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On The Minimal Index of the Generalized Minimal Ruled Surfaces in the Euclidean n-Space E^n

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On The Minimal Index of the Generalized Minimal Ruled Surfaces in the Euclidean n-Space E^n

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ABSTRACT

In this paper we have found the minimal index of the generalized minimal ruled surfaces defined in the Euclidean n-space E^n and new results were obtained for both developable and nondevelopable ruled surfaces. In addition a theorem given for the Riemann submanifold in [3] was defined for the generalized ruled surfaces in the Euclidean n-space E^n

I. INTRODUCTION

We shall assume throughout this paper that all manifolds, maps, vector fields, etc. ... are differentiable of class C^∞.

First of all, we give some properties of a general submanifold N of the Euclidean n--space E^n. Suppose that $\bar{D}$ is the Riemann connection of E^n, while D is the Riemann connection of N.

Then, if X,Y are the vector fields of N and if V is the second fundamental form of N, we have by decomposing $\bar{D}_X Y$ in a tangential and normal component.

$$\text{(I.1)} \qquad \bar{D}_X Y = D_X Y + V(X,Y).$$

The equation (I.1) is called Gauss equation. [1].

If ξ is any normal vector field on N, we find the Weingarten equation by decomposing $\bar{D}_X \xi$ in a tangential component and a normal component as

$$\text{(I.2)} \qquad \bar{D}_X \xi = -(A(\xi X)) + D^{\perp}{}_X \xi.$$

$A\xi$ determines at each point a self-adjoint linear map and $D^{\perp}$ is a metric connection in the normal bundle $\chi^{\perp}(M)$. We use the same notation $A\xi$ for the linear map and the matrix of the linear map.

Suppose that X,Y are vector fields on N, while ξ is a normal vector field, then if the standard metric tensor of E^n is denoted by $<,>$

$$X<Y,\xi> = <\bar{D}_X Y,\xi> + <Y,\bar{D}_X\xi> = 0 \tag{I.3}$$

or

$$<V(X,Y),\xi> = <Y,A\xi(X)>.$$

If $\xi_1, \xi_2, \ldots, \xi^n{}_{-k-f}$ constitute an orthonormal base field of the normal bundle $\chi^{\perp}(N)$, then the mean curvature vector H of N at the the point p is given by

$$H = \sum_{i=1}^{n-k-f} trA\xi_i / k + 1.\ \xi_i. \tag{I.4}$$

$\| H \|$ is the mean curvature. If $H = 0$ at each point p of N, then N is said to be minimal, [4].

II. (k+1) - DIMENSIONAL RULED SURFACES IN THE EUCLIDEAN n-SPACE E^n.

Suppose that the base curve r(s) of the (k+1)-dimensional ruled surface N in E^n is an orthogonal trajectory of the k-dimensional generating space $E_k(s)$, $(k \geq 1)$, which is spanned by the orthonormal base vectors $e_1(s), e_2(s), \ldots, e_k(s)$, then N can locally represented by

$$\varphi(s,u_1,u_2,\ldots, u_k) = r(s) + \sum_{i=1}^{k} u_i e_i(s),\ u_i \in IR,$$

$1 \leq i \leq k$, [4].

In this paper, we will say the generalized ruled surface instead of the $(k + 1)$ – dimensional ruled surface in the Euclidean n-space E^n.

DEFINITION II.1: Suppose that $\{e_o,e_1,\ldots,e_k\}$ is an orthonormal base field of a generalized ruled surface N, where e_o is the unit tangent vector of the orthogonal trajectories of the generating space $E_k(s)$. If

$$\text{rank } [e_o,e_1,\ldots, e_k, \bar{D}e_o e_1,\ldots, \bar{D}e_o e_k] = 2k-m$$

at each point p of N, then N will be called as m-developable. If $m = -1$, then the generalized ruled surface N is called as non-developable; If $m = k-1$, then N is called as total developable, [4].

Suppose that $\{e_o, e_1, \ldots, e_k\}$ is an orthonormal base field of the tangential bundle $\chi(N)$ and $\{\xi_1, \xi_2, \ldots, \xi_{n-k-1}\}$ is an orthonormal base field of the normal bundle $\chi^{\perp}(N)$. Then we have the following Weingarten equations:

$$\bar{D}_{e_o}\xi_j = \sum_{i=1}^{k} a^j{}_{oi}e_i + a^j{}_{oo}e_o + \sum_{m=1}^{n-k-f} b^j{}_{om}\,\xi_m$$

$$\bar{D}e_1\xi_j = \sum_{i=1}^{k} a^j{}_{1i}e_i + a^j{}_{1o}e_o + \sum_{m=1}^{n-k-1} b^j{}_{1m}\xi_m$$

(II.1) $\quad\ldots\quad\ldots\ldots\ldots\ldots\ldots\ldots$

$$\bar{D}e_k\xi_j = \sum_{i=1}^{k} a^j{}_{ki}e_i + a^j{}_{ko}e_o + \sum_{m=1}^{n-k-1} b^j{}_{km}\,\xi_m$$

$1 \leq j \leq n\text{-}k\text{-}1$. Moreover, because of (I.3) and since $V(e_i, e_n) = 0$, $1 \leq i, m \leq k$, we observe that

(II.2) $\quad <V(e_i,e_m), \xi_j> = <A\xi_j(e_i),e_m> = -\,a^j{}_{im} = 0$, [4].

