Commun. Fac. Sci. Univ. Ank. Series A_1
V. 39. pp. 9–14 (11990)

THE SETS OF HOMOTHETIC MAPPINGS

ISMET ARSLAN

Assistant Professor. Turkisch Air Force Academy, İstanbul, TURKEY

H. HILMI HACISALIHOĞLU

Professor of Mathematics The Faculty of Sciences Ankara University, Ankara, TURKEY

ABSTRACT

In this work, the homothetic Matrix Lie Group has been considered as an action group and the homothetic mapping sets have been obtained as a subset of mapping sets on E^n.

1. INTRODUCTION

Consider that G is a group and M is a differentiable manifold. As a consequence,

(a) The points on M coincide with elements of G

(b) $o : MxM \longrightarrow M$

$\quad (a,b) \longrightarrow aob^{-1}$

this operation is also differentiable in every where. (M, G) representation which has these two axioms is called a Lie Group [1].

If

$$\{[a_{ij}]_{nxn} \mid a_{ij} \in IR\}$$

is a submanifold of matrix space and a group with respect to matrix multiplication, then this group is defined as a matrix Lie Group [2].

Let M, $\bar{M}$ be n-dimentional C^∞ — manifolds and

$$\varphi : M \xrightarrow{\text{diffeomorphism}} \bar{M},$$

such that

$$\varphi_* : TM \longrightarrow T\bar{M}, \quad \forall\, x, y \in T_M(p)$$

and

$$< \varphi_*(x), \varphi_*(y)> |_{\varphi(p)} = c^2 < x, y> |_p, \text{ where } c^2 \text{ is a constant.}$$

The transformation φ which satisfies above equality is defined as a Homothetic Transformation [3].

Since homothetic transformations are free of metric choice, there is no need to any specialization in the metric.

If A is an orthogonal nxn matrix and $k = cI_n$ is a scalar matrix, then

$$H = kA,$$

is called a homothetic matrix.

The set of homothetic transformations (H(M)) is a group with respect to the operation of composition of functions. The set of homothetic matrices ($\mathcal{H}$(M)) which corresponds to the set of homothetic transformations (H(M)) is also a group with respect to matrix multiplication. Thus, the set (H(M)) which corresponds to the set ($\mathcal{H}$ (M)) is a group isomorphism [4].

The set of homothetic matrices ($\mathcal{H}$ (M)) is also a Matrix Lie Group [4].

2. MAPPING ON $\mathcal{H}$ (E^n)

Definition (Homothetic mapping): Let E^n be an n-dimentional C^∞ — manifold and (U, ψ) be a coordinate neighborhood. Then, there exist such functions;

$$f_x = \{h_1 |_x, h_2 |_x, \ldots, h_n |_x ; x\}, \forall\, x \in \psi\,(U),\ f_x \in \mathcal{H}\, B(E^n),$$

$$h_i \,|_x = \sum_{k=1}^{n} c\ a_{ki} \frac{\partial}{\partial x_k} \Big|_x .$$

The linear mapping (f_x) is called a homothetic mapping on E^n.

Theorem 1: $\{B(E^n)\ (E^n, GL\,(n,IR))\}$ is given as a main fibre set. Then, the following transformation exists:

$$V \subset E^n,\ \psi : \pi^{-1}(V) \longrightarrow V x GL(n,IR).$$

By means of above transformation, homothetic mapping converges to a homothetic matrix. In other words, every homothetic matrix indicates a homothetic mapping.

Proof: Let

$$f_x \in \mathcal{H}\, B(E^n) \ni \{h_1 |_x, h_2 |_x, \ldots, h_n |_x ; x\}$$

then, one obtains that

$$f_x \to \psi(f_x) = (x, [x_{ki}]),\ h_i|_x = \sum_{k=1}^{n} x_{ki} \frac{\partial}{\partial x_k}|_x .$$

In fact,

$$\mathcal{H}\, B(E^n) \subset B(E^n).$$

Thus, one can say that $[x_{ki}] \in GL\,(n, IR)$.

$$f_x : IR^n{}_1 \xrightarrow{\text{Lineer}} T_{E^n}(x)$$

$$f_x P = [x_{ki}]\, p$$

$$f_x P = \begin{bmatrix} \sum_{i=1}^{n} x_{1i} p_i \\ \cdot \\ \cdot \\ \cdot \\ \sum_{i=1}^{n} x_{ni} p_i \end{bmatrix} = \left(\sum_{i=1}^{n} x_{1i} p_i \right) \frac{\partial}{\partial x_i}\Big|_x + \ldots + \left(\sum_{i=1}^{n} x_{ni} p_i \right) \frac{\partial}{\partial x_n}\Big|_x$$

$$f_x P = \left(\sum_{i=1}^{n} ca_{1i} p_i \right) \frac{\partial}{\partial x}\Big|_x + \ldots + \left(\sum_{i=1}^{n} ca_{ni} p_i \right) \frac{\partial}{\partial x_n}\Big|_x .$$

