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ON THE 1-PARAMETER LORENTZIAN MOTIONS

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ABSTRACT

Birman and Nomizu studied trigonometry on the Lorentzian plane [1—2], and Yaglom defined rotation and motion for that plane [6].

In this paper we studied the l-parameter motion on the Lorentzian plane and obtained the properties of this motion resembling to the Euclidean plane.

INTRODUCTION

1– Parameter motion on the Euclidean plane is known very well [4]. The velocities are defined in Section 1 and relations between them are obtained in the sense of Lorentz. Section 2 includes centrodes and their properties. Accelerations are studied in the last section.

Lorentzian plane is a real two-dimensional vector space which is equipped with the inner product

$$<x, y>_L = x_1y_1 - x_2y_2$$

where $x = (x_1, x_2)$, $y = (y_1, y_2) \in R^2$. The Lorentzian plane is represented by L_2 or for the sake of shortness only by L. And we will use notation "LM", which will appear frequently in this paper, instead of "In the sense of Lorentz". Also L/L', will be used as the motion of L according to L′ where L and L′ are moving and fixed Lorentzian planes, respectively.

I. 1- PARAMETER MOTIONS ON THE LORENTZIAN PLANE

I. 1. Derivative Formulas

Let $\{O, \vec{\iota}_1, \vec{\iota}_2\}$ and $\{O', \vec{\iota}_1{}', \vec{\iota}_2{}'\}$ be moving and fixed coordinate frames of L and L′, respectively. Thus

$$\vec{OO'} = \vec{u} = \vec{\iota}_1 u_1 + \vec{\iota}_2 u_2, \ (u_1, u_2 \in R). \tag{1}$$

At the initial time $t = t_0$, let's consider O and O′ are coincident. So we obtain;

$$\left.\begin{aligned} \vec{\iota}_1 &= \vec{\iota}'_1 \operatorname{ch} \varnothing + \vec{\iota}'_2 \operatorname{sh} \varnothing \\ \vec{\iota}_2 &= \vec{\iota}'_1 \operatorname{sh} \varnothing + \vec{\iota}'_2 \operatorname{ch} \varnothing \end{aligned}\right\} \tag{2}$$

where $\varnothing$ is LM rotational angle.

Definition I.1.1: If the functions $u_1 = u_1(t)$, $u_2 = u_2(t)$ and $\varnothing = \varnothing(t)$ have the same domain as $t_0 \leq t \leq t_\nu$ then 1-parameter LM motion of **L** on **L′** is defined.

From (1) and (2) the LM derivative formulas of the **L/L′** are obtained as follows

$$\left.\begin{aligned} \dot{\vec{\iota}}_1 &= \vec{\iota}_2 \dot{\varnothing} \\ \dot{\vec{\iota}}_2 &= \vec{\iota}_1 \dot{\varnothing} \\ \dot{\vec{u}} &= \vec{\iota}_1 (\dot{u}_1 + u_2 \dot{\varnothing}) + \vec{\iota}_2 (\dot{u}_2 + u_1 \dot{\varnothing}) \end{aligned}\right\} \tag{3}$$

where "." denotes the derivation with respect to "t".

I. 2 Velocities

Definition I.2.1: Let $X = \vec{\iota}_1 x_1 + \vec{\iota}_2 x_2$ be a moving point of **L**. The velocity of X with respect to **L** is known as LM relative velocity of X. And it is shown by $\vec{V}_r$.

By the definition above,

$$\vec{V}_r = \vec{\iota}_1 \dot{x}_1 + \vec{\iota}_2 \dot{x}_2. \tag{4}$$

Definition I.2.2: Let X be a fixed point of L. The velocity of X according to L′ will be known as LM sliding velocity of X. And it is shown by $\vec{V}_f$.