Therefore we can obtain the matrix $A\xi_j$ as follows:

$$\text{(II.3)}\qquad A\xi = -\begin{bmatrix} a^j{}_{oo} & a^j{}_{o1} & \ldots & a^j{}_{ok} \\ a^j{}_{o1} & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ a^j{}_{ok} & 0 & \ldots & 0 \end{bmatrix}$$

Moreover, the Riemannian curvature $K(e_i,e_o)$ of N at a point p of N is given by

$$K(e_i,e_o) = -<\bar{D}e_ie_o, \bar{D}e_ie_o>|_p$$

$$\text{(II.4)}\qquad = -\sum_{j=1}^{n-k-1} (a^j{}_{oi})^2,\ 1 \leq i \leq k,\ [4].$$

The mean curvature vector H of the $(k+1)$– dimensional ruled surface N is given by

$$H = V(e_o,e_o)\ /\ k+1,\ [4].$$

III. ON THE MINIMAL INDEX OF THE MINIMAL GENERALIZED RULED SURFACES IN THE EUCLIDEAN n-SPACE E^n

Let S_{k+1} be the set of all real symmetric matrices of order $k+1$. In this paper we consider it as a vector space over the real field IR.

Now we define an inner product of any two elements A,B in S_{k+1} by

(III.1) $$<A,B> = tr(AB) / k+1.$$

then we have

(III.2) $$\| A \| = \sqrt{<A,A>} = \sqrt{tr(A^2) / k+1} \text{ and } \| I_{k+1} \| = 1.$$
where I_{k+1} denotes the unit matrix in S_{k+1}. [3].

Let m be a linear map from S_{k+1} to IR defined as

(III.4) $$m(A) = trA / k + 1.$$

Then the kernel of m can be represented by

(III.5) $$\ker m = \{A \| trA = 0, \forall A \in S_{k+1}\}.$$

In addition, since

$$<A, I_{k+1}> = m(A), \forall A \in S_{k+1}$$

we can write

$$S_{k+1} = \ker m \oplus IRI_{k+1}.$$

Let $\{\xi_1, \xi_2, \ldots, \xi_{n-k-1}\}$ be an orthonormal base field of $\chi^{\perp}(N)$. Then we can write

$$\xi = \sum_{j=1}^{n-k-1} a_j \xi_j$$

for all $\xi \in \chi^{\perp}(N)$.

Let the linear map m: $T_N(p) \longrightarrow IR$ be defined by

(III.6) $$\bar{m}(\xi) = \sum_{j=1}^{n-k-1} a_j m A_{\xi_j}, \quad \forall \xi \in T_N(p)$$

and Ψp: $T_N(p) \longrightarrow S_{k+1}$ is defined by

(III.7) $$\Psi p(\xi) = \sum_{j=1}^{n-k-1} a_j A_{\xi_j}, \forall \xi \in T_N(p).$$

DEFINITION III.1 (M–index$_p$ N):

The dimension of Ψp ($\ker \bar{m}$) is called the minimal index of the generalized ruled surface N at point p of N and denoted by

$$\dim \Psi p (\ker \bar{m}) = M\text{–index}_p N, \quad [3].$$

LEMMA III.1:

$$\dim(\ker m) = k(k+3)/2.$$

Proof: Since $\dim S_{k+1} = k^2 + 3k + 2/2$, we have

$$\dim(\ker m) = \dim S_{k+1}{}^{-1} = k(k+3)/2.$$

LEMMA III.2: Let $\overline{N}$ be a Riemannian submanifold in the Eucliden n-space E^n. Then

$$M\text{-index}_p N \leq \min\{\dim(\ker\overline{m}), \dim(\ker m)\}, \ [3].$$

THEOREM III.1: Let $\overline{N}$ be a $(k+1)$ – dimensional (generalized) ruled surface in E^n and $\{e_0, e_1, \ldots, e_k\}$ be an orthonormal base field of N. Then N is total developable iff

$$\overline{D}_{e_i} e_0 = 0, \quad 1 \leq i \leq k, \quad [2].$$

COROLLARY III.1: If N is total developable, then

$$K(e_i, e_0) = 0, \quad 1 \leq i \leq k.$$

Proof: From the Theorem III.1 and the equation II.4, the proof of corollary is clear.