Using the above equality, we can write

$$[x_{ki}] = [ca_{ki}],\ 1 \leq i,\ k \leq n$$

$$[x_{ki}] \in \mathcal{H}\, (E^n)$$

or, in other way,

$$[ca_{ki}] \in \mathcal{H}\, (E^n) \text{ is given}$$

if

$$[ca_{ki}] \in \mathcal{H}\, (E^n)$$

then

$$[ca_{ki}] \in G\ L\ (n, IR) \quad .$$

Thus,

$$\exists\, f'_x \in B\,(E^n) \ni f'_x = \{h'_1|_x, \ldots, h'_n|_x; x\}$$

where

$$h_i|_x = \sum_{k=1}^{n} ca_{ki} \frac{\partial}{\partial x_k}|_x \quad .$$

Finally, we can write

$$f'_x \in \mathcal{H}\, B(E^n) \quad .$$

Theorem 2: Let x be an any point on the n-dimentional Enclidean space E^n. If φ is a homothetic transformation of E^n then there is a radial transformation r of E^n and a rotation g around x and a sliding t(or another sliding t′) of E^n, such that

$$\varphi = t o r o g \text{ or } \varphi = r o g o t'.$$

Proof: Let an orthogonal system with initial point x at E^n be

$$\{x_1, x_1, \ldots, x_n\}$$

and a homothetic transformation be φ. Using the orthogonal system, homothetic motion, with matrix representation, will be,

$$\begin{bmatrix} y \\ 1 \end{bmatrix} = \begin{bmatrix} kA & B \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix}, \quad k = cI_n \in \sigma(n), \; A \in O(n), \; B \in \mathcal{R}^n{}_1$$

and using the fact that $D = \frac{1}{c} A^{-1} B$ one can obtain

$$\begin{bmatrix} y \\ 1 \end{bmatrix} = \begin{bmatrix} cI_n & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{c} k & D \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix}.$$

In the above equality, the first left matrix represents a scalar matrix $k = cI_n$, which gives us a radial transformation r. Second matrix defines a rotation around the point x and the third matrix indicates a sliding of E^n which is defined by

$D = \frac{1}{c} A^{-1} B$. So we can write that $\varphi = r o g o t$.

One can shows that the set of homothetic motions $\mathcal{H}(n)$ is a group with respect to the matrix multiplication.

Theorem 3:

For $x, y \in E^n$ and $f_x, f_y \in \mathcal{H} B(E^n)$ there is only one homothetic motion φ such as

$$\varphi (f_x) = f'_y.$$

Proof: Let

$$f_x = \{h_1|_x, h_2|_x, \ldots, h_n|_x; x\} \; ; \; f_y' = \{h'_1|_y, \ldots, h'_n|_y; y\} \in \mathcal{H}B(E^n)$$

where φ denotes the homothetic motion,

r denote the radial transformation,

g denotes the orthogonal transformation,

t denotes the sliding motion.

By using the theorem 2, one can write

$$\varphi = \text{torog}.$$

On the other hand, by using the technique given in [1], one obtains (in the following figure)

$$t(x) = y, \text{ when } t \in T(n),$$

similarly, for only one rog,

$$t_*^{-1}(h'_i) = (rog)_* (h_i)$$

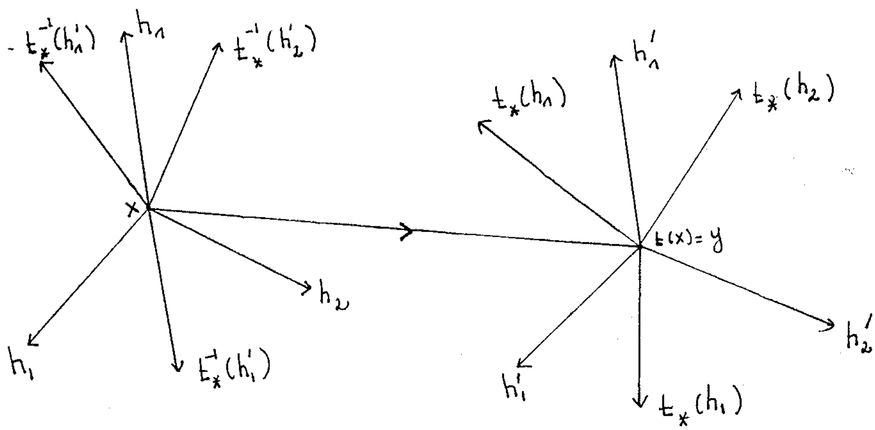
or

$$h'_i = t_*(rog)_* (h_i)$$
$$h'_i = (torog)_* (h_i)$$
$$h'_i = \varphi_* (h_i).$$

Thus we can write that

$$\varphi(\{h_1|_x,\ldots,h_n|_x;\ x\}) = \{\varphi_*(h_1|\varphi(x)),\ldots,\ \varphi_*(h_n|\varphi(x));\ \varphi(x)\}$$
$$= \{h'_1|_y,\ldots,h'_n|_y;y)\}$$
$$\varphi(f_x) = f'_y\,.$$

This result shows us the availability of a homothetic motion φ and its singularity.



(figure)

REFERENCES

1. HACISALİHOĞLU, H. Hilmi., *Yüksek Diferensiyel Gometriye Giriş* Fırat University, Elazığ, 1980.

2. AUSLANDER, L., *Differential Geometry* A Harper International Edition Harper Row. New–York, 1971.

3. GUGGENHEIMER, H.W., *Differential Geometry of Liberal Arts.* Universty of Minnesota, 1963.

4. ARSLAN, I., *Homotetik Matris Lie Grupları* Master dissertation, Ankara 1980.