \left(\frac{M}{T+\delta'} \log^{[p-2]}(n+k)\right)^{-(n+k)/\rho}$$

$$\cdot \left(\frac{M}{T+\delta} \log^{[p-2]}(n+k)\right)^{(n+k)/\rho}$$

for some positive $\delta' > \delta$.

The last sum on the right hand side being the sum of a convergent series, we have for all $n \geq N$,

$$\|\alpha_n, \rho, T+\delta\| \leq \left(\frac{M}{T+\delta} \log^{[p-2]}\chi_n\right)^{n/\rho} + \mu \text{ for each } \delta > 0,$$

μ being a finite constant depending only on T, δ' δ, ρ.
It follows that

$$\limsup_{n\to\infty} \frac{\|\alpha_n, \rho, T+\delta\|^{1/n}}{(\log^{[p-2]}\chi_n)^{1/p}} > \left(\frac{M}{T}\right)^{1/\rho} \text{ for each } \delta > 0.$$

So $\{\alpha_n\}$ satisfies (3.1). Hence it will form a proper basis in $\Gamma_{(p,1)}(\rho,T)$ only when $\{\alpha_n\}$ is a basis in $\Gamma_{(p,1)}(\rho,T)$.

But α_n's are clearly linearly independent and so it is enough to show that $\{\alpha_n\}$ spans $\Gamma_{(p,1)}(\rho,T)$.

Let $f(z) = \Sigma\, a_n e_n \in \Gamma_{(p,1)}(\rho,T)$. Form the equations

$$(3.4) \quad a_0 = c_0,\ a_n = c_n + \sum_{k=1}^{n} c_{n-k}\, h_{n-k,k}$$

These equations determine c_n uniquely in terms of the a_n's and yield $f(z) = \Sigma_n\, c_n\alpha_n$ provided we can justify the step by showing that $\Sigma_n\, |c_n|\, \|\alpha_n, \rho, T+\delta\|$ is convergent for each $\delta > 0$.

Fix $\delta > 0$ and write $\| f \|$ to denote $\| f, \rho, T+\delta \|$. Putting $\beta_n(z) = z^n \lambda_n(z)$, $n = 1,2,\dots$, it is clear that the convergence of

$$\sum_{n=1}^{\infty} |c_n| \, \|\alpha_n(z)\| \text{ will follow from that of } \sum_{n=1}^{\infty} |c_n| \, \|z^n\|$$

$$+ \sum_{n=1}^{\infty} |c_n| \, \|\beta_n\| .$$

Since

$$(3.5) \quad |c_n| \leq |a_n| + \sum_{k=1}^{n} |c_{n-k}| \, |h_{n-k,k}|$$

we see that the series $\sum_{n=1}^{\infty} |c_n| \, \|z^n\|$ is dominated by

$$\sum_{n=1}^{\infty} |a_n| \, \|z^n\| + \sum_{n=1}^{\infty} \{ \|z^n\| \sum_{k=1}^{n} |c_{n-k}| \, |h_{n-k,k}| \}$$

which is equal to $\sum_{n=1}^{\infty} |a_n| \, \|z^n\| + \sum_{n=1}^{\infty} \{ |c_n| \sum_{k=n+1}^{\infty} |h_{n,k-n}| \, \|z^k\| .$

So $\sum_{n=1}^{\infty} |c_n| \, \|z^n\| \leq \sum_{n=1}^{\infty} |a_n| \, \|z^n\| + \sum_{n=1}^{\infty} \{ |c_n|$

$$\sum_{k=n+1}^{\infty} |h_{n,k-n}| \, \|z^k\| \} \leq \sum_{n=1}^{\infty} |a_n| \, \|z^n\| + \sum_{n=1}^{\infty} |c_n| \, \|\beta_n\|$$

Since $\Sigma_n a_n z^n \in \Gamma_{(p,1)}(\rho,T)$, the above shows that for the required convergence of $\Sigma_n |c_n| \, \|\alpha_n\|$, we need only prove the convergence of $\Sigma_n |c_n| \, \|\beta_n\|$.

Now chose a $\delta' > \delta$ and two positive numbers N' and N'' such that

$$(3.6) \quad |a_n| \leq \exp \{ -n \exp^{[-1]} \left(\frac{M}{T+\delta'} \log^{[p-2]} \chi_n \right)^{1/\rho} \}$$

for all $n \geq N' = N'(\delta')$
and

$$(3.7) \quad \left(\frac{M}{T+\delta'} \log^{[p-2]} n\right)^{1/\rho} > 2$$

for all $n \geq N'' = N''(\delta')$.