TEOREM III.2: If the $(k+1)$–dimensional m–developable ruled surface N is minimal, then N is necessarily a submanifold of an E^{2k-m}, [4].

Now we have the following theorems about the minimal index of the generalized minimal ruled surface N in E^n.

THEOREM III.3: Let N be a generalized ruled surface in E^n and $\{\xi_1, \xi_2, \ldots, \xi_{n-k-1}\}$ be an orthonormal base field of $\chi^{\perp}(N)$. If N is a minimal ruled surface, then $M\text{-index}_p N \leq k, \ \forall\, p \in N$.

Proof: Since, for all $\xi \in \chi^{\perp}(N)$, $\xi = \sum_{j=1}^{n-k-1} a_j \xi_j$, $a_j \in \mathbb{R}$, we have

$$\overline{m}(\xi) = \sum_{j=1}^{n-k-1} a_j \operatorname{tr} A_{\xi_j}.$$

Since N is minimal, we find $\overline{m}(\xi) = 0$ and $\ker\overline{m} = \chi^{\perp}(N)$.

Moreover, since

$$\Psi_p(\xi) = \sum_{j=1}^{n-k-1} a_j A_{\xi_j}, \quad \forall \xi \in \ker\overline{m},$$

we get

$$\Psi_p(\ker\overline{m}) = Sp\ \{A_{\xi_1}, A_{\xi_2}, \ldots, A_{\xi_{n-k-1}}\}.$$

Since $trA_{\xi_j} = 0$, $1 \leq j \leq n-k-1$, the dimension of the vector space spanned by symmetric matrices in the form of A_{ξ_j} is equal to k. So that we can find

$$\dim \Psi_p(\ker\overline{m}) \leq k$$

or

$$M\text{–}index_p N \leq k.$$

COROLLARY III.2:

$$\dim\ (\ker\overline{m}) = n\text{–}k\text{–}1.$$

From the Corollary III.2, Lemma III.1 and Lemma III.2, we have the following corollary.

COROLLARY III.3:

$$M\text{–}index_p N \leq \min\ \{k,\ k\text{–}n\text{–}1,\ k(k+3)/2\}.$$

COROLLARY III.4: Let N be a minimal hyperruled surface in E^n. Then

$$M\text{–}index_p N = 0 \text{ or}$$

$$M\text{–}index_p N = 1.$$

Proof: Let N be a hyperruled surface in E^n and ξ be an unit bormal vector field of N. Then $\chi^{\perp}(N) = Sp\{\xi\}$ and since N is minimal, we get

$$\ker\overline{m} = Sp\ \{\xi\}.$$

Therefore, because of the definition of Ψ, we observe that

$$\Psi_p(\ker\overline{m}) = Sp\ \{A_\xi\}.\ \forall\ p \in N.$$

Case 1: If N is total developable,

$$K(e_i,e_o) = -a^2_{oi} = 0, \qquad 1 \leq i \leq n-2.$$

Thus we get $A_\xi = 0$. This implies that

$$\dim \Psi_p\ (\ker\overline{m}) = M\text{–}index_p N = 0 \qquad \forall p \in N.$$

Case 2: Let N be nondevelopable. Then $A_\xi \neq 0$. This implies that

$$\dim \Psi_p(\ker \overline{m}) = \text{M-index}_p N = 1. \qquad \forall p \in N.$$

THEOREM III.4: Let N be a hyperruled surface in E^n. Then

N is minimal and $\text{M-index}_p N = 0$ iff N is a hyperplane, $\forall p \in N$.

Proof: Let $\{e_0, e_1, \ldots, e_{n-2}\}$ be an orthonormal base field of $\chi(N)$ and ξ be an unit normal vector field of $\chi^{\perp}(N)$. Since N is minimal, we have

$$\ker \overline{m} = \text{Sp}\,\{\xi\} \text{ and } \Psi_p(\ker \overline{m}) = \text{Sp}\,\{A_\xi\}.$$

Moreover, since $\text{M-index}_p N = 0$ by hypothesis, we get $A_\xi = 0$. Therefore from the Weingarten equation

$$\bar{D}e_j\, \xi = - A_\xi\,(e_j) + b_j\, \xi, \qquad 0 \leq i \leq n\text{-}2$$

we observe that

$$<\bar{D}e_i \xi,\ \xi> = b_i = 0.$$

This means that

$$\bar{D}e_i\, \xi = 0, \qquad 0 \leq i \leq n\text{-}2.$$

The last equation implies that ξ is a parallel vector field with respect to N. Therefore N is a hyperplane in E^n.

