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ON REARRANGEMENTS OF INFINITE SERIES

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ABSTRACT

In this paper, we proved the converse of Riemann's theorem and then applied it to Cauchy product series of alternating series of real terms. Moreover, we showed that the concept of unconditionally convergence of infinite series can be replaced by the boundedness of sequence of partial sums of every rearranged series of series.

1. INTRODUCTION

In this paper, the word permutation will be used to denote any function $p : N \to N$, whose domain and range is N the natural numbers, and that is also one to one. The set of all permutations will be denoted by the symbol S(N). Let $\sum_{n=1}^{\infty} a_n$ be a series of real terms. Put

$$N_+ = \{n \in N: a_n \geq 0\}, \qquad N_- = \{n \in N: a_n < 0\}.$$

A series $\Sigma\, a_n$ is said to be of the type (α) or (β), if $\sum_{n \in N_+} a_n = \infty$ and $\sum_{n \in N_-} |a_n| < \infty$, or $\sum_{n \in N_+} a_n < \infty$ and $\sum_{n \in N_-} a_n = -\infty$, respectively, And the same series is said to be of the type (γ) if $\sum_{n \in N_+} a_n = \infty$ and $\sum_{n \in N_-} a_n = -\infty$ [3].

We call that Σa_n is said to be of the type (γ_0) if it is of the type (γ) and $a_n \to 0$ as $n \to \infty$.

The following theorem, discovered by Riemann in 1849, can be found in most standart text books on Advanced Calculus, for example [1],

p. 368. Theorem 1.1. Suppose a series of real terms $\sum_{n=1}^{\infty} a_n$ if of the type (γ_0). Suppose further that x and y are numbers in the closed interval $[-\infty, \infty]$ with $x \leq y$. Then, there exists $p \in S(N)$ for which

$$\liminf_{n\to\infty} \sum_{i=1}^{n} a_{p(i)} = x \text{ and } \limsup_{n\to\infty} \sum_{i=1}^{n} a_{p(i)} = y.$$

This result was generalized by Öztürk [4]. Besides it has been characterized in [6], [7] the permutation functions preserving convergence and even divergence of series, by defining the absolute equivalence of permutations.

Definition 1.1. A series $\sum_{n=1}^{\infty} a_n$ of real or complex numbers is said to be unconditionally convergent if $\sum_{n=1}^{\infty} a_{p(n)}$ is convergent for each $p \in S(N)$ [5].

Theorem 1.2. A series of real or complex numbers is unconditionally convergent if and only if it is absolutely convergent. [5].

The purpose of this paper is to prove the converse of Theorem 1.2, and to apply it to Cauchy product series of alternating series, (and in addition to these, to show that convergence of $\sum_{n=1}^{\infty} a_{p(n)}$ for each $p \in S(N)$ in Definition 1.1 can be replaced by boundedness of sequence $\left(\sum_{i=1}^{n} a_{p(i)}\right)$ for each $p \in S(N)$.

2. We shall now prove the following theorems.

Theorem 2.1. Suppose x and y are numbers in the closed interval $[-\infty, \infty]$ with $x \leq y$. If there exists $p \in S(N)$ for which

$$\liminf_{n\to\infty} \sum_{i=1}^{n} a_{p(i)} = x \text{ and } \limsup_{n\to\infty} \sum_{i=1}^{n} a_{p(i)} = y,$$

then, Σa_n is of the type (γ_0).

Proof. Suppose that the series $\sum_{n=1}^{\infty} a_n$ is not of the type (γ_0). In this case, there are two cases to consider for the sequence (a_n),

a) (a_n) has the terms with the same sign after a certain step or,

b) (a_n) has infinitely many positive real numbers and infinitely many negative real numbers.

In the case of (a), the series is either absolutely convergent or $\sum_{n=1}^{\infty} a_n = +\infty$. Because its sequence of partial sums is monotone after a certain step. This contradicts the hypothesis of the theorem. In the case of (b), the following cases occur.

(i) $\sum_{n\in N_+} a_n < \infty, \quad \sum_{n\in N_-} a_n = -\infty, \; a_n \to 0$ as $n \to \infty$,

(ii) $\sum_{n\in N} |a_n| < \infty$

(iii) $\sum_{n\in N_+} a_n < \infty, \quad \sum_{n\in N_-} a_n = -\infty, \; a_n \not\to 0$ as $n \to \infty$

(iv) $\sum_{n\in N_+} a_n = \infty \quad \sum_{n\in N_-} a_n > -\infty, \; a_n \to 0$ as $n \to \infty$,

(v) $\sum_{n\in N_+} a_n = \infty, \quad \sum_{n\in N_-} |a_n| < \infty, \; a_n \not\to 0$ as $n \to \infty$

(vi) $\sum_{n\in N_+} a_n = \infty \quad \sum_{n\in N_-} a_n = -\infty, \; a_n \not\to 0$ as $n \to \infty$.