We note that (3.6) is possible since $\Sigma\, a_n e_n \in \Gamma_{(p,1)}(\rho, T)$. Choose $N_* = \max(N, N', N'')$ where $N = N(\delta')$ is as defined in (3.3). So $N_* = N_*(\delta')$. The inequalities (3.5), (3.6) and (3.3) now give for $n \geq N_*$

$$|c_n| \leq \exp\left\{-n \exp^{[-1]}\left(\frac{M}{T+\delta'} \log^{[p-2]} \chi_n\right)^{1/\rho}\right\}$$

$$+ \sum_{k=1}^{n} |c_{n-k}| \exp\left\{-n \exp^{[-1]}\left(\frac{M}{T+\delta'} \log^{[p-2]} n\right)^{1/\rho}\right\}.$$

Now define positive numbers d_n as $d_0 = |a_0|$,

$$d_n = 1 + \sum_{k=1}^{n} d_{n-k},\ n \geq 1.$$

This gives

$$d_n - d_{n-1} = d_{n-1},\ n \geq 2.$$

From which we get $d_n = 2^{n-1}|d_1| = 2^{(n-1)}(1 + |a_0|)$

So

$$|c_n| = \exp\left\{-n \exp^{[-1]}\left(\frac{M}{T+\delta'} \log^{[p-2]} \chi_n\right)^{1/\rho}\right\} d_n$$

$$\text{or} \quad \frac{|c_n|}{\exp\left\{-n \exp^{[-1]}\left(\frac{M}{T+\delta'} \log^{[p-2]} \chi_n\right)^{1/\rho}\right\}} \leq d_n$$

$$= 2^{(n-1)}(1 + |a_0|) \quad \text{for } n \geq N_*$$

$$\text{Now} \sum_{n=1}^{\infty} |c_n|\, \|\beta_n\| = \sum_{n=1}^{\infty} |c_n|\, \|z^n \lambda_n(z), \rho, T+\delta\|$$

$$= \sum_{n=1}^{\infty} |c_n|\, \left\| \sum_{k=1}^{\infty} h_{n,k} z^{n+k}, \rho, T+\delta \right\|$$

$$= \sum_{n=1}^{\infty} |c_n| \sum_{k=1}^{\infty} |h_{n,k}| \exp\{(n+k)$$

$$\cdot \exp^{[q-2]}\left(\frac{M}{T+\delta}\log^{[p-2]}\chi_{n+k}\right)^{1/\rho}\}$$

We shall split this double summation as

$$\sum_{n=1}^{N_*-1}\sum_{k=1}^{N_*-1} + \sum_{n=1}^{N_*-1}\sum_{k=N_*}^{\infty} + \sum_{n=N_*}^{\infty}\sum_{k=1}^{\infty}$$

The first series is finite. The second series is dominated by the convergent series

$$\sum_{n=1}^{N_*-1} |c_n| \sum_{k=N_*}^{\infty} |h_{n,k}| \exp\{(n+k) \exp^{[-1]}\left(\frac{M}{T+\delta}\log^{[p-2]}\chi_{n+k}\right)^{1/\rho}\}$$

$$\leq \sum_{n=1}^{N_*-1} |c_n| \sum_{k=N_*}^{\infty} \exp\{-(n+k)\ \exp^{[-1]}\left(\frac{M}{T+\delta'}\log^{[p-2]}{}_{n+k}\right)^{1/\rho}$$

$$\cdot \exp\{(n+k)\ \exp^{[-1]}\left(\frac{M}{T+\delta}\log^{[p-2]}n+k\right)^{1/\rho}\}$$

$$= \sum_{n=1}^{N^*-1} |c_n| \sum_{k=N_*}^{\infty} \frac{\exp\{(n+k)\exp^{[-1]}\left(\frac{M}{T+\delta}\log^{[p-2]}n+k\right)^{1/\rho}\}}{\exp\{(n+k)\exp^{[-1]}\left(\frac{M}{T+\delta'}\log^{[p-2]}n+k\right)^{1/\rho}\}}$$

$$\leq N_*C \sum_{k=N_*}^{\infty} \exp\ [n+k)\ \{\exp^{[-1]}\left(\frac{M}{T+\delta}\log^{[p-2]}n+k\right)^{1/\rho}$$

$$-\exp^{[-1]}\left(\frac{M}{T+\delta'}\log^{[p-2]}n+k\right)^{1/\rho}\}\ ]$$

$$\left[\therefore \sum_{n=1}^{N_*-1} |c_n| \leq (N_*-1)\max|c_n| = N_*C\right]$$

which is convergent since $\delta' > \delta$.

Consider the third series,

i.e. $$\sum_{n=N_*}^{\infty} |c_n| \sum_{k=1}^{\infty} |h_{n,k}| \exp\left\{(n+k)\exp^{[-1]}\left(\frac{M}{T+\delta}\log^{[p-2]}n+k\right)^{1/\rho}\right\}$$

$$\leq \sum_{n=N_*}^{\infty} 2^{n-1}(1+|a_0|)\exp\left\{-n\exp^{[-1]}\left(\frac{M}{T+\delta'}\log^{[p-2]}n\right)^{1/\rho}\right\}$$

$$\cdot \sum_{k=N^*}^{\infty} \left[\exp\left\{(n+k)\exp^{[-1]}\left(\frac{M}{T+\delta}\log^{[p-2]}n+k\right)^{1/\rho}\right\}\right.$$

$$\left.\cdot \exp\left\{-(n+k)\exp^{[-1]}\left(\frac{M}{T+\delta'}\log^{[p-2]}n+k\right)^{1/\rho}\right\}\right].$$