Now, let N be a hyperplane and ξ be an unit normal vector field of N. Then, since $\{e_0, e_1, \ldots, e_{n-2}\}$ is an orthonormal base field of $\chi(N)$, we get.

$$\bar{D}e_i\, \xi = 0, \qquad 0 \leq i \leq n\text{-}2.$$

Therefore, from the Weingarten equation,

$$A_\xi = 0.$$

This means that N is minimal.

Since N is minimal $\ker \overline{m} = \text{Sp}\{\xi\}$ and $\Psi_p(\ker \overline{m}) = \text{Sp}\,\{A_\xi\}$.

Moreover, since $A_\xi = 0$ we get

$$\text{M-index}_p N = 0.$$

THEOREM III.5: Let N be a $(k + 1)$ -dimensional ruled surface in E^n. If N is minimal and $\text{M-index}_p N = 0$, $\forall p \in N$, then N is a submanifold of E^{k+1}

Proof: Since N is minimal, we get $\ker\bar{m} = \chi^{\perp}(N)$ and $\Psi_p(\ker\bar{m}) = Sp\,\{A\xi_1, \ldots, A\xi_{n-k-1}\}$. Moreover since

$$\dim \Psi_p(\ker\bar{m}) = M\text{–}index_p N = 0$$

by hypothesis, we find $A\xi_j = 0$, $1 \leq j \leq n-k-1$. Therefore, from the corollary III.1, N is total developable. In this case we have the Theorem III.2. So that N is a submanifold of E^{k+1}

Let F be a normal subbundle of $\chi^{\perp}(N)$ spanned by the normal vector field $V(e_o, e_i)$, $1 \leq i \leq k$. Then we have the following lemma.

LEMMA III.3:

N is m–developable iff the normal subbundle F of $\chi^{\perp}(N)$ is $(k-m-1)$–dimensional, [4].

THEOREM III.6: Let N be a generalized minimal ruled surface in E^n. If N is m–developable, then

$$M\text{–}index_p N \leq k-m-1, \ \forall\, p \in N.$$

Proof: Since N is minimal, we have

$$\Psi_p(\ker\bar{m}) = Sp\,\{A\xi_1, \ldots, A\xi_{n-k-1}\}$$

and from Lemma III.3

$$\dim F = k-m-1.$$

Suppose that $\{\xi_1, \xi_2, \ldots, \xi_{n-k-1}\}$ is an orthonormal base field of $\chi^{\perp}(N)$ such that $\}\xi_1, \xi_2, \ldots, \xi_{k-m-1}\}$ constitute an orthonormal base field of F.

If we consider the equation (II.2) in this case, we get

$$<v(e_i, e_o), \xi_r) = -\,a^r_{oj} = 0, \quad 1 \leq i \leq k,\ k-m \leq r \leq n-k-1.$$

Since $H = 0$, we have $trA\xi_j = 0$, $1 \leq j \leq n-k-1$. Therefore we find

$$A\xi_{k-m} = \ldots = A\xi_{n-k-1} = 0.$$

The last equation implies that

$$\Psi_p(\ker\bar{m}) = Sp\,\{A\xi_1, A\xi_2, \ldots, A\xi_{k-m-1}\}.$$

Thus we can find

$$M\text{–}index_p N \leq k-m-1.$$

That completes the proof of theorem.

COROLLARY III.5: If N is a (k+1)-dimensional minimal ruled surface in E^n and N is m-developable, then.

$$\text{M-index}_p N \leq \min \{k-m-1,\ n-k-1,\ k(k+3)/2\}.$$

ÖZET

Bu çalışmada E^n, n-boyutlu Öklid uzayında tanımlı genelleştirilmiş minimal regle yüzeylerinin minimal indeksleri hesaplandı ve yüzeyin açılabilir olması veya açılabilir olmaması halleri için yeni sonuçlar elde edildi.

Ayrıca, [3] de Riemann altmanifoldlarının minimal indeksleri ile ilgili olarak verilen bir teorem E^n de (k+1)-boyut-lu (genelleştirilmiş) regle yüzeyler için ifade edilir.

REFERENCES

[1] **Chen, B.Y.,** "Geometry of Submanifolds" Marcel Dekker, New York, 1973.

[2] **Kuruoğlu, N-Keleş, S.,** "Properties of generalized Ruled Surfaces in the Euclidean n-Space E^n and Massey's Theorem" Journal of the Faculty of Sciences and Arts of Karadeniz University, Series MA (to appear).

[3] **Otsuki, T.,** "A Theory of Riemannian Submanifolds" Kodai Math. Sem. Rep. 20 (1968).

[4] **Thas, C.,** "Minimal Monosystems" The Yokohoma Mathematical Journal, Vol. XXVI, No. 2, 1978.