It is clear that all of the cases mentioned above contradict hypothesis of the theorem. Thus, it must be of the type (γ_0), completing the proof.

Hence, we can write the following result by combining Theorem 1.1 and Theorem 2.1.

Corollary 2.2. Let $\sum_{n=1}^{\infty} a_n$ be a series of real numbers. Then, for any $x, y \in [-\infty, \infty]$ with $x \leq y$, there exists a permutation $p \in S(N)$ such that

$$\liminf_{n\to\infty} \sum_{i=1}^{n} a_{p(i)} = x \text{ and } \limsup_{n\to\infty} \sum_{i=1}^{n} a_{p(i)} = y$$

if and only if it is of the type (γ_0).

Theorem 2.3. Assume $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ and $\sum_{n=1}^{\infty} (-1)^{n+1} b_n$ be two alternating series of the type (γ_0) such that $n^s a_n = O(1)$ and $n^z b_n = O(1)$, where $\frac{1}{2} < s, z \leq 1$. Then for any $x, y \in [-\infty, \infty]$ with $x \leq y$, there exists $p \in S(N)$ for which

$$\liminf_{n\to\infty} \sum_{i=1}^{n} c_{p(i)} = x \text{ and } \limsup_{n\to\infty} \sum_{i=1}^{n} c_{p(i)} = y$$

where $\sum_{n=1}^{\infty} c_n$ is the series of Cauchy product of the given series.

Proof. Since, for each $n \in N$,

$$c_n = \sum_{i=1}^{n} (-1)^{i+1} a_i (-1)^{n-i+2} b_{n-i+1}$$

$$= (-1)^{n+1} \sum_{i=1}^{n} a_i b_{n-i+1},$$

$\sum_{n=1}^{\infty} c_n$ is an alternating series. On the other hand, it is clear that

$$\sum_{n\in N_+} c_n = \sum_{k=1}^{\infty} c_{2k-1} = \sum_{k=1}^{\infty} \sum_{i=1}^{2k-1} a_i b_{2k-i}$$

and

$$\sum_{n\in N_-} c_n = \sum_{k=1}^{\infty} -c_{2k} = \sum_{k=1}^{\infty} \sum_{i=1}^{2k} a_i b_{2k-i+1}.$$

We now want to show that Σc_n is of the type (γ_0). First of all, let us show that it is of the type (γ). It is easy to calculate that

$$s^+_n = \sum_{k=1}^{n} \sum_{i=1}^{2k-1} a_i b_{2k-i}$$

$$= \sum_{i=1}^{n} a_{2i-1} \sum_{k=i}^{n} b_{2(k-i)+1} + \sum_{i=1}^{n-1} a_{2i} \sum_{k=i}^{n-1} b_{2(k-i)+2}$$

and

$$s^-_n = \sum_{i=1}^{n} \sum_{k=i}^{2k} a_i b_{2k-i+1}$$

$$= \sum_{i=1}^{n} a_{2i-1} \sum_{k=i}^{n} b_{2(k-i)+2} + \sum_{i=1}^{n} a_{2i} \sum_{k=i}^{n} b_{2(k-i)+1}.$$

By considering above equalities, we can write $\lim s^{+}_{n} = \lim s^{-}_{n} = \infty$ since the given series are of the type (γ). Thus Σc_n is of the type (γ). Finally we show that $\lim_n c_n = 0$. Since $k^s a_k = O(1)$ and $k^z b = O(1)$, it can be written, for each n,

$$|c_n| = \sum_{k=1}^{n} a_k b_{n-k+1} = O(1) \sum_{k=1}^{n} \frac{1}{k^s (n-k+1)^z} .$$

Now, denote min $\{s, z\}$ by r. Then, we have

$$|c_n| = O(1) \sum_{k=1}^{n} \left(\frac{1}{k(n-k+1)}\right)^r = O(1) \frac{1}{(n+1)^r} \sum_{k=1}^{n} \left(\frac{1}{k} + \frac{1}{n-k+1}\right)^r.$$

On the other hand, by the fact that $|a+b|^r \leq |a|^r + |b|^r \ (\frac{1}{2} < r \leq 1)$

$$|c_n| = O(1) \frac{1}{(n+1)^r} \sum_{k=1}^{n} \left(\frac{1}{k^r} + \frac{1}{(n-k+1)^r}\right)$$

$$= O(1) \frac{1}{(n+1)^r} \sum_{k=1}^{n} \frac{1}{k^r}$$

If $r=1$ and $\frac{1}{2} < r < 1$, we obtain

$$|c_n| = O(1) \frac{1}{n+1} \sum_{k=1}^{n} \frac{1}{k} = O(1) \frac{\ln(n+1)}{n+1} \to 0 \text{ as } n \to \infty,$$

and

$$|c_n| = O(1) \frac{1}{(n+1)^r} \sum_{k=1}^{n} \frac{1}{k^r} = O(1) \frac{1}{(n+1)^{2r-1}} \to 0 \text{ as } n \to \infty,$$

respectively, which implies $c_n \to 0$ as $n \to \infty$. Thus Σc_n if of the type (γ_0). This step completes the proof of the theorem together with theorem 1.1.