Now consider the series

$$\sum_{k=N^*}^{\infty} \exp\left\{(n+k)\exp^{[-1]}\left(\frac{M}{T+\delta}\log^{[p-2]}n+k\right)^{1/\rho}\right\}$$

$$\cdot \exp\left\{-(n+k)\exp^{[-1]}\left(\frac{M}{T+\delta'}\log^{[p-2]}n+k\right)^{1/\rho}\right\}$$

$$= \sum_{k=N_*}^{\infty} \left[\frac{\frac{M}{T+\delta}\log^{[p-2]}n+k}{\frac{M}{T+\delta'}\log^{[p-2]}n+k}\right]^{\frac{(n+k)}{\rho}}$$

Case 1. For p = 2 we get

$$\sum_{k=N_*}^{\infty} \left[\frac{\frac{M}{T+\delta}(n+k)}{\frac{M}{T+\delta'}(n+k)}\right]^{\frac{n+k}{\rho}} = \sum_{k=N_*}^{\infty} \left(\frac{T+\delta'}{T+\delta}\right)^{\frac{n+k}{\rho}}$$

$$= \frac{\left(\frac{T+\delta'}{T+\delta}\right)^{\frac{n+N_*}{\rho}}}{1-\left(\frac{T+\delta'}{T+\delta}\right)^{1/\rho}} = \frac{T+\delta}{\delta-\delta'}\left(\frac{T+\delta'}{T+\delta}\right)^{\frac{n+N_*}{\rho}}$$

which is a convergent series.

Case 2. For $p > 2$ we get

$$\sum_{k=N_*}^{\infty} \left[\frac{\frac{M}{T+\delta} \log^{[p-2]}(n+k)}{\frac{M}{T+\delta'} \log^{[p-2]}(n+k)} \right]^{\frac{n+k}{\rho}} = \sum_{k=N_*}^{\infty} \left(\frac{T+\delta'}{T+\delta} \right)^{\frac{n+k}{\rho}}$$

is a convergent series.

Hence the third series is dominated by the series

$$\sum_{n=N_*}^{\infty} 2^{(n-1)}(1+|a_0|) \exp \left\{-n \exp^{[-1]} \left(\frac{M}{T+\delta'} \log^{[p-2]} n \right)^{1/\rho} \right\}$$

$$\cdot \sum_{k=N_*}^{\infty} \left(\frac{T+\delta'}{T+\delta} \right)^{\frac{n+k}{\rho}}$$

$$= \sum_{n=N_*}^{\infty} 2^n \left(\frac{1+|a_0|}{2} \right) \left(\frac{M}{T+\delta'} \log^{[p-2]} n \right)^{-n/\rho} . M_1 \left(\frac{T+\delta'}{T+\delta} \right)^{\frac{n+N_*}{\rho}}$$

$$= K \sum_{n=N_*}^{\infty} \frac{\left(\frac{M}{T+\delta'} \log^{[p-2]} n\right)^{-n/\rho}}{2^{-n}} \cdot \left(\frac{T+\delta'}{T+\delta} \right)^{\frac{n-N_*}{\rho}}$$

where $K = K(\delta, \delta')$

This is again dominated by the series

$$K \sum_{n=N_*}^{\infty} \frac{\left(\frac{M}{T+\delta'} \log^{[p-2]} n \right)^{-n/\rho}}{2^{-n}} \qquad \left(\because \frac{T+\delta'}{T+\delta} < 1 \right)$$

and is convergent due to the equation (3.7).

This completes the proof of the theorem.

4. Now it is of our interest to construct the proper Pincherle bases in $\Gamma_{(p,1)}(\rho,T)$. A direct application of Theorem 1 gives a general

method of construction of proper Pincherle bases from certain entire functions belonging to $\Gamma_{(p,1)}(\rho,T)$.

COROLLARY. Let $\varnothing$ be an entire function belongs to $\Gamma_{(p,1)}(\rho,T)$ having the power series expansion $\varnothing(z) = \sum_{n=o}^{\infty} t_n z^n$. If $t_o \neq 0$ and

$$\underset{(n+k)\to\infty}{\text{ltsup}} \quad (\log^{[p-2]}(n+k)) \left| \frac{t_{n+k}}{t_n} \right|^{\rho/(n+k)} \leq \frac{T}{M} \text{ for all } \delta > 0$$

and $k \neq 0$,

then the sequence $\{\alpha_n\}$ defined by

$$\alpha_n(z) = \frac{1}{t_n} \left[\varnothing(z) - \sum_{k=o}^{n-1} t_k z^k \right]$$

is a proper Pincherle basis in $\Gamma_{(p,1)}(\rho,T)$.

The proof follows on the lines of Arsove [1, Them 6] with the following values:

$$\alpha_n(z) = \frac{1}{t_n} \sum_{k=n+1}^{\infty} t_k z^{k-n}$$

and

$$R^k = \exp\left(k \exp^{[-1]} \left(\frac{M}{T+\delta} \log^{[p-2]} \chi_k\right)^{1/\rho}\right).$$

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