By the definition above we obtain $\vec{V}_f$ as follows

$$\vec{V}_f = \vec{\iota}_1 \{-\dot{u}_1 - (u_2 - x_2) \dot{\varnothing}\} + \vec{\iota}_2 \{-\dot{u}_2 - (u_1 - x_1) \dot{\varnothing}\}. \tag{5}$$

Definition I. 2.3: Let X be a moving point in **L**. The velocity of X according to **L′** is defined as LM absolute velocity. And it is shown by $\vec{V}_a$.

Theorem I.2.1: Let X be a moving point in **L** and $\vec{V}_r$, $\vec{V}_a$ and $\vec{V}_f$ the relative, absolute and sliding velocities of X, respectively. Then

$$\vec{V}_a = \vec{V}_f + \vec{V}_r \ .$$

The proof is obvious by using the definitions of velocities above.

Result I.2.1: Let X be a fixed point in **L**, then

$$\vec{V}_a = \vec{V}_f \ .$$

II. CENTRODES

Definition II. 1: Let $\varnothing$ be the LM rotation angle of **L / L′**. Then,

$$\frac{d\varnothing}{dt} = \dot{\varnothing}$$

will be defined as angular velocity of the LM motion.

We assume that $\dot{\varnothing} \neq 0$ for the LM motion. That is the LM motion is not only a translation.

Now we will investigate the points, at which the $\vec{V}_f$ is vanish for every $t \in [t_0, t_1]$. It gives us permission to obtain the concept of rotation pole for the LM motion.

Theorem II.1: If angular velocity is not zero, then there is a unique point whose sliding velocity is zero for every $t \in [t_0, t_1]$.

Proof: If $\vec{V}_f = \vec{0}$ then using (5) we obtain the unique point $P = (p_1, p_2)$ such as

$$p_1 = u_1 + \frac{\dot{u}_2}{\dot{\varnothing}} \ , \ p_2 = u_2 + \frac{\dot{u}_1}{\dot{\varnothing}} \ . \qquad (6)$$

So that the point P is fixed in the two **L** and **L′** planes at the same time. As a result, we can give the following definition.

Definition II. 2: The point P, obtained from Theorem II. 1 is defined as the rotation pole or the instantanious rotation pole centre of the 1-parameter lorentzian motion **L / L′**.

Theorem II. 2: Let P be rotation pole of **L / L′** and X be a moving point of **L′** then $\vec{PX}$ and $\vec{V}_f$ are LM perpendicular vectors to each other.

Proof: By using (5) we obtain;

$$\dot{u}_1 = (p_2 - u_2)\,\dot{\varnothing} \quad \text{and} \quad \dot{u}_2 = (p_1 - u_1)\,\dot{\varnothing}\ .$$

Therefore a new expression is obtained for $\vec{V}_f$ as

$$\vec{V}_f = \{(x_2 - p_2)\,\vec{\iota}_1 + (x_1 - p_1)\,\vec{\iota}_2\}\,\dot{\varnothing}.$$

On the other hand

$$\vec{PX} = (x_1 - p_1)\,\vec{\iota}_1 + (x_2 - p_2)\,\vec{\iota}_2\ ,$$

it is clear that,

$$<\vec{PX},\ \vec{V}_f>_L = 0\ .$$

Result II. 1: In a **L / L′** LM motion, the focus of X point of **L** is an orbit that it's normals pass thiough the rotation pole P.

Theorem II. 3: Let X be a moving point in **L** and P be rotation pole of the **L / L′** motion, then

$$\|\vec{V}_f\|_L = |\dot{\varnothing}|\ .\ \|\vec{PX}\|_L\ .$$

Definition II. 3: The orbit of rotation pole P, for each $t \in [t_0, t_1]$, of the **L** plane is named as movable pole curve. And the orbit of P on the **L′** plane is named as stable pole curve. And they are shown as (P) and (P′), respectively.

Theorem II. 4: The velocities of (P) and (P′) are the same for each $t \in [t_0, t_1]$.