We shall now show that the convergence of the series $\sum_{k=1}^{\infty} a_{p(k)}$ in Definition 1.1 can be replaced by the boundedness of the sequence $\left(\sum_{k=1}^{n} a_{p(k)}\right)$ which is weaker than the convergence of $\left(\sum_{k=1}^{n} a_{p(k)}\right)$ as a result of the following theorem.

Theorem 2.4. A series $\sum_{k=1}^{\infty} a_k$ of real terms is absolutely convergent if and only if the sequence $\left(\sum_{k=1}^{n} a_{p(k)}\right)$ is bounded for every $p \in S(N)$.

Proof. Necessity. Let Σa_k of real terms be an absolutely convergent series having sum S. Then, since, for every $p \in S(N)$, $\sum_{k=1}^{\infty} a_{p(k)}$ also converges absolutely and has sum S (see [2]), the sequence $\left(\sum_{k=1}^{n} a_{p(k)}\right)$ converges to S and, so that it is also bounded for every $p \in S(N)$.

Sufficiency. Let $\left(\sum_{k=1}^{n} a_{p(k)}\right)$ is bounded for every $p \in S(N)$. Suppose that on contrary, $\sum_{k=1}^{\infty} |a_k| = \infty$. Then Σa_k is either conditionally convergent or divergent. If it is conditionally convergent, then it also of the type (γ_0), so that, by Riemann's theorem, there exists a permutation $p \in S(N)$ such that $\sum_{k=1}^{\infty} a_{p(k)} = \infty$, that is, $\lim \sum_{k=1}^{n} a_{p(k)} = \infty$, which contradicts our assumption. Suppose now that the series $\sum_{k=1}^{\infty} a_k$ is divergent. Then, for (a_k), there exist two cases to consider:

a) (a_k) has the same signs after a certain step

or,

b) (a_k) has infinitely many positive numbers and infinitely many negative numbers.

In the first case, $\Sigma a_k = \pm \infty$. In the second case, the series is of type (β) or of the type (α) of of the type (γ). If it is either of the type (β) or of the type (α), $\Sigma a_k = \pm \infty$. If it is of the type (γ), then $\sum_{k=1}^{\infty} a_{p(k)} = \infty$ for some $p \in S(P)$. In fact, let us denote the series of positive numbers and the negative numbers of Σa_k by $\sum_{k=1}^{\infty} a_k^{+}$ and

$\sum_{k=1}^{\infty} a_k^-$, respectively. Then, there exists on $m_1 \in N$ such that

$$\sum_{k=1}^{m_1} a_k^+ > 1 - a_1^-$$

since $\sum_{k=1}^{\infty} a_k^+ = \infty$. Now chose a number m_2 such that

$$m_1 < m_2 \text{ and } \sum_{k=1}^{m_2} a_k^+ > 2 - a_1^- - a_2^-.$$

By continuing in this way, in general, we can establish the positive integers m_v provided that $m_v > m_{kv-1}$ $(v = 2, 3, ...)$ for which

$$\sum_{k=1}^{m_v} a_k^+ > v - a_1^- - a_2^- - ... - a_v^-.$$

Thus $(a_1^+, ..., a^+_{m_1}, a_1^-, a^+_{m_1+1}, ..., a^+_{m_2}, a_2^-, ..., a^+_{m_{v-1}+1}, ..., a^+_{m_v}, a_v^-, ...)$

is a rearrangement of (a_k). Now if we denote its sequence of partial sums by (s_n), then (s_{m_v+v}) is a subsequence of (s_n) and, for each $v \in N$

$$s_{m_v+v} = \sum_{k=1}^{m_v} a_k^+ + \sum_{k=1}^{v} a_k^- > v.$$

Therefore $\sup_n s_n = \infty$. This contradicts our assumption, too. Hence,

$\Sigma |a_k| < \infty$, completing the proof of the theorem.

By considering Theorem 1.2 and the above theorem we can give the following main result which introduces an alternative result for unconditionally convergence (Def. 1.1).

Corollary 2.5. A series Σa_k of real terms is unconditionally convergent if and only if the sequence $\left(\sum_{k=1}^{n} a_{p(k)}\right)$ is bounded for every $p \in S(N)$.

ÖZET

Bu çalışmada, Riemann Teoremi'nin tersi ispat edilerek, reel terimli alterne serilerin Cauchy çarpımları serisine uygulanmıştır. Ayrıca, serilerin şartsız yakınsaklık kavramının, yeniden düzenlenmiş serilerin kısmi toplamlar dizisinin sınırlılılığı ile değiştirilebileceği gösterilmiştir.

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