Proof: The point P is the solution of the equation $\vec{V}_f = \vec{O}$. So the equality given at the Theeren I.2.1. becames

$$\vec{V}_a = \vec{V}_r$$

which completes the proof of the theorem.

Result II. 2: During the motion **L/L′**, (P) and (P′) roll, without sliding, upon each other.

III. ACCELERATIONS

In this section we will define LM relative, absolute, sliding and Coriolis acceleration vectors. Mentioned vectors above will be represented $\vec{b}_r$, $\vec{b}_a$, $\vec{b}_f$ and $\vec{b}_c$, respectively.

Definition III. 1: Let L and L′ be movable and fixed Lorentzian planes, respectively and X be a moving point in L and $\vec{V}_r$ be the relative velocity vector of X. So derivating $\vec{V}_r$ according to t, we obtain LM relative acceleration vector $\vec{b}_r$ as:

$$\vec{b}_r = \dot{\vec{V}}_r = \vec{\iota}_1 \ddot{x}_1 + \vec{\iota}_2 \ddot{x}_2$$

where $X = x_1 \vec{\iota}_1 + x_2 \vec{\iota}_2$ and $\vec{V}_r = \vec{\iota}_1 \dot{x}_1 + \vec{\iota}_2 \dot{x}_2$.

$\dot{\vec{V}}_a = \vec{b}_a$ is the LM absolute acceleration vector of X according to the fixed Lorentzian plane L′.

Now let's consider that X is a fixed point in L, then the acceleration vector of X according to L′ is named LM sliding acceleration vector of X and

$$\vec{b}_f = \dot{\vec{V}}_f = -\vec{\iota}_1 \{\dot{p}_2 \dot{\varnothing} + (p_1 - x_1)\, \dot{\varnothing}^2 + (p_2 - x_2)\, \ddot{\varnothing}\} - \vec{\iota}_2 \{\dot{p}_1 \dot{\varnothing} + (p_2 - x_2)\, \dot{\varnothing}^2 + (p_1 - x_1)\, \ddot{\varnothing}\}.$$

Now let X be a moving point in the moving Lorentzian plane **L**, then

$$\vec{b}_a = \dot{\vec{V}}_a = (\vec{V}_f + \vec{V}_r)^{\cdot} = \dot{\vec{V}}_f + \dot{\vec{V}}_r$$

and so

$$\vec{b}_a = -\vec{\iota}_1 \{p_2 \dot{\varnothing} + (p_1 - x_1) \dot{\varnothing}^2 + (p_2 - x_2) \ddot{\varnothing}\} - \vec{\iota}_2 \{p_1 \dot{\varnothing} + (p_2 - x_2) \dot{\varnothing}^2 + (p_1 - x_1) \ddot{\varnothing}\} + 2\dot{\varnothing} \{\vec{\iota}_1 \dot{x}_2 + \vec{\iota}_2 \dot{x}_1\} + \vec{\iota}_1 \ddot{x}_1 + \vec{\iota}_2 \ddot{x}_2$$

where

$$\vec{b}_c = 2\dot{\varnothing} \{\vec{\iota}_1 \dot{x}_2 + \vec{\iota}_2 \dot{x}_1\} \tag{7}$$

will be named LM Coriolis acceleration vector of X. So we can give the following theorem.

Theorem III. 1: Let X be a moving point in **L** then,

$$\vec{b}_a = \vec{b}_f + \vec{b}_c + \vec{b}_r \ .$$

Result III. 1: If X a fixed point of **L** in the **L/L′** motion then,

$$\vec{b}_a = \vec{b}_f \ .$$

Theorem III. 2: The LM $\vec{b}_c$ Coriolis acceleration vector and $\vec{V}_r$ relative velocity vector are perpendicular to each other.

Proof: As we know from (4) and (7)

$$\vec{V}_r = \vec{\iota}_1 \dot{x}_1 + \vec{\iota}_2 \dot{x}_2 \ ,$$

$$\vec{b}_c = 2\dot{\varnothing} \{\vec{\iota}_1 \dot{x}_2 + \vec{\iota}_2 \dot{x}_1\} \ .$$

So it is obvious that

$$< \vec{V}_r, \vec{b}_c >_L = 0 \ .$$

Theorem III. 3: Let X be a moving point in **L** and $\vec{b}_c = \vec{O}$ then **L / L′** motion is only a slide and vice versa.

Proof: Because of $\vec{b}_c = \vec{O}$ then

$$2\dot{\varnothing} \{\vec{\iota}_1 \dot{x}_2 + \vec{\iota}_2 \dot{x}_1\} = \vec{0}$$

and then,

$$\dot{\varnothing} = 0 \ .$$

So $\varnothing$ is constant. That is $\mathbf{L}/\mathbf{L}'$ must be a slide.

The other side of the theorem is obvious.

The point at which $\vec{b}_f = \vec{O}$ provides us the LM acceleration pole concept for $\mathbf{L}/\mathbf{L}'$ motion.

Theorem III. 4: If $\dot{\varnothing}^4 - \ddot{\varnothing}^2 \neq 0$ and the LM pole point at a "t" time is $P = (p_1, p_2)$, then at the same "t" time the LM acceleration pole point's coordinates are

$$x_1 = p_1 + \frac{(\dot{p}_2 \dot{\varnothing}^2 - \dot{p}_1 \ddot{\varnothing})\, \varnothing}{\dot{\varnothing}^4 - \ddot{\varnothing}^2}, \quad x_2 = p_2 + \frac{(\dot{p}_1\, \varnothing^2 - p_2\, \ddot{\varnothing})\, \dot{\varnothing}}{\dot{\varnothing}^4 - \ddot{\varnothing}^2} .$$

Proof: By the explanation before the theorem, $\vec{b}_f$ must be zero. So

$$-\vec{\iota}_1 \{\dot{p}_2 \dot{\varnothing} + (p_1 - x_1) \dot{\varnothing}^2 + (p_2 - x_2) \ddot{\varnothing}\} - \vec{\iota}_2 \{\dot{p}_1 \dot{\varnothing} + (p_2 - x_2) \dot{\varnothing}^2 + (p_1 - x_1) \ddot{\varnothing}\} = 0$$

and then

$$\left.\begin{aligned} \dot{p}_2\, \dot{\varnothing} &= (x_1 - p_1)\, \dot{\varnothing}^2 + (x_2 - p_2)\, \ddot{\varnothing} \\ \dot{p}_1\, \dot{\varnothing} &= (x_2 - p_2)\, \dot{\varnothing}^2 + (x_1 - p_1)\, \ddot{\varnothing} \end{aligned}\right\} \qquad (8)$$

is obtained. Since the coefficient determinant of (8) is $\dot{\varnothing}^4 - \ddot{\varnothing}^2$ and different from zero, we have the solution of the system as

$$x_1 = p_1 + \frac{(\dot{p}_2 \dot{\varnothing}^2 - \dot{p}_1 \ddot{\varnothing})\, \dot{\varnothing}}{\dot{\varnothing}^4 - \ddot{\varnothing}^2}, \quad x_2 = p_2 + \frac{(\dot{p}_1 \dot{\varnothing}^2 - \dot{p}_2 \ddot{\varnothing})\, \dot{\varnothing}}{\dot{\varnothing}^4 - \ddot{\varnothing}^2} .$$

ÖZET

Birman ve Nomizu Lorentz düzlemi üzerinde trigonometri çalıştılar [1–2]. Yaglom bu düzlem üzerinde dönmeyi ve hareketi tanımladı [6].

Biz bu çalışmada Lorentz düzlemi üzerinde l-parametreli hareketleri çalıştık ve Öklid düzlemindeki l-parametreli hareketler için var olan özelliklerin benzerlerini Lorentzian hareketler için elde ettik.

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